



## RAYCHAUDHURI EQUATIONS IN TWO DIMENSIONS FLAT SPACE BACKGROUND

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**Abstract:** The kinematics of deformation in two dimensions in flat space background is investigated by solving the evolution equation for expansion scalar, shear, and rotation (i.e., ESR variables). The evolution equation i.e., Raychaudhuri equations along with the procedure to solve these equations are discussed explicitly. We analyze the initial conditions of the above-mentioned kinematical quantities (i.e., expansion scalar, shear, and rotation (vorticity)). The conditions of convergence and divergence of expansion scalar are also examined in detail.

**Keywords:** Deformation vector, expansion scalar, shear, rotation, Raychaudhuri equations.

### I. Introduction

To understand the phenomenon of deformation [1, 2] in a two-dimension medium, one needs to analyze the behavior of kinematical quantities which are expansion scalar, shear, and rotation (i.e., ESR variables) [3, 4]. In this paper, first, there is a representation of the deformation vector in terms of ESR variables. While analyzing the kinematics of such deformation we confront a local, first-order, non-linear, and coupled set of equations for the above-mentioned kinematical quantities known by the Raychaudhuri equations [1, 3]. The Raychaudhuri equations are well recognized in the study of space-time singularity in gravitation and cosmology [5]-[12]. Its analysis for expansion leads to the focusing of geodesics [2]. The Indian physicist Amal Kumar Raychaudhuri independently discovered these equations [1]. These equations are also used in the proof of singularity Theorem [7,13,14-17] (through the notion of focusing on geodesics).

In this paper, efforts have been made to highlight another way of solving Raychaudhuri equations in an explicitly different manner. The plan of this paper is as follows. After introduction in section I, the Raychaudhuri equations in two-dimensional curved spaces are derived in section II. In section III, we acquire the solution of Raychaudhuri equations by implementing the techniques of variable separable method and the method of solving

homogeneous differential equation [6,15]. Subsequently, we focus on the initial values of the variables. Further, in section I, we will emphasize exploring the condition of convergence and divergence of expansion scalar. Finally, the results obtained are summarized in Section V.

## II: The Evolution of Raychaudhuri Equation

The Raychaudhuri equations can be derived by looking at the evolution of the deformation vector [3]. A deformation vector can be represented by  $\xi^j$  which characterizes the vector joining any two infinitesimally separated points of the medium [3]. The time rate of change of deformation vector is

$$\frac{d\xi^i}{dt} = B_j^i(t)\xi^j + O(\Delta t^2) \quad (1)$$

Here  $B_j^i$  is an arbitrary second rank tensor specifying the time evolution of the deformation vector. We differentiate equation (1) once again with time and using equation (1) subsequently, we obtain

$$\ddot{\xi}(i) = (B_j^i + B_k^i + B_j^k)\xi^j \quad (2)$$

The general expression for the evolution tensor can be described in terms of expansion scalar ( $\theta$ ), shear ( $\sigma_{ij}$ ) and rotation ( $\omega_{ij}$ ) as,

$$B_{ij} = \frac{1}{2}\theta\delta_{ij} + \sigma_{ij} + \omega_{ij}$$

Here isotropic expansion is trace scalar, shear is a symmetric traceless tensor and rotation is an antisymmetric tensor as described below [3],

$$\frac{1}{2}\theta\delta_{ij} = \begin{pmatrix} \frac{1}{2}\theta & 0 \\ 0 & \frac{1}{2}\theta \end{pmatrix} \quad \sigma_{ij} = \begin{pmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_1 \end{pmatrix} \quad \omega_{ij} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

Here  $\sigma_1$  and  $\sigma_2$  are the shear components while  $\omega$  is the only rotation component [3]. The governing equation for evolution tensor further reads as,

$$\dot{B}_k^i B_j^k = 0$$

which leads to the following set of four equations for kinematic variables,

$$\frac{d\theta}{ds} + \frac{\theta^2}{2} + 2(\sigma_1^2 + \sigma_2^2 - \omega^2) = 0 \quad (3)$$

$$\frac{d\sigma_1}{ds} + \theta\sigma_1 = 0 \quad (4)$$

$$\frac{d\sigma_2}{ds} + \theta\sigma_2 = 0 \quad (5)$$

$$\frac{d\omega}{ds} + \theta\omega = 0 \quad (6)$$

These are the Raychaudhuri equations for expansion scalar, shear, and rotation in two dimensions.

### III: The Solution of Raychaudhuri Equation

The Raychaudhuri equations. (3) -(6) are non-linear differential equation. We apply the method of variable separable to solve these equations as discussed below:

From equations (4) and (5), we obtain

$$\frac{d\sigma_1}{d\sigma_2} = \frac{-\theta\sigma_1}{-\theta\sigma_2} \Rightarrow \frac{d\sigma_1}{d\sigma_2} = \frac{\sigma_1}{\sigma_2} \quad (7)$$

we acquire the condition  $\sigma_1 = C_1\sigma_2$  and after solving equations (5) and (6) in the same way, we arrived at the following equations

$$\begin{aligned} \sigma_2 &= C_2\omega \\ \sigma_1 &= C_1 C_2\omega \end{aligned} \quad (8)$$

Equation (3) can be rewritten to the following equation after the replacement of values from equation (8)

$$\frac{d\theta}{ds} + \frac{\theta^2}{2} + 2C_3\omega^2 = 0 \quad (9)$$

$$\text{With, } C_3 = C_1C_2^2 + C_2^2 - 1$$

Here  $C_1$ ,  $C_2$  and  $C_3$  are arbitrary integral constants. The equation (3) can be rewritten in the following form

$$\frac{d\theta}{ds} = -\left(\frac{\theta^2 + 4C_3\omega^2}{2}\right) \quad (10)$$

The equation (10) along with equation (6), we have,

$$\frac{d\omega}{d\theta} = \left(\frac{2\theta\omega}{\theta^2 + 4C_3\omega^2}\right) \quad (11)$$

Here we acquire a homogeneous differential equation (11) in terms of expansion scalar and rotation which can be solved specifically [6] to obtain the value of expansion scalar in terms of rotation as below:

$$\theta = \pm \sqrt{\frac{\omega}{C_4} + 4C_3\omega^2} \quad (12)$$

where  $C_4$  is a new constant of integration. As we have  $\frac{d\omega}{ds} = -\theta\omega$  and after using equation (12)

$$\frac{d\omega}{ds} = -\omega \sqrt{\frac{\omega}{C_4} + 4C_3\omega^2}$$

Integrating both the sides leads to,

$$\omega = \frac{4C_4}{(s+C_5)^2 - 16C_3C_4^2} \quad (13)$$

Here  $C_5$  is another constant. The expansion scalar and component of shear [16] may then be obtained accordingly as below:

$$\theta = \frac{2(s+C_5)}{(s+C_5)^2 - 16C_3C_4^2} \quad (14)$$

$$\sigma_1 = \frac{4C_1C_2C_4}{(s+C_5)^2 - 16C_3C_4^2} \quad (15)$$

$$\sigma_2 = C_2\omega, \quad (16)$$

$$\sigma_2 = \frac{4C_2C_4}{(s+C_5)^2 - 16C_3C_4^2} \quad (17)$$

The initial conditions on ESR variables with  $s=0$ , reads as below:

$$\theta_0 = \frac{2C_5}{C_5^2 - 16C_3C_4^2} \quad (18)$$

$$\sigma_{10} = \frac{4C_1C_2C_4}{C_5^2 - 16C_3C_4^2} \quad (19)$$

$$\sigma_{20} = \frac{4C_2C_4}{C_5^2 - 16C_3C_4^2} \quad (20)$$

$$\omega_0 = \frac{4C_4}{C_5^2 - 16C_3C_4^2} \quad (21)$$

The integration constants  $C_1, C_2$  may however be written in terms of the initial values of ESR variables as below:

$$C_1 = \frac{\sigma_{10}}{\sigma_{20}} \quad (22)$$

$$C_2 = \frac{\sigma_{20}}{\omega_0} \quad (23)$$

$$C_3 = \frac{\sigma_{10}^2 + \sigma_{20}^2 - \omega_0^2}{\omega_0^2} \quad (24)$$

$$C_4 = \frac{\omega_0}{\theta_0^2 - 4(\sigma_{10}^2 + \sigma_{20}^2 - \omega_0^2)} \quad (25)$$

$$C_5 = \frac{2\theta_0}{\theta_0^2 - 4(\sigma_{10}^2 + \sigma_{20}^2 - \omega_0^2)} \quad (26)$$



The Raychaudhuri equations are solved here in an explicitly different manner as compared to [3]. The initial values of the variables have been attained by putting  $s = 0$  in terms of five arbitrary constants. The convergence of expansion scalar for various physically combination of integration parameters or initial values of ESR variables is analyzed and presented in Table I.

Table I: The conditions for convergence and divergence of kinematical quantities.

$s = 4\sqrt{C_3 C_4} - C_3$		$\theta$	$\sigma_1$	$\sigma_2$	$\omega$
1	$\sqrt{C_3 C_4} > 0$	$+\infty$			
2	$\sqrt{C_3 C_4} < 0$	$-\infty$	-	-	-
3	$C_1 C_2 C_4 > 0$	-	$+\infty$	-	-
4	$C_1 C_2 C_4 < 0$	-	$-\infty$	-	-
5	$C_2 C_4 > 0$	-	-	$+\infty$	-
6	$C_2 C_4 < 0$	-	-	$-\infty$	-
7	$C_4 > 0$	-	-	-	$+\infty$
8	$C_4 < 0$	-	-	-	$-\infty$

This clearly indicates the role of different constants and initial conditions on the convergence and divergence of the expansion scalar.

#### IV. CONCLUSIONS

In this paper, we have used a different technique to solve these equations than the techniques used in [3]. The Raychaudhuri equations in the background of flat two-dimensional space are solved by applying variable separable method. The equations (18) – (21) show the variation of all kinematic variables while Table I represents the condition of convergence and divergence of expansion scalar, shear tensor and rotation tensor with various conditions on concerned parameters. It is observed that rotation defines the convergence of expansion scalar while shear always assist it. The results obtained are consistent with those obtained in [3]. We also intend to report on the analytical solutions of Raychaudhuri Equations in three dimensions in flat space background as derived in [3].

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## References

- [1] Carroll, (2004). Space time and Geometry: An Introduction to General Relativity. San Francisco: Addison Wesley ISBN 0-8053-8732-3.
- [2] Ciufolini, I. and J. A. (1995). Wheeler *Gravitation and Inertia*, Princeton University Press, Princeton USA.
- [3] Dasgupta, A., Nandan, H. and Kar, S. (2007). Kinematics of deformable media. *Annals of Physics* 323(2008) 1621(doi: 10.1016/j.aop.2008.01.006) ar XiV:0709.0582.
- [4] Dasgupta, A., Nandan, H. and Kar, S. (2009). Kinematics of flows on curved, deformable media. *Int. J. Geom. Meth. Mod. Phys.* 6645-666.
- [5] Ellis, G.F.R. (1971). *General Relativity and Cosmology*. International School of Physics, Enrico Fermi-Course XLVII, Academic Press, New York.
- [6] Simmons, F. George. (2010). *Differential Equation with Applications and Historical Notes*. Tata Mc Graw Hill Edition.
- [7] Hawking, S.W. (1966). Occurrence of singularities in open universe. *Phys rev. Lett.* 15(1965) 689; Singularities in the universe, *ibid* 17444.
- [8] Hawking, S.W. and Ellis. G.F.R. (1973). *The large-scale structure of space-time*. Cambridge University Press, Cambridge, UK.
- [9] Joshi, P.S. (1997). *Global aspects in gravitation and cosmology*. Oxford University Press, Oxford, UK.
- [10] Kar, S. (2006). Focusing of branes in warped backgrounds *Indian Journal of Physics*, 80(8) 73; gr-qc/0702131 and references therein.
- [11] Kar, S. and Gupta S. Sen. (2007). The Raychaudhuri equations: *A brief review*. *Pramana*, 6949, gr-qc/0611123 and references therein.
- [12] Landau, L. and Lifshitz. E. M. (1986). *Theory of Elasticity*. Butterworth-Heinemann, Pergamom Press.
- [13] Penrose, R. (1965). Gravitational collapse and space-time singularities. *Phys. Rev. Lett* 14.
- [14] Poission, E. (2004). *A relativist's toolkit: the mathematics of black hole mechanics*. Cambridge University Press.
- [15] Raisinghania, M. D. (2015). *Advanced Differential Equations*. S. Chand & Company Pvt. Ltd. New Delhi 110055.
- [16] Ross, S. L. (2014). *Differential Equations*. John Wiley & Sons Publications, Wiley Publications.
- [17] Wald, R. M. (1984). *General Relativity*. University of Chicago Press, Chicago, USA.