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CONFORMALLY FLAT CHARGED FLUID SPHERE WITH SPIN

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Abstract

The present paper provides solutions of Einstein-Maxwell- Cartan fields equations for static conformally flat charged fluid sphere. Various physical and geometrical features have been found and discussed. Constants appearing in the solution have been calculated using boundary conditions.

Key words

Metric, conformally flat, fluid sphere, field equations, Ricci tensor, Weyl tensor.

1 INTRODUCTION:

In recent years various relativistists have focused their attention towards the study of charged fluid sphere with conformal flatness in Einstein – Cartan theory [8-13]. Chang [2] has found some conformal flat interior solutions of the Einstein-Maxwell equations for a charged stable static sphere. These solutions satisfy physical condition inside the sphere. Singh and Yadav [9] have solved the E-C equations for a charged fluid sphere by deriving a general set of differential equations which the function $\nu(r)$ & $\lambda(r)$ of the metric coefficients must satisfy and have obtained the solution by adopting a technique similar to that of Alder [1] for an uncharged fluid sphere in general relativity. The relevant differential equation reduces to Euler's equation which may be treated as a generalization of the equation of Wyman [14]. Unlike general relativity they have shown that pressure p is discontinuous at the boundary of the fluid sphere. Raychaudhuri [7], Nduka [5] have taken the E-C equations in a form so as to preserve the charge conservation principle. With this formulation Raychaudhuri has investigated the possibility of bounce in the presence of a magnetic field for Bianchi type-I universe with p=0 and p= ρ . Yadav and Sinha [12] have also found some solutions for static charged dust sphere with conformal flatness. Whereas Yadav and Prasad [13] have obtained general solution representing conformally flat non- static spherically symmetric perfect fluid distribution in Einstein – Cartan theory.

In this paper, we have taken the static conformally flat charged perfect fluid sphere with spin. We have solved the Einstein-Maxwell-Cartan field equations by a different technique using Hehl's (1974) approach by assuming effective mass density $\overline{\rho}\,(=\rho-2\pi K^2)$ to be constant. We have also found pressure, gravitational mass density and spin density for the distribution and the physical constants appearing in the solution have been obtained by matching the solution with Reissner-Nordstrom metric at the boundary.

2 THE FIELD EQUATIONS AND THEIR SOLUTIONS:

As has been already pointed out by Hehl [4], the Maxwell field does not couple to torsion. It is easy to see that if it did not couple to torsion, one would have to sacrifice the charge conservation principle. If one tries to obtain the electromagnetic field from the variational principle with the usual Lagrangian, it is found that gauge invariance is last. The variational principle gives

$$F^{\mu\nu}_{/\nu} = J^{\mu}$$

Or if $Q^{\nu}_{\alpha\beta}$ be the anti-symmetric part of the affinity

$$F^{\mu\nu}_{;\nu} + Q^{\mu}_{\alpha\nu} F^{\alpha\nu} = J^{\mu}$$

So that

$$\frac{1}{(-g)^{1/2}}\left[J^{\mu}(-g)^{1/2}\right]_{,\mu} = \frac{1}{(-g)^{1/2}}\left[Q^{\mu}_{\alpha\nu} F^{\alpha\upsilon}(-g)^{1/2}\right]_{,\mu}$$

Or,

$$\left[J^{\mu} - Q^{\mu}_{\alpha\nu} F^{\alpha\upsilon}(-g)^{1/2}\right]_{,\mu} = 0$$

with

$$F_{\mu\nu} = A_{\mu/\nu} - A_{\nu/\mu} \neq A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu,\nu} - A_{\nu,\mu}$$

where a vertical line indicates co-variant differentiation with the non-symmetric affinity, a semi colon denotes the co-variant derivative with respect to the Christoffel symbols and a comma denotes partial derivative.

Thus it J^{μ} is identified as the charge current vector, then in general the charge conservation principle is violated. It therefore, seems more appropriate to adopt the prescription that the electromagnetic field tensor is to be defined by

$$F_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu,\nu} - A_{\nu,\mu}$$

and we have usual Maxwell field equations

$$F^{\mu\nu}_{;\nu} = J^{\mu}$$

$$\left[J^{\mu}(-g)^{1/2}\right]_{,\mu} = 0$$

The Einstein- Cartan- Maxwell field equations may be may be written as [6,7]

(2.1) $$R_{ij} - \frac{1}{2} R g_{ij} = -8\pi t_{ij}$$

(2.2) $$\left[(-g)^{1/2} F^{ij}\right]; j = (-g)^{1/2} j^{i}$$

(2.3) $$\left[F_{ij}; k\right] = 0$$

where R_{ij} is the Ricci tensor of asymmetric connection and also the energy momentum tensor t_{ij} is not symmetric, $F_{\mu\nu}$ is the electromagnetic field tensor and J^{μ} is current four vector (we have set C and the gravitational constant also to be equal to unity).

We use here the static spherically symmetric metric

(2.4) $$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

where ν & λ are functions of r only.

For the system under study the symmetric energy momentum tensor $\overline{T_j^i}$ splits into two parts viz. T_j^i and E_j^i for matter and electromagnetic field respectively as

(2.5) $$\overline{T_j^i} = T_j^i + E_j^i$$

where

(2.6) $$T_j^i = (\rho + p)u_j u^i - p\delta_j^i$$

with

(2.7) $$u_i u^i = 1$$

Here p is pressure, ρ is density of matter and u^i is the velocity vector of matter.

We use co-moving co-ordinates so that

$$u^i = \delta_4^i$$

The non-vanishing components of T_i^j are

(2.8) $$T_4^4 = \rho, T_1^1 = T_2^2 = T_3^3 = -p$$

Thus the Einstein-Cartan-Maxwell field equations are

(2.9) $$e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) - \frac{1}{r^2} = -8\pi\overline{\rho} - E^2$$

(2.10) $$\frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) = -8\pi\overline{p} + E^2$$

(2.11) $$e^{-\lambda}\left[\frac{1}{4}\nu'\lambda' - \frac{1}{4}\nu'^2 - \frac{1}{2}\nu'' - \frac{1}{2}\left(\frac{\nu' - \lambda'}{r}\right)\right] = -8\pi\overline{\rho} - E^2$$

where following Hehl [3, 4] we have defined effective pressure $\overline{p}$ and effective density $\overline{\rho}$ as

(2.12) $$\overline{p} = p - 2\pi K^2., \overline{\rho} = \rho - 2\pi K^2$$

with

(2.13) $$K = He^{-\nu/2}$$

Here H is a constant of integration and dashes denote differentiation w.r.t. r

To solve (2.2), we get

(2.14) $$E^2 = \frac{Q^2(r)}{r^4}$$

where Q(r) represents the total charge contained within the sphere of radius r i.e.

(2.15) $$Q(r) = 4\pi\int \rho_e r^2 dt$$

where ρ_e is the charge density.

Eliminating $\overline{p}$ form (2.9) & (2.10), we get

(2.16) $$e^{-\lambda}\left[\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} - \frac{1}{r^2} - \frac{1}{2r}(\lambda' + \nu') + \frac{e^{\lambda}}{r^2}\right] = 2E^2$$

For the spherically symmetric metric (2.4), the non-vanishing components of Weyl tensor are

$$C_{1212} = -\frac{1}{12}r\upsilon' + \frac{1}{12}r\lambda' - \frac{1}{6}e^{\lambda} + \frac{1}{6} - \frac{1}{24}r^2\lambda'\upsilon' + \frac{1}{24}r^2\upsilon'^2 - \frac{1}{12}r^2\nu''$$

$C_{1313} = Sin^2\theta C_{1212}$

$C_{1414} = 2\frac{e^{\nu}}{r^2} \quad C_{1212}$

$C_{2323} = -2\,Sin^2\theta\, e^{-\lambda}\, r^2 C_{1212},$

$C_{2424} = -e^{-v-\lambda} C_{1212}$

$C_{3434} = Sin^2\theta\, e^{v-\lambda}\, C_{1212}$

For conformal flatness vanishing of the Weyl tensor provides us

(2.17) $$\frac{e\lambda}{r^2} - \frac{1}{r^2} - \frac{\upsilon'^2}{4} + \frac{\upsilon'\lambda'}{4} - \frac{\upsilon''}{2} - \frac{1}{2r}\left(\lambda' - \upsilon'\right) = 0$$

Multiplying equation (2.17) by $e^{-\lambda}$ and then adding the result to (2.16), we get.

(2.18) $$E^2 = -e^{-\lambda}\left(\frac{\lambda'}{2r} + \frac{1}{r^2}\right) + \frac{1}{r^2}$$

It is clear from these equations that it is $\overline{p} = p - 2\pi K^2$ and not the p which is continuous across the boundary r = r_0 of the fluid sphere. The continuity of $\overline{p}$ across the boundary ensures that of $\upsilon' \exp\upsilon$. Further with $\overline{p}$ =p- $2\pi K^2$ & $\overline{\rho} = \rho - 2\pi K^2$ replacing p and ρ respectively, we are assured that the metric co-efficients are continuous across the boundary. Hence we shall apply the usual boundary conditions to the solutions of equations (2.9) (2.10) & (2.11).

The exterior metric is taken as Reissiner –Nordstrom metric given by

(2.19) $$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q_b^2}{r^2}\right)dt^2 - \left(1 - \frac{2M}{r} + \frac{Q_b^2}{r^2}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

when $Q_b = Q(r_b)$ and M is the total mass of the fluid sphere.

Equation (2.9), (2.10), (2.11), (2.13), (2.14) and (2.18) are six equations in seven unknowns $\lambda, \mu, \rho, p, K, Q(r)$ and E^2. Thus the system is indeterminate. To make the system determinate we require one more relation or condition. For this we will specify the effective mass density $\overline{\rho}$ or the charge density distribution ρ_e or the effective pressure distribution $\overline{p}$ or spin density K and solve for the remaining equations. The solution of (2.18) is

(2.20) $e^{-\lambda} = 1 + Ar^2 + B(r)$ where A is the integration constant and B(r) is

(2.21) $$B(r) = 2r^2\int\frac{E^2}{r}dr$$

Inserting (2.20) into (2.9) we get

(2.22) $$\frac{B'}{r} + \frac{B}{r^2} = -8\pi\overline{\rho} - E^2 - 3A$$

Now differentiating (2.22) we have

(2.23) $$\frac{d\left(E^2\right)}{dr} + 2\frac{E^2}{r} = -\frac{8\pi}{3}\frac{d\overline{\rho}}{dr}$$

The E^2 can be determined from the equation (2.23) if the $\overline{\rho}$ have been predetermined.

We consider the uniform effective mass density i.e.

$$\overline{\rho} = \text{Constant} = C$$

Then from (2.23) we get

(2.24) $$\frac{d\left(E^2\right)}{dr}+2\frac{E}{r}=0$$

Integrating equation (2.24), we get the expressions for the electric field and charge as

(2.25) $$E^2=\frac{\alpha}{r^2}, Q^2=\alpha r^2$$

Using equation (2.12) matter density ρ is given by

(2.26) $$8\pi\rho=16\pi^2H^2e^{-\nu}+C_1, C_1=8\pi C$$

Equations (2.21) and (2.25) yield

(2.27) $$B(r)=-\alpha$$

Then from (2.20) and (2.27) we get

(2.28) $$e^{-\lambda}=1-\alpha+Ar^2$$

Equation (2.9) may be written as

(2.29) $$8\pi\bar{\rho}=e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)-\frac{1}{r^2}+E^2$$

From (2.29) and (2.25) we get

(2.30) $$8\pi\bar{\rho}=e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)-\frac{1}{r^2}+\frac{\alpha}{r^2}$$

$$\Rightarrow 8\pi\bar{\rho}r^2=e^{-\lambda}\left(1-r\lambda'\right)-1+\alpha$$

$$\Rightarrow 1-\alpha+8\pi\bar{\rho}r^2=\frac{d}{dr}(re^{-\lambda})$$

Integrating we get

(2.31) $$e^{-\lambda}=\alpha_1-\frac{r^2}{R^2}$$

where

(2.32) $$\frac{1}{R^2}=\frac{8\pi\bar{\rho}}{3}, \alpha_1=1-\alpha, A=-\frac{1}{R^2}$$

Eliminating $\bar{p}$ between equations (2.10) and (2.11) we get

(2.33) $$e^{-\lambda}\left(\frac{\upsilon'^2}{4}+\frac{\upsilon''}{2}-\frac{\left(\upsilon'+\lambda'\right)}{2r}-\frac{1}{r^2}-\frac{\nu'\lambda'}{4}\right)+\frac{1}{r^2}=2E^2$$

Substituting the value of λ and E^2 from equations (2.31) and (2.25) in (2.33) we get

(2.34) $$\upsilon''+\frac{\upsilon'^2}{2}+\left(\frac{r}{r^2-\alpha_1R^2}-\frac{1}{r}\right)\upsilon'=\frac{-2\alpha R^2}{\left(r^2-\alpha_1R^2\right)}$$

Using the transformation

(2.35) $$x=r^2,$$

$$\beta=\frac{e^{\frac{\nu}{2}}}{\sqrt{x}}$$

we get the equation (2.34) into the form

(2.36) $$\frac{d^2\beta}{dx^2}+\left[\frac{1}{x}+\frac{1}{2(x-\alpha_1 R^2)}\right]\frac{d\beta}{dx}=\frac{R^2}{2x^2(x-\alpha_1 R^2)}\beta$$

On letting

(2.37) $$y=\frac{2\mu}{(\alpha_1 R^2)^{\frac{1}{2}}}\tan^{-1}\left(\frac{x}{\alpha_1 R^2}-1\right)^{\frac{1}{2}}$$

where μ is a constant, equation (2.37) is reduced to the form

(2.38) $$\mu^2\frac{d^2\beta}{dy^2}+\frac{R}{4}\beta=0$$

For $r^2\geq\alpha_1 R^2$,

Finally we get the solution of equation (2.34) as

(2.39) $$e^{\nu}=r^2\left(L\sin\psi+M\cos\psi\right)$$

where

(2.40) $$\psi=\frac{1}{\sqrt{1-\alpha}}\tan^{-1}\left(\frac{r^2}{\alpha_1 R^2}-1\right)^{\frac{1}{2}}$$

It is obvious that for real solution 1-$\alpha>0$ or equivalently $\alpha_1>0$ and L and M are constants of integration.

Now using equations (2.10), (2.12), (2.26), and (2.39), pressure and density are given by

(2.41) $$8\pi p=16\pi^2K^2+\frac{2\alpha_1}{r^2}-\frac{3}{R^2}-\frac{2}{\sqrt{\alpha_1}}\left(\frac{1}{r^2}-\frac{1}{R^2}\right)$$
$$\tan\left(\psi\sqrt{\alpha_1}\right)\left(L-M\tan\psi\right)/\left(L\tan\psi+M\right)$$

(2.42) $$8\pi\rho=16\pi^2K^2+C_1$$

Also spin density K is given by

(2.43) $$K=\frac{H}{r(L\sin\psi+M\cos\psi)}$$

For $\frac{r^2}{\alpha_1 R^2}<1$, the final solution of equation (2.34) is found to be

(2.44) $$e^{\nu}=r^2\left(L\sin\psi+M\cosh\psi\right)^2$$

Also pressure and density in this case are given by

(2.45)

$$8\pi p=16\pi^2K^2+\frac{2\alpha_1}{r^2}-\frac{3}{R^2}-\frac{2}{\sqrt{\alpha_1}}\left(\frac{1}{r^2}-\frac{1}{R^2}\right)\frac{1}{\tan\left(\psi\sqrt{\alpha_1}\right)}\times\frac{L+M\tanh\psi}{L\tanh\psi+M}\quad(2.46)$$

$$8\pi\rho=16\pi^2K^2+C_1$$

Also spin density K in this case is obtained as

(2.47) $$K=\frac{H}{r\left(L\sinh\psi+M\cosh\psi\right)}$$

The constants appearing in this solution can be evaluated by matching the solution with Reissner-Nordstrom metric at the boundary as already discussed in this section. Thus the constants $\alpha, L, M\ \&\ H$ are given by.

(2.48) $$\alpha_1 - \frac{r_b^{\,2}}{R_b^{\,2}} = 1 - \frac{2M}{r_b} + \frac{Q_b^{\,2}}{r_b^{\,2}}, \alpha_1 = 1 - \alpha.$$

(2.49) $$r_b^{\,2}\left(L \sinh\psi_b + M \cosh\psi_b\right)^2 = 1 - \frac{2M}{r_b} + \frac{Q_b^{\,2}}{r_b^{\,2}}$$

(2.50) $$2r_b\left(L \sinh\Psi_b + M \cosh\psi_b\right)^2 + 2r_b^2\left(L \sinh\Psi_b + M \cosh\Psi_b\right)\left(L \cosh\Psi_b + M \cosh\Psi_b\right)\Psi_b$$

$$= \left(\frac{2M}{r_b^{\,2}} - \frac{2Q_b^{\,2}}{r_b^{\,3}}\right),$$

(2.51) $$H^2 = \frac{\left(8\pi\rho_b - \frac{3}{R_b^{\,2}}\right)}{16\pi^2} . r_b^{\,2}\left(L \sinh\psi_b + M \cos\psi_b\right)^2$$

where r_b, R_b, ρ_b and ψ_b are values of r, R, ρ and ψ at the boundary

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