



$J\beta$ -CLOSED SETS IN TOPOLOGICAL SPACES

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Abstract: In this paper, a new class of sets called $J\beta$ -closed sets is introduced and its properties are studied. The relationships among closed, α -closed, β -closed and their generalized closed sets are investigated. Several examples are provided to illustrate the behavior of these new class of sets.

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1. Introduction

Many investigations related to generalized closed and open sets have been published in various forms of open and closed sets. In 1937, Stone [13] introduced the notion of regular open sets. In 1965, Njastad [9] introduced the concept of α -open sets. In 1968, the notion of π -open sets were introduced by Zaitsev [20] which are weaker form of regular open sets in topological spaces. Velicko [18] proposed δ -open sets which are stronger than open sets. In 1970, Levine [5] initiated the study of so called generalized closed (briefly g -closed) sets. In 1983, Abd EI-Monsef et al [1] introduced the notion of β -open sets. In 1993, Palaniappan and Rao [10] introduced the concept of $rg\beta$ -closed sets. Maki et. al [7] introduced the notion of $g\alpha$ -closed sets. In 1994, Maki et. al [6] introduced the notion of αg -closed sets. In 1995, Dontchev [2] introduced the notion of $g\beta$ -closed sets. In 1996, Dontchev [3] introduced the notion of δg -closed sets. In 2000, Veera Kumar [15] introduced the notion of g^* -closed and \hat{g} -closed sets. In 2003, Veera Kumar [16, 17] introduced the notion of $g^\#$ -closed sets. In 2008, Jafari [4] introduced the notion of $*g$ -closed sets. In 2010, Viswanathan [19] introduced the notion of $g^*\alpha$ -closed sets. Sarsak and Rajesh [12] introduced the concept of $\pi g\beta$ -closed sets. In 2012, Sudha [14] introduced the notion of δg^* -closed sets. In 2016, Pious [11] introduced the notion of regular*-open sets. In 2019, Meenakshi et. al [8] have introduced a class of new sets namely η^* -open sets which is placed between the classes of δ -open and open sets.

2. Preliminaries

Throughout this paper, spaces (X, \mathfrak{T}) , (Y, σ) , and (Z, γ) (or simply X , Y and Z) always mean topological spaces. Let A be a subset of a space X . The closure of A and interior of A are denoted by $cl(A)$ and $int(A)$ respectively. A subset A is said to be **regular open** [13] (resp. **regular closed** [13]) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). The finite union of regular open sets is said to be **π -open** [20]. The complement of a π -open set is said to be **π -closed** [20].

Definition 2.1. A subset A of a topological space (X, \mathfrak{T}) is said to be

(i) **α -open** [9] if $A \subset int(cl(int(A)))$.

(ii) **β -open [1]** if $A \subset \text{cl}(\text{int}(\text{cl}(A)))$.

The complement of a α -open (resp. β -open) set is called **α -closed** (resp. **β -closed**). The intersection of all α -closed (resp. β -closed) sets containing A , is called **α -closure** (resp. **β -closure**) of A , and is denoted by **$\alpha\text{-cl}(A)$** (resp. **$\beta\text{-cl}(A)$**). A subset A of a topological space (X, \mathfrak{T}) is said to be **clopen** if it is both open and closed in (X, \mathfrak{T}) .

Definition 2.2. A subset A of a topological space (X, \mathfrak{T}) is said to be **generalized closed** (briefly **g-closed**) [5] if $\text{cl}(A) \subset U$ whenever $A \subset U$ and $U \in \mathfrak{T}$.

The **generalized closure** of A is defined as the intersection of all g-closed sets in X containing A and is denoted by **$\text{cl}^*(A)$** . The **generalized interior** of A is defined as the union of all g-open sets in X contained in A and is denoted by **$\text{int}^*(A)$** .

Definition 2.4. The **δ -interior** of a subset A of X is the union of all regular open sets of X contained in A and is denoted by **$\delta\text{-int}(A)$** . The subset A is called **δ -open [18]** if $\delta\text{-int}(A) = A$. i.e. a set is δ -open if it is the union of regular open sets, the complement of a δ -open set is called **δ -closed**. Alternatively, a set $A \subset X$ is δ -closed if $A = \delta\text{-cl}(A)$, where $\delta\text{-cl}(A)$ is the intersection of all regular closed sets of (X, \mathfrak{T}) containing A .

Definition 2.5. Let (X, \mathfrak{T}) be a topological space. A subset A of (X, \mathfrak{T}) is called **regular*-open [11]** (or **r^* -open**) if $A = \text{int}(\text{cl}^*(A))$. The complement of a regular*-open set is called **regular*-closed**. The union of all regular*-open sets of X contained in A is called **regular*-interior** and is denoted by **$r^*\text{-int}(A)$** . The intersection of all regular*-closed sets of X containing A is called **regular*-closure** is denoted by **$r^*\text{-cl}(A)$** .

Definition 2.6. A subset A of a topological space (X, \mathfrak{T}) is called **η^* -open [8]** set if it is a union of regular*-open sets (r^* -open sets). The complement of a η^* -open set is called **η^* -closed**. A subset A of a topological space (X, \mathfrak{T}) is called **η^* -Interior** of A is the union of all η^* -open sets of X contained in A and is denoted by **$\eta^*\text{-int}(A)$** . The intersection of all η^* -closed sets of X containing A is called as the **η^* -closure** of A and is denoted by **$\eta^*\text{-cl}(A)$** .

Remark 2.7.

- (i) regular open $\Rightarrow \pi$ -open $\Rightarrow \delta$ -open $\Rightarrow \eta^*$ -open \Rightarrow open $\Rightarrow \alpha$ -open \Rightarrow s-open $\Rightarrow \beta$ -open
- (ii) regular closed $\Rightarrow \pi$ -closed $\Rightarrow \delta$ -closed $\Rightarrow \eta^*$ -closed \Rightarrow closed $\Rightarrow \alpha$ -closed \Rightarrow s-closed $\Rightarrow \beta$ -closed
- (iii) regular open $\Rightarrow \pi$ -open $\Rightarrow \delta$ -open $\Rightarrow \eta^*$ -open \Rightarrow open $\Rightarrow \alpha$ -open \Rightarrow p-open $\Rightarrow \beta$ -open
- (iv) regular closed $\Rightarrow \pi$ -closed $\Rightarrow \delta$ -closed $\Rightarrow \eta^*$ -closed \Rightarrow closed $\Rightarrow \alpha$ -closed \Rightarrow p-closed $\Rightarrow \beta$ -closed

Remark 2.8. For every subset U of X ,

- (i) $\beta\text{-cl}(U) \subset \text{s-cl}(U) \subset \alpha\text{-cl}(U) \subset \text{cl}(U) \subset \eta^*\text{-cl}(U) \subset \delta\text{-cl}(U) \subset \pi\text{-cl}(U) \subset r\text{-cl}(U)$.
- (ii) $\text{g-cl}(U) \subset \text{cl}(U) \subset \eta^*\text{-cl}(U) \subset \delta\text{-cl}(U) \subset \pi\text{-cl}(U) \subset r\text{-cl}(U)$.
- (iii) $\beta\text{-cl}(U) \subset \text{p-cl}(U) \subset \alpha\text{-cl}(U) \subset \text{cl}(U) \subset \eta^*\text{-cl}(U) \subset \delta\text{-cl}(U) \subset \pi\text{-cl}(U) \subset r\text{-cl}(U)$.

Definition 2.9. A subset A of a topological space (X, \mathfrak{T}) is said to be

- (1) **δ -generalized closed** (briefly **$\delta\text{g-closed}$**) [3] if $\delta\text{-cl}(A) \subset U$ whenever $A \subset U$ and $U \in \mathfrak{T}$.
- (2) **$\text{g}\beta$ -closed [2]** if $\beta\text{-cl}(A) \subset U$ whenever $A \subset U$ and $U \in \mathfrak{T}$.
- (3) **α -generalized closed** (briefly **$\alpha\text{g-closed}$**) [6] if $\alpha\text{-cl}(A) \subset U$ whenever $A \subset U$ and $U \in \mathfrak{T}$.
- (4) **δg^* -closed [14]** if $\delta\text{-cl}(A) \subset U$ whenever $A \subset U$ and U is g-open in X .
- (5) **generalized α -closed** (briefly **$\text{g}\alpha$ -closed**) [7] if $\alpha\text{-cl}(A) \subset U$ whenever $A \subset U$ and U is α -open in X .
- (6) **generalized* α -closed** (briefly **$\text{g}^*\alpha$ -closed**) [19] if $\alpha\text{-cl}(A) \subset U$ whenever $A \subset U$ and U is $\text{g}\alpha$ -open in X .
- (7) **$\hat{\text{g}}$ -closed [16]** if $\text{cl}(A) \subset U$ whenever $A \subset U$ and U is semi-open in X .
- (8) ***g-closed [4]** if $\text{cl}(A) \subset U$ whenever $A \subset U$ and U is $\hat{\text{g}}$ -open in X .
- (9) **g^* -closed [15]** if $\text{cl}(A) \subset U$ whenever $A \subset U$ and U is g-open in X .
- (10) **$\text{g}^\#$ -closed [17]** if $\text{cl}(A) \subset U$ whenever $A \subset U$ and U is αg -open in X .

- (11) **$\pi g\beta$ -closed** [12] if $\beta\text{-cl}(A) \subset U$ whenever $A \subset U$ and U is π -open in X .
- (12) **$rg\beta$ -closed** [10] if $\beta\text{-cl}(A) \subset U$ whenever $A \subset U$ and U is regular open in X .
- (13) **g -open** (resp. **δg -open, $g\beta$ -open, αg -open, δg^* -open, $g\alpha$ -open, $g^*\alpha$ -open, \hat{g} -open, $*g$ -open, g^* -open, $g^\#$ -open, $\pi g\beta$ -open, $rg\beta$ -open) set if the complement of A is g -closed (resp. δg -closed, $g\beta$ -closed, αg -closed, δg^* -closed, $g\alpha$ -closed, $g^*\alpha$ -closed, \hat{g} -closed, $*g$ -closed, g^* -closed, $g^\#$ -closed, $\pi g\beta$ -closed, $rg\beta$ -closed).**

3. $J\beta$ -closed Sets

In this section a new class of generalized closed sets, called $J\beta$ -closed sets are introduced. The relations between $J\beta$ -closed sets and various existing closed sets are investigated.

3.1 Definition A subset U of a topological space (X, \mathfrak{T}) is said to be **$J\beta$ -closed** if $\beta\text{-cl}(U) \subset M$ whenever $U \subset M$ and M is η^* -open in (X, \mathfrak{T}) .

3.2 Proposition. Every g -closed set is $J\beta$ -closed but not conversely.

Proof. Let U be a g -closed set and M be any η^* -open set containing U in X . Since every η^* -open set is open, so M is open. Given that U is g -closed, $\text{cl}(U) \subset M$. Hence $\beta\text{-cl}(U) \subset \text{cl}(U) \subset M$, which implies that U is $J\beta$ -closed.

3.3 Proposition. Every g^* -closed set is $J\beta$ -closed but not conversely.

Proof. Let U be a g^* -closed set and M be any η^* -open set containing U in X . Since every η^* -open set is g -open, so M is g -open. Given that U is g^* -closed, $\text{cl}(U) \subset M$. Hence $\beta\text{-cl}(U) \subset \text{cl}(U) \subset M$, which implies that U is $J\beta$ -closed.

3.4 Proposition. Every δ -closed set is $J\beta$ -closed but not conversely.

Proof. Let U be a δ -closed set and M be any η^* -open set containing U in X . Since U is δ -closed, $\delta\text{-cl}(U) = U \subset M$. As $\beta\text{-cl}(U) \subset \delta\text{-cl}(U) \subset M$ and hence U is $J\beta$ -closed.

3.5 Proposition. Every δg -closed set is $J\beta$ -closed but not conversely.

Proof. Let U be a δg -closed set and M be any η^* -open set containing U in X . Since every η^* -open set is open, so M is open. Given that U is δg -closed, $\delta\text{-cl}(U) \subset M$. As $\beta\text{-cl}(U) \subset \delta\text{-cl}(U) \subset M$. We get $\beta\text{-cl}(U) \subset M$. Hence U is $J\beta$ -closed.

3.6 Proposition. Every δg^* -closed set is $J\beta$ -closed but not conversely.

Proof. Let U be a δg^* -closed set and M be any η^* -open set containing U in X . Since every η^* -open set is g -open, so M is g -open. Given that U is δg^* -closed, $\delta\text{-cl}(U) \subset M$. As $\beta\text{-cl}(U) \subset \delta\text{-cl}(U) \subset M$. We get $\beta\text{-cl}(U) \subset M$. Hence U is $J\beta$ -closed.

3.7 Proposition. Every αg -closed set is $J\beta$ -closed but not conversely.

Proof. Let U be a αg -closed set and M be any η^* -open set containing U in X . Since every η^* -open set is open, so M is open. Given that U is αg -closed, $\alpha\text{-cl}(U) \subset M$. As $\beta\text{-cl}(U) \subset \alpha\text{-cl}(U) \subset M$. We get $\beta\text{-cl}(U) \subset M$. Hence U is $J\beta$ -closed.

3.8 Proposition. Every $g\alpha$ -closed set is $J\beta$ -closed but not conversely.

Proof. Let U be a $g\alpha$ -closed set and M be any η^* -open set containing U in X . Since every η^* -open set is α -open, so M is α -open. Given that U is $g\alpha$ -closed, $\alpha\text{-cl}(U) \subset M$. As $\beta\text{-cl}(U) \subset \alpha\text{-cl}(U) \subset M$. We get $\beta\text{-cl}(U) \subset M$. Hence U is $J\beta$ -closed.

3.9 Proposition. Every $g\alpha^*$ -closed set is $J\beta$ -closed but not conversely.

Proof. Let U be a $g\alpha^*$ -closed set and M be any η^* -open set containing U in X . Since every η^* -open set is $g\alpha$ -open, so M is $g\alpha$ -open. Given that U is $g\alpha^*$ -closed, $\alpha\text{-cl}(U) \subset M$. As $\beta\text{-cl}(U) \subset \alpha\text{-cl}(U) \subset M$. We get $\beta\text{-cl}(U) \subset M$. Hence U is $J\beta$ -closed.

3.10 Proposition. Every $g^\#$ -closed set is $J\beta$ -closed but not conversely.

Proof. Let U be a $g^\#$ -closed set and M be any η^* -open set containing U in X . Since every η^* -open set is αg -open, so M is αg -open. Given that U is $g^\#$ -closed, $\text{cl}(U) \subset M$. As $\beta\text{-cl}(U) \subset \text{cl}(U) \subset M$. We get $\beta\text{-cl}(U) \subset M$. Hence U is $J\beta$ -closed.

3.11 Proposition. Every $*g$ -closed set is $J\beta$ -closed but not conversely.

Proof. Let U be a $*g$ -closed set and M be any η^* -open set containing U in X . Since every η^* -open set is \hat{g} -open, so M is \hat{g} -open. Given that U is $*g$ -closed, $\text{cl}(U) \subset M$. As $\beta\text{-cl}(U) \subset \text{cl}(U) \subset M$. We get $\beta\text{-cl}(U) \subset M$. Hence U is $J\beta$ -closed.

3.12 Proposition. Every \hat{g} -closed set is $J\beta$ -closed but not conversely.

Proof. Let U be a \hat{g} -closed set and M be any η^* -open set containing U in X . Since every η^* -open set is s -open, so M is s -open. Given that U is \hat{g} -closed, $\text{cl}(U) \subset M$. As $\beta\text{-cl}(U) \subset \text{cl}(U) \subset M$. We get $\beta\text{-cl}(U) \subset M$. Hence U is $J\beta$ -closed.

3.13 Proposition. Every $g^*\alpha$ -closed set is $J\beta$ -closed but not conversely.

Proof. Let U be a $g^*\alpha$ -closed set and M be any η^* -open set containing U in X . Since every η^* -open set is $g\alpha$ -open, so M is $g\alpha$ -open. Given that U is $g^*\alpha$ -closed, $\alpha\text{-cl}(U) \subset M$. As $\beta\text{-cl}(U) \subset \alpha\text{-cl}(U) \subset M$. We get $\beta\text{-cl}(U) \subset M$. Hence U is $J\beta$ -closed.

3.14 Proposition. Every $g\beta$ -closed set is $J\beta$ -closed but not conversely.

Proof. Let U be a $g\beta$ -closed set and M be any η^* -open set containing U in X . Since every η^* -open set is open, so M is open. Given that U is $g\beta$ -closed, $\beta\text{-cl}(U) \subset M$. Hence U is $J\beta$ -closed.

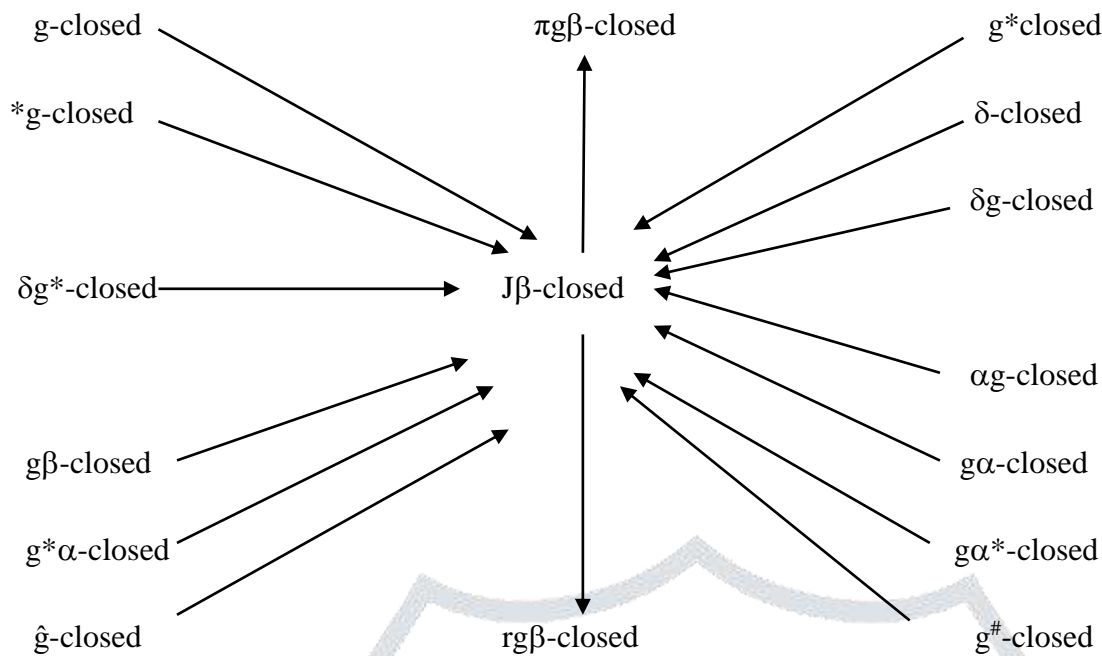
3.15 Proposition. Every $J\beta$ -closed set is $\pi g\beta$ -closed but not conversely.

Proof. Let U be a $J\beta$ -closed set and M be any π -open set containing U in X . Since every π -open set is η^* -open, so M is η^* -open. Given that U is $\pi g\beta$ -closed, $\beta\text{-cl}(U) \subset M$. Hence U is $\pi g\beta$ -closed.

3.16 Proposition. Every $J\beta$ -closed set is $rg\beta$ -closed but not conversely.

Proof. Let U be a $J\beta$ -closed set and M be any r -open set containing U in X . Since every r -open set is η^* -open, so M is η^* -open. Given that U is $J\beta$ -closed, $\beta\text{-cl}(U) \subset M$. Hence U is $rg\beta$ -closed.

3.17 Remark. From the above definitions, theorems and known results the relationship between $J\beta$ -closed sets and some other existing generalized closed sets are implemented in the following figure:



Where none of the implications is reversible as can be seen from the following examples:

3.18 Example. Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$. Then the subset $A = \{a\}$ is $J\beta$ -closed but not g -closed in (X, \mathfrak{T}) .

3.19 Example. Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{\phi, X, \{a\}\}$. Then the subset $A = \{b\}$ is $J\beta$ -closed but not δ -closed in (X, \mathfrak{T}) .

3.20 Example. Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{\phi, X, \{a, b\}\}$. Then the subset $A = \{a\}$ is $J\beta$ -closed but not δg^* -closed in (X, \mathfrak{T}) .

3.21 Example. Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{\phi, X, \{a\}, \{a, b\}\}$. Then the subset $A = \{b\}$ is $J\beta$ -closed but not δg -closed in (X, \mathfrak{T}) .

3.22 Example. Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{\phi, X, \{a\}, \{b, c\}\}$. Then the subset $A = \{a, b\}$ is $J\beta$ -closed but not $g\alpha^*$ -closed in (X, \mathfrak{T}) .

3.23 Example. Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{\phi, X, \{a\}, \{a, b\}\}$. Then the subset $A = \{b\}$ is $J\beta$ -closed but not $g^\#$ -closed in (X, \mathfrak{T}) .

3.24 Example. Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$. Then the subset $A = \{a\}$ is $J\beta$ -closed but not *g -closed in (X, \mathfrak{T}) .

3.25 Example. Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{\phi, X, \{a, b\}\}$. Then the subset $A = \{a\}$ is $J\beta$ -closed but not \hat{g} -closed in (X, \mathfrak{T}) .

3.26 Example. Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{\phi, X, \{a\}, \{a, b\}\}$. Then the subset $A = \{a, b\}$ is $J\beta$ -closed but not $g^*\alpha$ -closed in (X, \mathfrak{T}) .

3.27 Example. Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{\phi, X, \{a\}\}$. Then the subset $A = \{a, b\}$ is $J\beta$ -closed but not $g\alpha$ -closed in (X, \mathfrak{T}) .

3.28 Example. Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{\phi, X, \{a, b\}\}$. Then the subset $A = \{a, c\}$ is $J\beta$ -closed but not αg -closed in (X, \mathfrak{T}) .

3.29 Example. Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{\phi, X, \{a\}, \{a, b\}\}$. Then the subset $A = \{b\}$ is $J\beta$ -closed but not g^* -closed in (X, \mathfrak{T}) .

4. Properties of $J\beta$ -closed sets

4.1 Theorem. The finite union of $J\beta$ -closed sets is $J\beta$ -closed.

Proof. Let $\{U_i / i = 1, 2, 3, \dots, n\}$ be a finite class of $J\beta$ -closed sets of (X, \mathfrak{T}) . Let $U = \cup_{i=1}^n U_i$. Let M be a η^* -open set containing U . This implies $U_i \subset M$ for every i . By assumption $\beta\text{-cl}(U_i) \subset M$ for every i . This implies $\cup_{i=1}^n \beta\text{-cl}(U_i) \subset M$. Then $\beta\text{-cl}(\cup_{i=1}^n U_i) \subset M$. Thus $\beta\text{-cl}(U) \subset M$. Hence finite union of $J\beta$ -closed sets is $J\beta$ -closed in (X, \mathfrak{T}) .

4.2 Remark. The intersection of two $J\beta$ -closed sets need not be $J\beta$ -closed.

4.3 Example. Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{\phi, X, \{a\}\}$. Here $J\beta$ -closed sets are $\{\phi, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Then $\{a, c\} \cap \{a, b\} = \{a\}$ is not a $J\beta$ -closed set.

4.4 Theorem. For $x \in X$, then the set $X - \{x\}$ is $J\beta$ -closed or η^* -open.

Proof. Assume that $X - \{x\}$ is not η^* -open. So X is the only η^* -open set containing $X - \{x\}$. That is $\beta\text{-cl}(X - \{x\}) \subset X$. Then $X - \{x\}$ is a $J\beta$ -closed set in X .

4.5 Theorem. The subset S of X is $J\beta$ -closed in X if and only if $\beta\text{-cl}(S) - S$ contains no non empty η^* -closed set in X .

Proof. Necessity: Let F be a η^* -closed set in X such that $F \subset \beta\text{-cl}(A) - A$. Then $A \subset X - F$. Since A is a $J\beta$ -closed set and $X - F$ is η^* -open then $\beta\text{-cl}(A) \subset X - F$. (i.e $F \subset X - \beta\text{-cl}(A)$). Then $F \subset (X - \beta\text{-cl}(A)) \cap \beta\text{-cl}(A) - A$. Therefore $F = \phi$.

Sufficiency: Let us assume that $\beta\text{-cl}(A) - A$ contains no non empty η^* -closed set. Let $A \subset U$ and U be η^* -open. Suppose that $\beta\text{-cl}(A)$ is not contained in U , then $\beta\text{-cl}(A) \cap U^c$ is non empty η^* -closed set of $\beta\text{-cl}(A) - A$ which is a contradiction. Therefore $\beta\text{-cl}(A) \subset U$. Hence A is a $J\beta$ -closed set.

4.6 Theorem. If M is $J\beta$ -closed set in X and $M \subset N \subset \beta\text{-cl}(M)$, then N is $J\beta$ -closed set in X .

Proof. Let U be a η^* -open set in X , since M is $J\beta$ -closed, we have $\beta\text{-cl}(M) \subset U$. Let $M \subset N \subset \beta\text{-cl}(M) \subset U$. Since $N \subset \beta\text{-cl}(M)$, we have $\beta\text{-cl}(N) \subset \beta\text{-cl}(M)$. Then $\beta\text{-cl}(N) - N \subset \beta\text{-cl}(M) - M \subset U$. By the above theorem $\beta\text{-cl}(M) - M$ contains no non empty η^* -closed set in X . Therefore $\beta\text{-cl}(N) - N$ contains no non empty η^* -closed set in X . Hence N is a $J\beta$ -closed set in X .

4.7 Theorem. Let D be a $J\beta$ -closed set in (X, \mathfrak{T}) . Then D is β -closed if and only if $\beta\text{-cl}(D) - D$ is η^* -closed.

Proof. (Necessity): Let D be a β -closed set in X . Then $\beta\text{-cl}(D) = D$ and therefore $\beta\text{-cl}(D) - D = \phi$ which is a η^* -closed.

(Sufficiency): Let $\beta\text{-cl}(D) - D$ be a η^* -closed set. Since D is $J\beta$ -closed, By theorem $\beta\text{-cl}(D) - D$ does not contain a non-empty η^* -closed set which implies $\beta\text{-cl}(D) - D = \phi$. That is $\beta\text{-cl}(D) = D$. Hence D is β -closed.

4.8 Definition. Let $B \subset A \subset X$. Then B is $J\beta$ -closed relative to A if $\beta\text{-cl}_A(B) \subset M$, whenever $B \subset M$, M is η^* -open in A .

4.9 Theorem. Let $B \subset A \subset X$. and suppose that B is $J\beta$ -closed in X , then B is $J\beta$ -closed relative to A . The converse is true if A is β -closed in X .

Proof. Suppose that B is $J\beta$ -closed in X . Let $B \subset M$, M is η^* -open in A . Since M is η^* -open in A , $M = V \cap A$, where V is η^* -open in X . Hence $B \subset M \subset V$. Since B is $J\beta$ -closed in X , $\beta\text{-cl}_A(B) \subset M$. Hence $\beta\text{-cl}_A(B) \cap A \subset V \cap A$ which in turn implies that $\beta\text{-cl}_A(B) \subset V \cap A = M$. Therefore B is $J\beta$ -closed relative to A .
 Now to prove the converse, assume that $B \subset A \subset X$ where A is β -closed in X and B is $J\beta$ -closed relative to A . Let $B \subset M$, M is η^* -open in X . Then $A \cap M$ is η^* -open in A by the definition of subspace topology. Since $B \subset A$ and $B \subset M$, $B \subset A \cap M$. Since B is $J\beta$ -closed relative to A , $\beta\text{-cl}_A(B) \subset A \cap M$. Since $B \subset A$, $\beta\text{-cl}(B) \subset \beta\text{-cl}(A)$. Hence $\beta\text{-cl}(B) \subset A$. Therefore $\beta\text{-cl}(B) \cap A \subset \beta\text{-cl}(B)$ which implies $\beta\text{-cl}_A(B) = \beta\text{-cl}(B)$. Hence $\beta\text{-cl}(B) \subset A \cap M = M$. Thus B is $J\beta$ -closed in X .

5. Conclusion

The concept of new closed set namely $J\beta$ -closed set using η^* -open sets is introduced and studied. The relationship of $J\beta$ -closed sets using existing closed sets is established. Finally, some of their fundamental properties are studied. Hence I conclude that the defined set forms a topology which results in this work may be entered widely. The $J\beta$ -closed set can be used to derive a new decomposition of unity, closed map and open map, homeomorphism, closure and interior and new separation axioms. This idea can be extended to bitopological and fuzzy topological spaces.

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