JETIR.ORG ISSN: 2349-5162 | ESTD Year : 2014 | Monthly Issue
JOURNAL OF EMERGING TECHNOLOGIES AND INNOVATIVE RESEARCH (JETIR)
An International Scholarly Open Access, Peer-reviewed, Refereed Journal

SYMMETRY SOLUTION FOR THE BOUNDARY LAYER TWO-DIMENSIONAL FLOW OF POWER-LAW FLUID

[1]Bhavixa Bhagat, [2]M. G. Timol

[1]Research Scholar, [2]Retired Professor
[1]Department of Mathematics,
[1]Veer Narmad South Gujarat University, Surat-395007, Gujarat, India

Abstract: The symmetry solution of laminar, steady boundary layer two-dimensional flow of incompressible power-law fluids with appropriate boundary conditions is constructed using the Lie symmetry technique. The nonlinear partial differential equation of governing Non-Newtonian problem is turned into a nonlinear ordinary differential equation using the symmetry solution. The resultant ordinary differential equation with boundary conditions is converted to an initial value problem via one parameter of group transformations. A numerical solution for the initial value problem is calculated using Maple software by Fehlberg's fourth-fifth order Runge-Kutta technique. The characteristics of the velocity field for different physical parameters are also discussed by the graphical representation.

Keywords - Boundary Layer, Non-Newtonian Fluid, Symmetry solution, Runge-Kutta method

I. INTRODUCTION

The boundary layer theory of non-Newtonian fluid has been prominent notice in the study of fluid flow behaviour and the various characteristics of the fluid for many decennary and it is important being their existence in various manufacturing industries and engineering practices. Moreover, the nonlinearity of these equations fascinated mathematicians' s interest in evaluating the accuracy and power of the numerical and approximation techniques. This theory has been developed on different non-Newtonian fluid models. The most significant model is Ostwald-de Waele model that is also famous as the power-law model, which represents linear relation between the shear stress and shear rate for Newtonian fluids, while nonlinearity between the shear stress and shear rate for the non-Newtonian fluids. The power-law fluids are classified as pseudoplastic (shear thinning), Newtonian, and dilatant (shear-thickening) fluids.

The governing boundary layer flow of pseudoplastic fluids has been first investigated by Schowalter [34] and solved numerically for both pseudoplastic and dilatant fluids by Acrivos et al. [2]. The similarity solutions for the boundary equations of non-Newtonian power-law fluids are obtained by Kapur and Srivastava [23], Lee et al. [24], Na and Hansen [29], Timol et al. [40], Pakdemirli et al. [31], Denier and Dabrowski [15], Bognar [11] and Bilige [8]. Approximate solution of boundary layer laminar flow past flat plate is derived by Lemieux et al. [25]. The velocity profiles under various geometry of boundary layer theory of jet flows are also discussed by Schlichting [35], Bickley [7], Gutfinger et al. [18], Kapur [21] and others. The boundary layer flow past the flat plate is the traditional theory in various fields of engineering. Most of the investigation has tackled the interior flow in pipelines, channels, or annuli; external flow past submerged bodies has been considered in a few cases only. In this type of flow, objects are surrounded by the fluid and flow is known as external flow, for examples automobiles, air around the airplane, flow around submarines etc.

In recent past years, Mayer [28] has developed similarity equations of power-law fluids over the flat plate and solved them by the integral method. Mosayebidorcheh [27] has solved the boundary layer flow of shear thinning fluid over flat plat by different transform methods. Patil et al. [32] have produced similarity solutions of boundary layer flow of the Reiner-Philippoff fluid model. The similarity solution of the Sisko fluid model is developed via the method of dimensional analysis by Surati et al. [39]. Magan et al. [26] have derived the analytic solution of a jet flow of the power-law fluid by the Lie symmetry method. El-Gamel and El-Senawy [17] have solved the Blasius equation over a semi-infinity flat plate numerically. Shukla et. al [38] have derived non-similar solutions of flow past the vertical flat plate of Powell-Eyring fluid. Ou and Chen [30] have worked on hypersonic flow past flat plate in the continuum region. Al-Ashhab [5] has discussed the properties of nonlinear power-law fluid using "Crocco" variables.

Lie symmetry approach is employed in the recent past year by many researchers to solve boundary layer equations of power-law fluids. This approach is invented by Sophus Lie in the nineteenth century to find all symmetries of differential equations of one-parameter group transformations that leave a given family of equations invariant, no ad-hoc hypotheses or general study of the equation under examination is required. Generally, all known exact integration methods for ordinary and also partial differential equations are combined in this method [10]. This method provides symmetry solutions also known as similarity solutions of governing partial differential equation (PDE) which reduce the independent variables of PDE at once and convert PDE into ordinary differential equation (ODE). From the physical standpoint, the boundary layer equations are particularly intriguing because they can admit a large number of symmetry solutions. Akgul et al. [4] have applied the Lie group method on squeezed flow over a porous surface. Darji and Timol [14] have developed an invariant solution and discussed the velocity profile of similarity function of MHD boundary layer flow of Williamson fluid via the deductive group method. RamReddy et al. [33] have investigated similarity solution of a boundary layer flow of a micropolar fluid through a vertical plate. Afify and Uddin [3] have derived group transformations of boundary layer double diffusive flow using Lie symmetry method and solved numerically by the Runge-Kutta method. Abd-el-Malek et al. [1] have discussed natural convective flow past vertical plate via Lie symmetry method. The detailed concept of the Lie Symmetry technique is found in the literature [6, 9, 10, 13, 16, 19, 20, 37].

In this article, first, we have transformed the governing steady, 2-dimensional boundary layer flow equations of non-Newtonian power-law fluid into dimensionless PDE in terms of stream function. Motivated by the research paper of Kapur et al. [23], we have derived the most general form of symmetry solution of governing PDE. This symmetry solution is not presumed and reduced the (n-1) number of independent variables of the PDE at once and converted into the ODE with appropriate boundary conditions. Using the resultant boundary value problem (BVP), we have discussed the flow past flat plate case and converted it into an initial value problem (IVP) using one-parameter of group transformation via the Lie symmetry method. The reduced IVP is numerically solved by the fourth-fifth order Runge-Kutta method and presented graphically with characteristics of velocity profiles using Maple software.

II. Problem Formulation

We examine the 2-dimensional steady flow of viscous fluid. The governing boundary flow is incompressible, laminar flow of a non-Newtonian power-law fluid. For such a flow, equations for the boundary layer can be written as [23]:

$$\frac{\partial \bar{u}_*}{\partial \bar{x}_*}+\frac{\partial \bar{v}_*}{\partial \bar{y}_*}=0, \tag{1}$$

$$\bar{u}_*\frac{\partial \bar{u}_*}{\partial \bar{x}_*}+\bar{v}_*\frac{\partial \bar{u}_*}{\partial \bar{y}_*}=\nu\frac{\partial}{\partial \bar{y}_*}\left(\left|\frac{\partial \bar{u}_*}{\partial \bar{y}_*}\right|^{n-1}\frac{\partial \bar{u}_*}{\partial \bar{y}_*}\right)+U_\circ\frac{dU_\circ}{d\bar{x}_*}, \tag{2}$$

where $\bar{u}_*$ and $\bar{v}_*$ appoint for fluid velocity components in the $\bar{x}_*$ and $\bar{y}_*$ directions, $U_\circ$ represents velocity in the $\bar{x}_*$ direction outside the boundary layer, and $\nu=\mu/\rho$, μ,ρ,n are the consistency, the density of the fluid and the flow behaviour index.

The boundary conditions of the governing problem are:

$$\bar{u}_*\left(\bar{x}_*,0\right)=0,\ \ \bar{v}_*\left(\bar{x}_*,0\right)=0,\ \ \bar{u}_*\left(\bar{x}_*,\infty\right)=U_\circ(\bar{x}_*). \tag{3}$$

The above governing equations can be made a non-dimensional form using the following quantities.

$$x=\frac{\bar{x}_*}{L},\ \ y=\left(Re\right)^{1/n+1}\frac{\bar{y}_*}{L},\ \ u=\frac{\bar{u}_*}{U_\varepsilon},\ \ v=\left(Re\right)^{1/n+1}\frac{\bar{v}_*}{U_\varepsilon},\ \ U=\frac{U_\circ}{U_\varepsilon}\ \ ,Re=\frac{U_\varepsilon{}^{2-n}L^n}{\nu}, \tag{4}$$

where L is the reference length, U_ε is the velocity of the main stream, and Re is the generalized Reynolds number.

Considering equation (1), we suggest a dimensionless stream function $\psi\left(x,y\right)$, such that

$$u=\frac{\partial\psi}{\partial y},\ \ v=-\frac{\partial\psi}{\partial x}. \tag{5}$$

Substituting (4) and (5) into equations (1)-(3), we get (see [23]):

$$\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right)-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial Y}\right)=\frac{\partial}{\partial y}\left[\left|\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y}\right)\right|^{n-1}\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y}\right)\right]+U(x)\frac{dU}{dx}. \tag{6}$$

Together with the boundary conditions

$$\frac{\partial\psi}{\partial y}\left(x,0\right)=0,\ \ \frac{\partial\psi}{\partial x}\left(x,0\right)=0,\ \ \frac{\partial\psi}{\partial y}\left(x,\infty\right)=U(x). \tag{7}$$

III. Lie Symmetry Approach

First, we derive the Lie symmetry generator and then find symmetry solution of equation (6) that converted PDE in equation (6) with boundary condition (7) into ODE using the Lie symmetry method.

3.1 Lie symmetry generator

We can rewrite the equation (6) as follows:

$$\Omega=\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right)-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial Y}\right)-\frac{\partial}{\partial y}\left[\left|\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y}\right)\right|^{n-1}\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y}\right)\right]-U(x)\frac{dU}{dx}=0. \tag{8}$$

Now, we consider the Lie symmetries of equation (8) characterized by the symmetry generator

$$G=\xi^1\left(x,y,U,\psi\right)\frac{\partial}{\partial x}+\xi^2\left(x,y,U,\psi\right)\frac{\partial}{\partial y}+\eta\left(x,y,U,\psi\right)\frac{\partial}{\partial\psi}+\tau\left(x,y,U,\psi\right)\frac{\partial}{\partial U}. \tag{9}$$

The symmetry generators are obtained by solving the determining equations which will be generated by the Lie's invariance condition

$$G^{(3)}\left(\Omega\right)\big|_{\Omega=0}=0, \tag{10}$$

where

$$G^{(3)}=G+\zeta_{[1]}\frac{\partial}{\partial\psi_x}+\zeta_{[2]}\frac{\partial}{\partial\psi_y}+\zeta_{[12]}\frac{\partial}{\partial\psi_{xy}}+\zeta_{[22]}\frac{\partial}{\partial\psi_{yy}}+\zeta_{[222]}\frac{\partial}{\partial\psi_{yyy}}+\tau_{[1]}\frac{\partial}{\partial U_x}, \tag{11}$$

is the third prolongation of G, subscripts of ψ and U represent partial derivative and

$$\zeta_{[i]} = D_i(\eta) - \psi_x D_i(\xi^1) - \psi_y D_i(\xi^2), \tag{12}$$

$$\zeta_{[ij]} = D_j(\zeta_{[i]}) - \psi_{ix} D_j(\xi^1) - \psi_{iy} D_j(\xi^2), \tag{13}$$

$$\tau_{[x]} = D_x(\tau) - U_x D_x(\xi^1), \tag{14}$$

where i and j stand for x, y. In equations (12)-(14), D represents the total derivatives with respect to x and y (see [6], [10]).

The determining equations are separated according to the derivative of ψ and U. Solving the determining equations and using the invariance condition of boundary conditions (7), gives the following five Lie symmetry generators:

$$G_1 = x\frac{\partial}{\partial x} - \frac{\psi}{n-2}\frac{\partial}{\partial \psi} - \frac{U}{n-2}\frac{\partial}{\partial U}, G_2 = \frac{\partial}{\partial x}, G_3 = y\frac{\partial}{\partial y} + \frac{2n-1}{n-2}\psi\frac{\partial}{\partial \psi} + \frac{n+1}{n-2}U\frac{\partial}{\partial U},$$
$$G_4 = \frac{\partial}{\partial \psi}, G_5 = \frac{1}{U}\frac{\partial}{\partial U}. \tag{15}$$

Therefore, the symmetry generator in Eq. (9) of PDE (6) is obtained by the following linear combination of point symmetries:

$$G = c_1 G_1 + c_2 G_2 + c_3 G_3 + c_4 G_4, \tag{16}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

3.2 Symmetry solution

To derive the symmetry solution of PDE (6), we consider the symmetry generator G given by Eq. (16). Then using the invariant condition of the Lie symmetry method, the characteristic equations are

$$\frac{dx}{c_1 x + c_2} = \frac{dy}{c_3 y} = \frac{d\psi}{\frac{\left[(2n-1)c_3 - c_1\right]}{(n-2)} + c_4} = \frac{dU}{\frac{\left[(n+1)c_3 - c_1\right]U^2 + c_5(n-2)}{U(n-2)}} \tag{17}$$

Take c_3=1in (17) without any loss of generality. Then the solving (17) gives the general symmetry solutions of PDE (7) as:

$$\psi = \left[x + \frac{c_2}{c_1}\right]^{\frac{2n-1-c_1}{c_1(n-2)}} F(\xi) - \frac{c_4(n-2)}{(2n-1)-c_1}, \; U = C\left[x + \frac{c_2}{c_1}\right]^{\frac{n+1-c_1}{c_1(n-2)}} - \left[\frac{c_5(n-2)}{(n+1)-c_1}\right]^{1/2}, \tag{18}$$

where F is an arbitrary function, C is an arbitrary constant and similarity variable is

$$\xi = \frac{y}{\left[x + \frac{c_2}{c_1}\right]^{1/c_1}}. \tag{19}$$

Since the stream function is prescribed up to an arbitrary constant, we may choose c_4=0 and also c_5=0 without any loss of generality.

Now substitution the derivatives of Eq. (18) in equation (6) yields the third-order nonlinear ODE:

$$\frac{d}{d\xi}\left[|F''|^{n-1} F''\right] + pFF'' + q\left[1 - (F')^2\right] = 0, \tag{20}$$

where prime designates the differentiation with respect to ξ and

$$p = \frac{2n-1-c_1}{c_1(n-2)}, \quad q = \frac{n+1-c_1}{c_1(n-2)}. \tag{21}$$

The corresponding boundary conditions in Eq. (7) also transform to

$$F(0) = 0, \quad F'(0) = 0, \quad F'(\infty) = 1. \tag{22}$$

The equation (20) is an ODE, if $(n+1)p - (2n-1)q \neq 0$,

IV. The Flow Past a Flat Plate

Here we examine the case of flow past a flat plate. If we put $q=0$, equation (20) reduces to the generalized Blasius problem

$$\frac{d}{d\xi}\left[|F''|^{n-1} F''\right] + \frac{1}{n+1}FF'' = 0. \tag{23}$$

Together with boundary conditions

$$F(0) = 0, \quad F'(0) = 0, \quad F'(\infty) = 1. \tag{24}$$

Thus, equation (23) is a nonlinear BVP with boundary conditions (24).

As the equations (18) and (19), the dimensionless velocities can be acquired by as a function of similarity variable:

$$u(x,y) = F'(\xi),$$
$$v(x,y) = \bar{v}\left[\xi F'(\xi) - F(\xi)\right], \tag{25}$$

where $\bar{v}(x) = \mathrm{Re}_x^{-1/n+1}/n+1$,

Now, we convert the BVP (23)-(24) into the IVP by the Lie symmetry method. Using the Lie symmetry generator of a differential equation, we can obtain one-parameter group transformations using Lie equations [20].

Thus, the group transformations are

$$\bar{\xi} = e^{(n-2/3)\alpha}\xi \text{ and } \bar{F} = e^{(2n-1/3)\alpha}F. \tag{26}$$

where "α" is the parameter of group transformation.

Under the transformations (26), equation (23) becomes invariant and transforms to

$$\frac{d}{d\bar{\xi}}\left[\left|\left(\bar{F}''\right)\right|^{n-1}\bar{F}''\right]+\frac{1}{n+1}\bar{F}\,\bar{F}''=0. \quad (27)$$

And boundary conditions (24) at zero, become

$$\bar{F}(0)=0, \quad \bar{F}'(0)=0. \quad (28)$$

Now, set missing initial condition is equal to the parameter of transformation as:

$$F''(0)=\alpha. \quad (29)$$

Using the transformation (26), Eq. (29) yields

$$e^{-\alpha}\bar{F}''(0)=\alpha, \quad (30)$$

which is independent of α.

Hence, we get $\bar{F}''(0)=1$. (31)

To determine α , from Eqs. (24) and (26) yield

$$\alpha=\left[\frac{1}{\bar{F}'(\infty)}\right]^{3/n+1}. \quad (32)$$

Thus, we get the following IVP

$$\frac{d}{d\xi}\left[\left|\left(F''\right)^{n-1}\right|F''\right]+\frac{1}{n+1}FF''=0. \quad (33)$$

Corresponding the initial conditions

$$F(0)=0, \quad F'(0)=0, \quad F''(0)=\alpha. \quad (34)$$

V. Results And Discussion

The numerical solution of governing problem is calculated by Runge-Kutta fourth-fifth order method using the Maple software. The unknown parameter α is calculated from equation (32). Here to solve initial value problem (33)-(34), we have to replace $F''(0)=\alpha$ by $\bar{F}''(0)=1$ (see Eqs. (26), (27) and (30)). The key element of this method is to choose suitable finite value of $\bar{\xi}_\infty$, say $\bar{\xi}_i$ such that value of $\bar{F}'(\bar{\xi}_i)\cong\bar{F}'(\bar{\xi}_\infty)$.

Table 1. Numerical Results of initial value problem governing by equations (27) and (33)

Power index (*n*)	$\bar{F}'(\bar{\xi}_i)$	$\bar{\xi}_i$	$F''(0)=\alpha$	ξ_i
0.3	1.507245	450	0.3879774602	769.5239949
0.6	1.824661	40	0.3238055491	67.70082964
0.7	1.900524	25	0.3220108789	40.85038050
0.8	1.968073	19	0.3235424931	29.83878051
0.9	2.029252	8.6	0.3271392541	12.95477306
1	2.085409	6.8	0.3320573784	9.819842777
1.5	2.317291	3.676	0.3647735251	4.348821857
2	2.501309	3.3613	0.3997906696	3.3613
3	2.796412	3.3328	0.4624333340	2.577280416
5	3.241218	3.57388	0.5554511610	1.985119017
10	4.004006	4.194686	0.6849884652	1.529401383
15	4.550713	4.691876	0.7526795277	1.369799411

Table 1 Shows the numerical calculation for $\bar{\xi}_i$ of the solution of equation (27) with initial conditions (28) and (31) for different values of *n*. The results in Table 1 indicate that the thickness ξ_i is rapidly decreasing until $n=3$. Table 1 also contains the correponding value of $\bar{F}'(\bar{\xi}_i)$ and the value of α for values of *n* running from 0.3 to 15.

The velocities profile of initial value problem (33) with (34), is displayed in Figs. 1, 2 and 3. Fig.1 represents the velocity components $u(x,y)=F'(\xi)$ parallel to the wall for some n-values (n=0.6, 1, 1.5, 3). Fig. 1 shows that as the value of flow consistency index increases $F'(\xi)$ assumes the value 1.0. The velocity components perpendicular to the wall are indicated in Fig. 2 by plotting $v(x,y)/\bar{v}$ for some various values of power law index *n*.

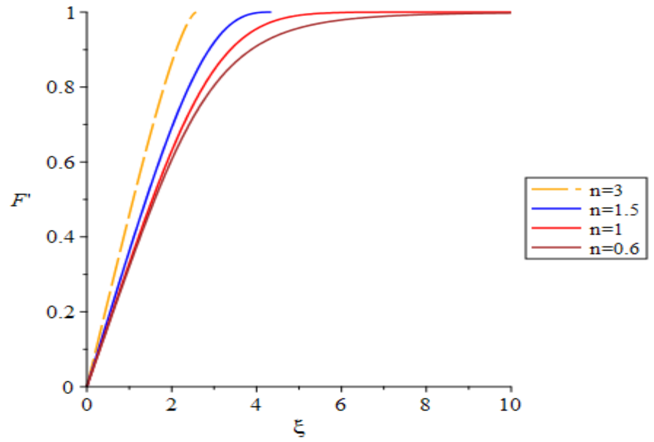

Figure 1. The velocity profiles of $u(x,y)=F'(\xi)$ for various values of n

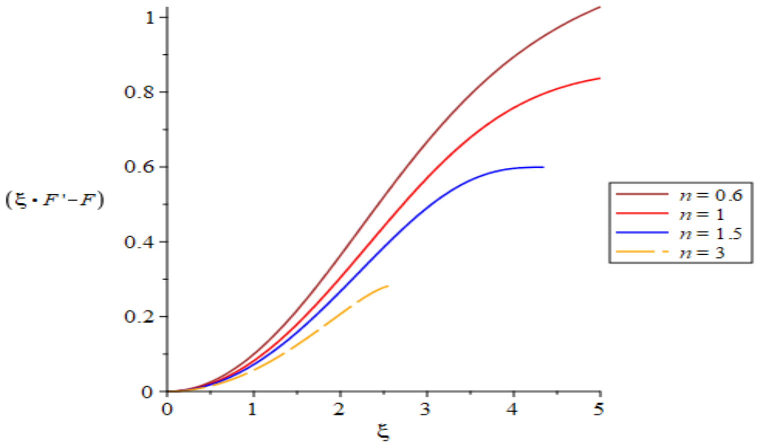

Figure 2. The velocity profiles of $v(x,y)/\bar{v}=\xi.F'(\xi)-F(\xi)$ for different values of n

Fig. 3 presents the cross-stream variation of the velocity gradient $F''(\xi)$ for different power low index n. It is shown that the $F''(\xi)$ decrease monotonically from $F''(0)$ at the wall to zero outside the viscous boundary layer.

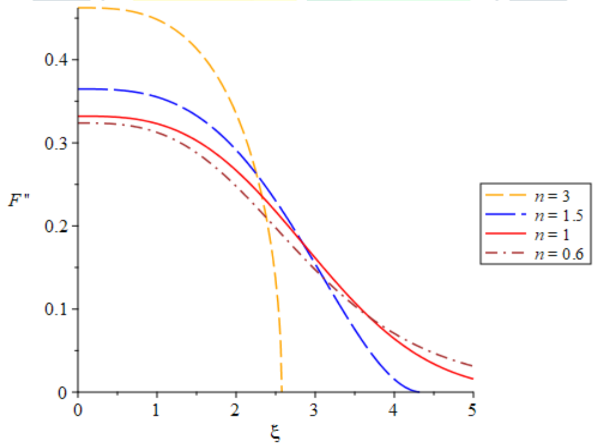

Figure 3. The variation of the velocity gradient $F''(\xi)$ across the stream

VI. Conclusion

In the present article, we have derived the general form of the symmetry solutions of two-dimensional boundary layer equations of non-Newtonian power-law fluids using Lie symmetry method. The derived symmetry solutions are reduced the independent variables of PDE at once of governing problem and convert that PDE into ODE with appropriate boundary conditions. Using the flow past flat plate case, the resultant BVP is transformed into IVP using one-parameter transformations, these transformations are generated by Lie symmetry generator. The relevant numerical solutions and graphs are presented the flow behavior past flat plate with different n-values. We hope that this research development is useful for the future researchers to investigate the flow behavior of different flows like two-dimensional stagnation point flow, flow past convergent channels with different surface geometry.

References

[1] Abd-el-Malek, M.B., Badran, N.A. and Amin, A.M. and Hanafy, A.M. 2021. Lie symmetry group for unsteady free convection boundary-layer flow over a vertical surface, Symmetry, 13(175): 1-11.

[2] Acrivos, A., Shah, M. J. and Petersen, E. E. 1960. Momentum and heat transfer in laminar boundary-layer flows of non-Newtonian fluids past external surfaces, A.I.Ch.E. Journal, 6 (2): 312 -317.

[3] Afify, A.A. and Uddin, Md. J. 2016. Lie Symmetry Analysis of a Double-Diffusive Free Convective Slip Flow with a Convective Boundary Condition Past a Radiating Vertical Surface Embedded in a Porous Medium, Journal of Applied Mechanics and Technical Physics, 57(5): 925-936.

[4] Akgul, M.B. and Pakdemirli, M. 2012. Lie group of a non-Newtonian fluid flow over a porous surface, Scientia Iranica, Transanctions B: Mechanical Engineering 19:1534-1540.

[5] Al-Ashhab, S. 2020. Properties of boundary-layer flow solutions for non-Newtonian fluids with non-linear terms of first and second-order derivatives, J. Eng. Math., 123: 29-39.

[6] Arrigo, D.J. 2015. Symmetry Analysis of Differential Equations, John Wiley & Sons, Inc., New Jersey.

[7] Bickley, W. G. 1937. The Plane Jet, Phil. Mag., 23: 727-731.

[8] Bilige, S. and Han, Y. 2018. Symmetry reduction and numerical solution of a nonlinear boundary value problem in fluid mechanics, International Journal of Numerical Methods for Heat & Fluid Flow, 28(3): 518-531.

[9] Bluman, G. W. and Anco, S. C. 2002. Symmetry and Integration Methods for Differential Equations. Springer-Verlag, New York.

[10] Bluman, G.W. and Kumei, S. 1989. Symmetries and Differential Equations, Applied Mathematical Science 81, Springer-Verlag, New York.

[11] Bognár, G. 2010. Similarity solution of a boundary layer flow for non-Newtonian fluids, Int. J. Nonlinear Sci. & Numerical Simulation, 10:1555-1566.

[12] Bognár, G. 2011. Analytic Solutions to a Boundary Layer Problem for Non-Newtonian Fluid Flow Driven by Power Law Velocity Profile, WSEAS Transactions on Fluid Mechanics, 6: 22-31.

[13] Chhabra, R.P. and Richardson J.F. 2008, Second Edition. Non-Newtonian Flow and Applied Rheology: Engineering Applications, Butterworth-Heinemann/IChemE.

[14] Darji, R.M. and Timol, M.G. 2014. On variance analysis of MHD boundary layer equations for non-Newtonian Williamson fluids, International Journal of Advances In Applied Mathematics and Mechanics (IJAAMM), 1(4): 10-19.

[15] Denier, J. P. and Dabrowski, P. P. 2004. On the boundary-layer equations for power-law fluids, Proceedings of the Royal Society A, 460(2051): 3143–3158.

[16] Dresner, L. 1999. Applications of Lie's Theory of Ordinary and Partial Differential Equations, Institute of Physics Publishing, Bristol.

[17] El-Gamel, M. and El-Shenawy, A. 2018. A numerical solution of Blasius equation on a semi-infinity flat plate, SeMA Journal, 75: 475-484.

[18] Gutfinger, C. and Shinar, R. 1964. Velocity distribution in two-dimensional laminar liquid-into-liquid Jets in power fluids, AIChE Journal, 10: 631-639.

[19] Hydon, P.E. (2000). Symmetry Methods for Differential Equations: A Beginner's Guide, Cambridge University Press, UK.

[20] Ibragimov, N.H. and Kovalev, V. F. 2009. Approximate and Renormgroup Symmetries. Beijing and Springer-verlag: Berlin, Germany. Bristol.

[21] Kapur, J.N. 1962. On the two-dimensional jet of an incompressible pseudo-plastic Fluid. Journal of the Physical Society of Japan, 17: 1303-1309.

[22] Kapur, J. N. 1963. Flows of Power-Law Fluids Past a Flat Plate with Uniform Suction and Injection, Journal of the Physical Society of Japan, 18(4): 578-582.

[23] Kapur, J.N. and Srivastava, R.C. 1963. Similar Solution of the Boundary Layer Equations for Power Law Fluids, Journal of Applied Mathematics and Physics (ZAMP), 14: 382-388.

[24] Lee, S. Y. and Ames, W. F. 1966. Similarity Solutions for Non-Newtonian Fluids, A.1.Ch.E. Journal, 12(4): 700-707.

[25] Lemieux, P. F., Dubey, R. N. and Unny, T. E. 1971. Variational Method for a Pseudoplastic Fluid in a Laminar Boundary Layer Over a Flat Plate, Journal of Applied Mechanics, Transactions ASME, 345-349.

[26] Magan, A.B., Mason, D.P. and Mahomed, F.M. 2017. Analytical solution in parametric form for the two-dimensional liquid jet of a power-law fluid, International Journal of Non-Linear Mechanics, 93: 53-64.

[27] Mosayebidorcheh, S. (2013). Solution of the Boundary Layer Equation of the Power-Law Pseudoplastic Fluid Using Differential Transform Method, Mathematical Problems in Engineering Journal, Hindawi Publishing Corporation, 2013, Article ID 685454: 1-8.

[28] Myers T. G. 2010. An approximate solution method for boundary layer flow of a power law fluid over a flat plate, International Journal of Heat and Mass Transfer, 53(11-12): 2337-2346.

[29] Na, T.Y. and Hansen A.G. 1967. Similarity Solutions of a Class of Laminar Three-Dimensional Boundary Layer Equations of Power Law Fluids, Int. J. Non-Linear Mechanics 2: 373-385.

[30] Ou J. and Chen J. (2019). DSMC data-improved numerical simulation of hypersonic flow past a flat plate in near-continuum regime, Computers and Fluids, 194, Article 104308: 1-12.

[31] Pakdemirli, M., Yurusoy, M. and Kucukbursa, A. 1995. Symmetry groups of boundary layer equations of a class of non-Newtonian fluids, Int. J. Non-Linear Mechanics, 31(3): 267-276.

[32] Patil V.S., Patil N.S. and Timol, M.G. 2015. A remark on similarity analysis of boundary layer equations of a class of non-Newtonian fluids, International Journal of Non-Linear Mechanics, 71: 127-131.

[33] RamReddy, Ch., Pradeepa, T. and Srinivasacharya D. 2015. Similarity Solution for free convection flow of a micropolar fluid under convective boundary condition via Lie scaling group transformations, Advances in High Energy Physics, Article Id 650813: 1-16.

[34] Schowalter W.R. (1960). The Application of Boundary-Layer Theory to Power-Law Pseudoplastic Fluids: Similar Solutions, A.I.Ch.E. Journal, 6 (1): 24-28.

[35] Schlichting, H. (1933), Laminar Strahlauspreitung, ZAMM, 13: 260- 263.

[36] Shanker R., Kumar P., Aneja A. and Ashutosh S. (2017). Boundary-Layer Flows of Non-Newtonian Power Law Fluids, International Journal of Scientific & Engineering Research, 8(10): 456-461.

[37] Sheshadri, R. and Na T.Y. (1985). Group Invariance in Engineering Boundary Value Problems, Springer-Verlag, New York.

[38] Shukla, H., Surati, H. and Timol M.G. (2019). Local non-similar solution of Powell-Eyring fluid flow over a vertical flat plate, Applications and Applied Mathematics: An International Journal (AAM), 14(2): 973-984.

[39] Surati, H. C., Shukla, H. and Timol, M.G. 2016. On the Transformation of Non-Newtonian Boundary Value Problem to Initial Value Problem, International Journal of Engineering Science Invention Research & Development, 3(3): 180-189.

[40] Timol, M.G. and Kalthia N.L. (1986). Similarity solutions of three-dimensional boundary layer equations of non-Newtonian fluids, Int. J. Non-Linear Mechanics, 21(6): 475-481.