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Logistic Growth Model and its Extention with Verification in aspect of India's Population

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ABSTRACT

Population growth model tries to predict the population of an organism that reproduces according to fixed rules. Depending on how many times and how often it reproduces, how many new organisms it produces each time and how often it reproduces, the model can predict what the India's Population will be at a given time.

Populations do not usually grow in unlimited factors stop the population increase. Two limiting factors are lack of resources and mortality. Limiting factors have the greatest effect on large populations that have grown rapidly. As population is growing due to overcrowding and limitation of resources, the birth rate decreases and death rate increases with the population size. The present paper provides logistic growth model and its extension with verification in aspects of India's population.

Key words: Population, Ecology, Continuous growth, Logistic growth, Crowding effects.

INTRODUCTION

A fundamental problem in ecology is that of growth, whether it is the growth of a cell, an organ, a plant, a human or population etc. Single species models are relevance to laboratory studies in particular but, in the real world, can reflet a telescoping of effects which influence the population dynamics[2, 5]. Human population grow is much the same way as the population of any other organism. Therefore, the principle of population growth is applied to human. The population biology or mathematical ecology deals with the increase and fluctuation of populations. The fundamental study of the problem in ecology is not of recent origin. In fact, Lotka and Voltera were early pioneers developing foundation work in this field [11, 12]. The book by Nisbet and Gurney [9] is comprehensive account of mathematical modelling in population dynamics: a good elementary introduction is given in the book Edelstein-Keshet [3]. If organisms can't find enough of the resources they need to grow and reproduce they will have fewer or no young and the rate of population growth goes down. If many in the population die due to predators or disease, population growth is also reduced. If lack of resources such as food or water cause a high mortality rate, it also limits growth, but the mechanism in this case is different from a lack of food simply leading to fewer births

According to Ahsan and Kapur, continuous growth can generate enormous numbers in short time. It would be useful to understand that as the population grows exponentially with it, the demand for resources such as water, food, fertilizers etc. grow exponentially [5, 8]. Hence, even small reduction in exponential growth rates are important contributions towards efforts to maintain the human population within the earth's carrying capacity.

The principle of exponential growth for human populations was first produced by Malthus an English Clergyman and political economist in his famous book 'An essay on the principle of population' published in 1798 [7]. Malthus achieved notoriety through this work for publishing that human population grows at a (geometrical) rate that is faster than the (arithmetical) rate of growth of supply of commodities necessary for life. First of all, here we discussed continuous growth model.

1.1 FORMULATION OF THE CONTINUOUS GROWTH MODEL:

Let us take the following assumptions for the formulating the population growth model:

i) The population is sufficiently large

ii) Population is homogeneous

iii) There are no limitation to growth.

i.e., no limitations of food, space and so on. Population changes only by the occurrence of birth and deaths.

Let us assume that population be denoted by N(t), which is always an integer. Also, let us assume that during a small unit time period, a percentage 'b' of the population is newly born and a percentage 'd' of the population dies. Then, new population is

$$N(t+\Delta t) = N(t) + b\ N(t)\Delta t - d.N(t)\Delta t$$

$$\Rightarrow N(t+\Delta t) - N(t) = [b\ N(t) - d\ N(t)]\Delta t$$

$$\Rightarrow \frac{N(t+\Delta t) - N(t)}{\Delta t} = (b-d)N(t)$$

Which implies that,

$$\Rightarrow \frac{\Delta N(t)}{\Delta t} = rN(t) \qquad \ldots (1.1.1)$$

where $r = b - d$ is a positive constant

from our assumption that average rate of change of the population over an interval of time is proportional to the size of the population. Using the instantaneous rate of change to approximate the average rate of change, we have the following differential equation model

$$\frac{dN}{dt} = rN, \qquad \ldots (1.1.2)$$

With, $N(t_0) = N_0, \quad t_0 \le t$ … (1.1.3)

SOLVING THE MODEL:

On separating the variables of (1.1.2), we have

$$\frac{dN}{N} = rdt,$$ on integrating, we have

$$Log\, N = rt + c, \qquad \dots (1.1.4)$$

where c is a integration constant.

Applying the conditions given by (1.1.3), the equation (1.1.4) is

$c = \log N_0 - rt_0$

Putting the value of c in (1.1.4), we get

$$\log \frac{N}{N_0} = r(t - t_0) \qquad \dots (1.1.5)$$

Finally, exponentiating both sides of the equation (1.1.5), we get the solution

$$N(t) = N_0 e^{r(t-t_0)} \qquad \dots (1.1.6)$$

Equation (1.1.6) is known as Malthusian model of population growth which shows that population grows exponentially with time. Again, equation (1.1.6) gives the population size at any time t.

If the net growth rate $r > 0$, N(t) grows exponentially without any bound (Fig 1). For $r < 0$, $N(t) \to 0$ $at\ t \to \infty$ implying that the population is ultimately driven extinction. Both these out-comes are extreme and are not found to occur in the nature.

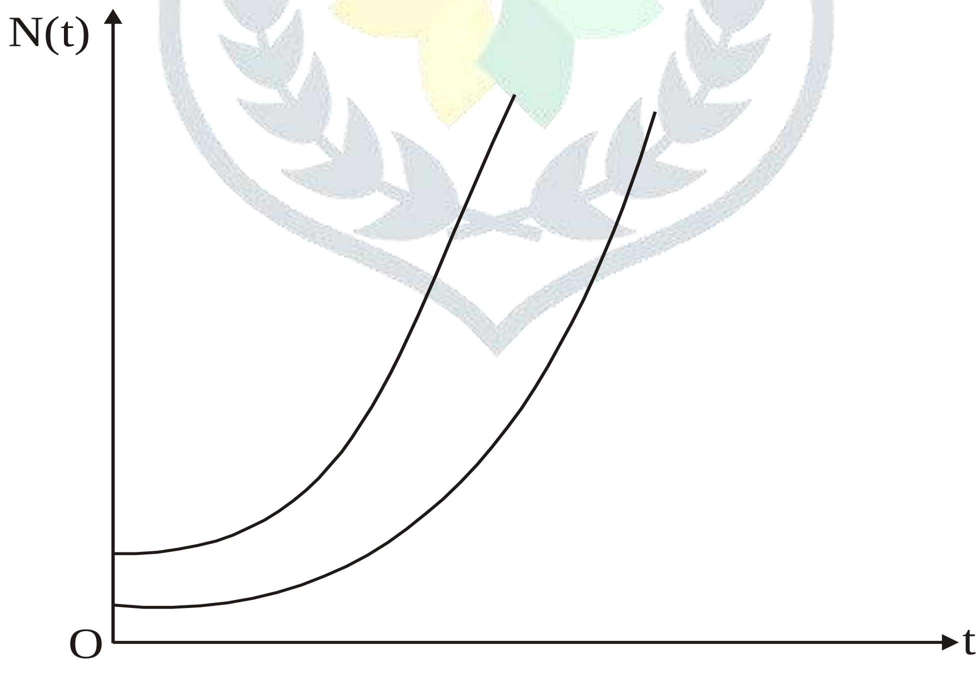

Fig 1: Exponential Growth Model

GRAPHICAL REPRESENTATION:

Using equation (1.1.6) we have, $N(t) = N_0 e^{rt}$

Then, we have the following observations:

(I) the population grows, exponentially if r > 0, decays exponentially if r < 0 and remains constant if r = 0 (Fig 2)

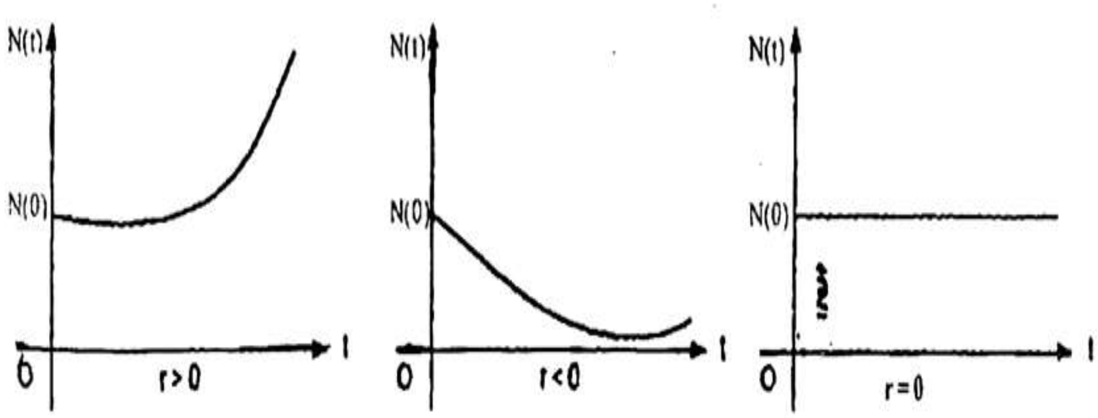

Figure - 2

(II) If r > 0, the population will become double its present size at time t_d where

$$2N(0) = N_0 e^{t_d} \quad or \quad e^{t_d} = 2$$

$$\Rightarrow t_d = \frac{1}{r} log2 = (0.6931) r^{-1}$$

Here, t_d is called the doubling period of the population and it may be noted that this doubling period is independent of N(0). It depends only on r and is such that greater the value of r i.e. greater the difference between births and death rates, the smaller is the doubling period.

(III) If r < 1, the population will become half its present size in time t_d, where $\frac{1}{2} N(0) = N_0 e^{rt_h} \quad \Rightarrow e^{rt_h} = \frac{1}{2}$

i.e. $t_h = \frac{1}{r} log \frac{1}{2} = -(0.69314) r^{-1}$

Clearly, t_h is also independent of N(0) and since r < 0, $t_h > 0$. t_h may be called the half life (period) of the population and it decreases as the excess of death rate over birth rate increases.

VERIFYING THE MODEL:

Since, $\log \frac{N}{N_0} = r(t - t_0)$, our model predicts that if we plot $\log \frac{N}{N_0}$ versus t – t_0, a straight line passing the origin with slope r should result. However, if we plot the population data for the India for several years, the model does not fit very well, especially in the later years. In fact, census [1] for the population of India in 2011 census was 1,210,193,422 and in 1991 it was 846,427,039 putting these values in (1.1.6), we get

$$\frac{1,210,193,422}{846,427,039} = e^{r}(2011-1991)$$

$$\therefore r = \frac{1}{20}\log\frac{1210193422}{846427039}$$

$$r = 0.018$$

That is during the 20 years period for 1991 to 2011, population of India was increasing at the average rate of 1.8% per year. We can use this information with equation (1.1.6) to predict the population of India in 2020 in this case

$t_0 = 2011$, $N_o = 1210193422$ & $r = 0.018$

$$N(2020) = 1210193422e^{0.018*(2020-2011)}$$

$$= 1,448,601,539$$

But in July 2020 (est.), the population of India was 1,380,004,385. Therefore, our prediction is off the mark by approximately 5 %. We can probably live with that magnitude of error, but let us look in the distant future. Our model predicts that the population of the India will be 40.30 billion in the 2200; a population that for exceeds current estimate of the maximum sustainable population of the entire planet.

Also, due to Malthus 1798 but actually suggested earlier, by Euler is pretty unrealistic. However, if we consider the past and predicted growth estimates for the total world population from the 17th to 21st centuries it is perhaps less unrealistic as seen in the following Table – I.

Table - 1

Date	Mid 17th Century	Mid 19th Century	1918 – 27	1960	1974	1987	1999	2010	2022
Population (billions)	0.5	1	2	3	4	5	6	7	8

Hence, we are forced to conclude that our model have some limitations for a long period.

LIMITATIONS:

Under some ideal conditions, when the availability of space, food and other resources do not inhibit growth, many biological populations are observed to grow initially at an approximately exponential rate. After some time, when the population size becomes considerably large, there is a lack of food, space and other resources; also, there is a population due to over- crowding. All these consequences are collectively called "Crowding effects". The crowding effect forces the growth rate to decline. These considerations make it clear that the growth rate r cannot be constant, but must depend on the size or density of the population. When r is a decreasing function of N, the model is said to describe a process of 'feedback' or 'compensation'.

1.2. LOGISTIC GROWTH MODEL (Verhulst Model)

Under some ideal condition, when the availability of space to grow, plenty of food and other resources do not inhibit growth, may biological populations are observed to grow initially at an approximately exponential rate. After some time, when the population size becomes considerably large, there is a lack of food, space and other resources; Also there is pollution due to overcrowding [4, 8, 11]. All these consequences are collectively called "Crowding effects".

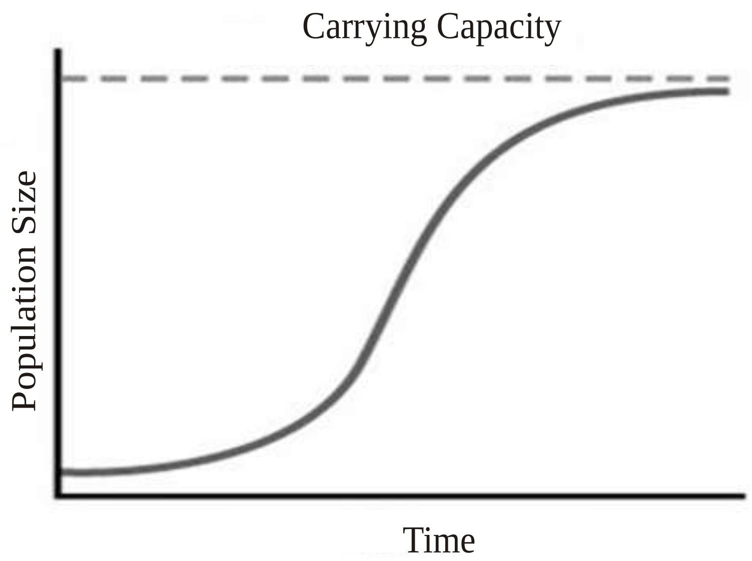

Figure – 3 : Logistic growth curve

As a population is growth due to overcrowding and limitations of resources, the birth rate b decreases and death rate d increases with the population size N. The growth curve defined by such a population follows S-shaped or sigmoid pattern when density is plotted versus time. Such simple or ideal growth form is called 'Logistic' and the corresponding growth equation is known as logistic equation [4, 6, 7, 10]. The sigmoid curve was first suggested to describe the growth of human populations by P.F. Verhulst in 1836.

FORMULATION OF THE MODEL:

Let us assume that r to be positive and let $r = r_1\left(1-\frac{N}{K}\right)$, $r_1 > 0$ and $K > 0$ are constant. Equation becomes

$$\frac{dN}{dt} = r_1 N\left(1-\frac{N}{K}\right), \quad N(0) = N_0 \qquad \ldots (1.2.1)$$

Equation (1.2.1) is known as Verhulst's famous logistic equation and also called logistic growth in population. In this model the per capita birth rate is

$$r = r_1\left(1-\frac{N}{K}\right),$$

It depends on N. The constant K is the carrying capacity of the environment, which is usually determined by the available sustaining recourses.

GRAHICAL REPRESENTATION:

There are two steady states or equilibrium states for (1.2.1), namely N = 0 and N = K, that is where $\frac{dN}{dt} = 0. N = 0$ is unstable since linearization about it $\frac{dN}{dt} \approx r_1 N$ and so N grows exponentially from any initial value. The other equilibrium N = K is stable: linearization about it give $\frac{d(N-K)}{dt} \approx -r_1(N-K)$ and so $N \to K \text{ as } t \to \infty$. the carrying capacity K determines the size of the stable steady state population while r_1 is a measure of the rate at which it is reached, that is, it is a measure of the dynamics.

Here, the horizontal lines are the equilibrium solutions N(t) = 0 and N(t) = K. if the initial population level $N_0 > K$, N(t) monotonically decreases towards K. the upper curve depicts this situation. Also, the lower curve, with its characteristic 'Sigmoid' or 'Ogive' shape known as 'Logistic growth curve'.

Solution of the Model

Equation (1.2.1) can be rewritten as

$$K\frac{dN}{dt} = r_1 NK - r_1 N^2 = r_1 N(K-N)$$

on separating the variables, we get

$$\frac{KdN}{N(K-N)} = r_1 dt$$

Which can also be written as

$$\left[\frac{1}{N} + \frac{1}{K-N}\right] dN = r_1 dt \qquad \ldots (1.2.2)$$

Integrating (1.2.2) and solving, we get

$$\log N - \log(K-N) = r_1 t + C$$

$$\log\frac{N}{K-N} = r_1 t + C \qquad \ldots (1.2.3)$$

Where C is integration constant. Using (1.2.1), we get

$$C = \log\frac{N_0}{K-N_0}$$

Substituting the value of C in (1.2.3), we get

$$\log\frac{N}{K-N} = \log e^{r_1 t} + \log\frac{N_0}{K-N_0} = \log\frac{N_0 e^{r_1 t}}{K-N_0}$$

which gives

$$\frac{N}{K-N} = \frac{N_0 e^{r_1 t}}{K-N_0}$$

$$\Rightarrow [(K-N_0) + N_0 e^{r_1 t}]N = KN_0 e^{r_1 t}$$

$$\Rightarrow N = \frac{K}{1+\left(\frac{K-N_0}{N_0}\right)e^{-r_1 t}} \qquad \dots (1.2.4)$$

Hence,

$$N(t) = \frac{K}{1+K_1 e^{-r_1 t}} \qquad \dots (1.2.5)$$

where $K_1 = \frac{K-N_0}{N_0}$, a constant.

This is known as the Verhulst's formula.

Taking limit $t \to \infty$ in (1.2.5), We get (since $r_1 > 0$)

$$N_{max} = \text{Lim}\, N(t) \to K \quad as\ t \to \infty = K \qquad \dots (1.2.6)$$

This shows that there is a limit to growth of N as desired by biological facts and tends to indicate the correctness of our model.

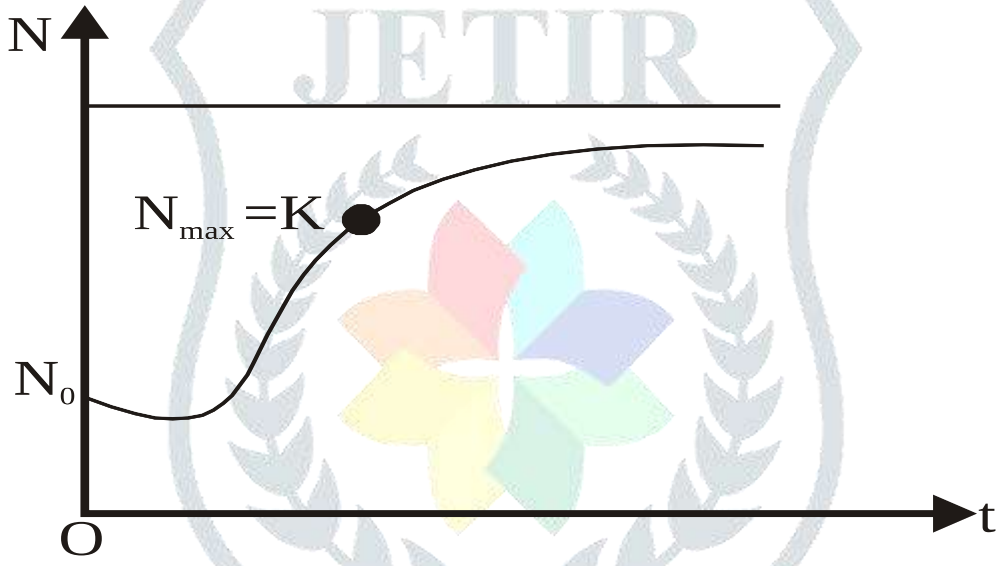

Fig 4: Solution curve of the logistics growth equation.

Interpretation:

Equation (1.2.5) represents the size of the population at any time t. From (1.2.5) it is also clear that $N(t) \to K$ as $t \to \infty$. therefore, a population that satisfies the logistic equation is not like a naturally growing population, it does not grow without bound, but approaches the finite limiting population K as $t \to \infty$., but since $\frac{dN}{dt} > 0$ in this case, therefore, population is steadily increasing.

Now, differentiating (1.2.1) w.r.to t, we have

$$\frac{d^2N}{dt^2} = [\frac{dN}{dt} - \frac{2N}{K}\frac{dN}{dt}] = \frac{r_1}{K}[K - 2N]\frac{dN}{dt}$$

Putting the value of $\frac{dN}{dt}$ from (1.2.1), We have

$$\frac{d^2N}{dt^2} = \frac{r_1^2}{K^2}N(K - N)(K - 2N)$$

Now, we shall discuss following cases:

1) If K – 2N > 0, we get K – N > N > 0. Then $\frac{d}{dt}(\frac{dN}{dt}) > 0$. Therefore, the increase $\frac{dN}{dt}$ increases with time. Hence, there is an accelerated growth population in the range 0 < N < $\frac{K}{2}$.

2) If $\frac{K}{2} < N < K$, then K – 2N < 0 and K – N > 0, therefore, $\frac{dN}{dt}$ is a decrease function of time. Hence, there is a retarded growth of the population in $\frac{K}{2} < N < K$.

1.3. EXTENSION OF SOLUTION OF LOGISTIC GROWTH MODEL

To apply equation (1.2.5), suppose that at time t = 1 and time t = 2, the values of N are N_1 and N_2 respectively, then from equation (1.2.5), we obtain

$$N_1 = \frac{K}{1+(K/N_0-1)e^{-r_1}}, N_2 = \frac{K}{1+(K/N_0-1)e^{-2r_1}}$$

or,

$$\frac{1}{K(1-e^{-r_1})} = \frac{1}{N_1} - \frac{e^{-r_1}}{N_0} \quad \ldots (1.3.1)$$

$$\frac{1}{K(1-e^{-2r_1})} = \frac{1}{N_2} - \frac{e^{-2r_1}}{N_0} \quad \ldots (1.3.2)$$

To find K and r_1 in terms of N_0, N_1, N_2; dividing equations (1.3.1) by (1.3.2), we have

$$e^{-r_1} = \frac{N_0(N_2-N_1)}{N_2(N_1-N_0)} \quad \ldots (1.3.3)$$

Putting these values in (1.3.1), we have

$$\frac{1}{K} = \frac{N_1^2 - N_0N_2}{N_1(N_0N_1 - 2N_0N_2 + N_1N_2} \quad \ldots (1.3.4)$$

The value of equation (1.3.3) and (1.3.4) can be used to write equation (1.2.5) in terms of N_0, N_1 & N_2. The limiting value of N is

$$N_{max} = \lim_{n\to\infty} N(t) = \frac{N_1(N_0N_1 - 2N_0N_2 + N_1N_2)}{N_1^2 - N_0N_2} \quad \ldots (1.3.5)$$

Verification of the Model

The population of India for the year 1971 – 2011 is given in the table 1.2. Using this data find:

(a) The Theoretically Max. Population of India.

(b) The prediction of population of India in 2031 & 2051.

Table 2: Population of India [1]

Years	Populations (in millions)
1971	548.1
1981	683.3
1991	846.3
2001	1028.7
2011	1210.10

Solution :

Let t = 0, 1, 2 corresponds to the years 1971, 1991, 2011 respectively. Then

$N_0 = 548.1$, $N_1 = 846.3$, $N_2 = 1210.1$

(a) Putting the values in equation (1.3.5), we have

$N_{max} = 2579.63$ (millions)

(b) Putting the values of N_0, N_1& N_2 in equation (1.3.3), we have

$e^{-r_1} = 0.55$. Putting $e^{-r_1} = 0.55$ & $K = N_{max} = 2579.63$

we get $N = \dfrac{2579.633}{1 + 3.70(0.55)^t}$

Since year 2031 and 2051 corresponds to t = 3 & t = 4,

Thus, putting t = 3 & t = 4

$N_3 = 1596.71$ (millions) & $N_4 = 1927.15$ (millions)

which is the approximate population of India in 2031 and 2051 respectively.

1.4. Conclusion:

In conclusion we found that the predicted carrying capacity of population of India is 2579.63 (millions). Population growth rate of any country depends on the several vital coefficients. Also, according to this model population growth rate of India is 1.8% per annum. Based on this model we also find that the population of India is expected to be 1576.91 (millions) in the year 2031 (a half of it carrying capacity). Thus, they must be re-evaluated every few years to enhance the determination of variation in the population growth.

In order to derive more benefits from models of population growth, one should subdivide population in to different age groups for effective capture, analyze and planning purpose. Other models can be developed by subdividing the population into males and females, since the reproduction rate in a population usually depends more on the numbers of females then on the numbers of males.

The government should work towards development of educations, industrialization of rural areas of country. This will have an effect in improving its absorptive capacity (K) development through population growth measure.

Finally, the government should also setup the dissemination of civic education on birth control methods to enable it to manage its resources allocation through efficient and effective population growth rate measurement principle. However, present attempts appear to provide acceptable predictions for the population growth of our country.

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