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GENERAL CRITERION FOR SOLVABILITY OF THE GENERALIZED PELL'S EQUATION

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ABSTRACT: *If d is a positive integer, not perfect square, Lagrange was the first to show that Pell's equation has an infinite number of solutions, and hence these solutions are non-trivial. It is a descriptive study in which the proposition is verified by the use of theorems as well as a number theoretic approach. The goal of this paper is to investigate and obtain solutions of the generalized Pell's equation. The study methods includes a review and debate of previously available documents to reach the solvability of generalized Pell's equation. There are many techniques for solving Pell's equations, but we observed the techniques are convergence of the continued fraction technique, PQa technique, LMM technique, and Brute-Force Search technique in the solutions of generalized Pell's equation.*

Keywords: *continue fraction, Diophantine equation, Pell's equation, integer, rational.*

INTRODUCTION

The study of Diophantine equations is to find the integral or rational solutions of polynomial equations. In Mathematics, polynomial equation usually involving two or more unknowns such that the only integers or rational solutions are studied. The term Diophantine relates to Diophantus of Alexandria, a Hellenistic Mathematician from 3^{rd}century, who studied this equation and was one of the first Mathematicians to use symbols in algebra (Austin 1981). Aryabhatta (Ansari 1977) was one of the first ancient Indian Mathematicians to develop an approach to solving first degree Diophantine equations.

A. Thue (1909) proved the important result on congruence equations by using bounds. Suppose a, b are integers with $1 < b < a$ and $gcd(a, b) = 1$. Then the congruence $bx \equiv y(mod\ a)$ has a solution (x, y) satisfying $|x| \leq \sqrt{a}$ and $|y| \leq \sqrt{a}$, where x and y are non-zero integers. The equation $F(x, y) = N$, where F is an integral binary form, N is a non-zero integer and x, y are integers, is called the Thue equation. This equation is of small degree can be solved over the integers using algorithms that are implemented in computer algebra programs (Bilu & Hanrot 1996). A. Baker (1968) comprehensive research of linear forms in the logarithms of algebraic numbers resulted in the first effective proof of Thue's discoveries. A. Thue (1909) devised a method in the theory of Diophantine approximations to solve the problem of approximating algebraic numbers by rational numbers. One can find a value $\vartheta = \vartheta(n)$ so that for every n^{th} degree algebraic number α, the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{\vartheta+\varepsilon}} \tag{1}$$

has a finite solutions in rational for some $\varepsilon > 0$ and infinite solutions for any $\varepsilon < 0$. A. Thue proved that $\vartheta \leq \frac{n}{2} + 1$. Thue's technique is based on the properties of a specific polynomial $f(x, y)$ with integer coefficients in two variables x, y, and the fact that there exist two solutions of (1) for $\vartheta \leq \frac{n}{2} + 1$ with sufficiently large values of q. By extending, Thue's technique to any polynomial about any number of variables which is related to the polynomial $f(x, y)$ and use of the large number of solutions of (1). Roth (1955) obtained the approximation of the size of ϑ which is called the Thue–Siegel–Roth theorem and states that $\vartheta = 2$ for any $n \geq 2$. Thue's technique can be extended to the case of algebraic approximation. Thue proved the following theorem.

Theorem (Thue 1909)

Let N be a non-zero integer and $f = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be an irreducible polynomial with integral coefficients and degree of $n \geq 3$. Consider a homogeneous polynomial

$$F(x, y) = y^n f\left(\frac{x}{y}\right) = a_n x^n + a_{n-1} x^{n-1} y + \cdots + a_1 x y^{n-1} + a_0 y^n$$

Then the equation $F(x, y) = N$ has either no solution or only a finite number of solutions in integers.

When the degree of F is $n = 2$ the above theorem is in contrast situation (Dickson 1957). For instance, if $F(x, y) = x^2 - dy^2$, where d is positive integer, not perfect square, then for non-zero integer N, the quadratic Diophantine equation of the form

$$x^2 - dy^2 = N \tag{2}$$

has either no or infinitely many integral solutions. The equation (2) is referred to as the generalized Pell's equation (Dickson 1957), which is named after John Pell, a Mathematician who worked in the 17^{th} century to find the integer solutions to equations of this type.

When $N = 1$, equation (2) becomes

$$x^2 - dy^2 = 1 \tag{3}$$

It is known as the classical Pell's equation (Barbeau 2003, Niven *et al.* 1991) and Brahmagupta (598–670) and Bhaskara (1114–1185) were the first to study of this equations (Arya 1991). Lagrange (1736-1813), not Pell, developed the theory. When d is a positive, not perfect square, Lagrange was the first to show that Pell's equation has an infinite number of solutions in (Niven *et al.*1991). In fact, if (x_1, y_1) is initial solution of (3), then n^{th} positive solution is (x_n, y_n) defined by $x_n + y_n\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^n$, for integer $n > 1$ (Niven *et al.*1991).

Example

Suppose $F(x, y) = x^3 - 3y^3$ is irreducible and therefore, every equation $x^3 - 3y^3 = N,\ N \in \mathbb{Z}$, has only finitely many integral solutions. This contrasts with Pell's equation $x^2 - 3y^2 = 1$ that has infinitely many solutions.

Matthews (2000) considered the solutions of (2) for $d > 0$. For the solvability of (2) with $gcd(x, y) = 1$ a necessary condition is that the congruence $u^2 \equiv d(mod\ Q_0)$ shall be solvable, where $Q_0 = |N|$. Mollin (2001) also gave a formula for the solutions to both equations $x^2 - dy^2 = c$ and $x^2 - dy^2 = -c$ using the ideals $I = [Q, P + \sqrt{d}]$, where $d > 0$, not perfect square. Mollin et *al.* (2002) considered the solutions to the Diophantine equation $ax^2 - by^2 = c$ in continued fraction of $\sqrt{a^2 b}$ and they explored criteria for the solvability of $ax^2 - by^2 = c$ where a, b, c are positive integers. Mollin et *al.* (1994)

considered the equation $x^2 - dy^2 = -3$. Several researchers, including Kaplan and Williams (1986), Lenstra (2002), Tekcan (2004), and others, looked at particular specific Pell's equations and their integer solutions.

Theorem (Niven *et al.*1991)

Every Pell's equation $x^2 - dy^2 = 1$ has a non-trivial solution. Moreover, it has infinitely many solutions.

Proof

Since d is positive integer, not a perfect square, $\sqrt{d}$ is irrational. Then there are infinitely many distinct fractions $\frac{a}{b}$ such that $\left|\sqrt{d} - \frac{a}{b}\right| < \frac{1}{a^2}$ which satisfy

$$\begin{aligned}
|a^2 - db^2| &= \left|a - \sqrt{d}b\right|\left|a + \sqrt{d}b\right| \\
&= a\left|\sqrt{d} - \frac{a}{b}\right|\left|a + \sqrt{d}b\right| \\
&< \frac{\left|a + \sqrt{d}b\right|}{b} \\
&\le \frac{a}{b} + \sqrt{d} \\
&\le 2\sqrt{d} + 1
\end{aligned}$$

Thus, $a^2 - db^2 = N$ has infinitely many solutions such that $N > 0$ and $\frac{a}{b} \in \mathbb{Q}$.

If irrationality of $\sqrt{d} \Rightarrow N \neq 0$. Then there are only finitely many solutions, N^2 possibilities for the residues of the pairs (a, b) modulo $|N|$. Thus, we can select two distinct fractions $\frac{a_1}{b_1}, \frac{a_2}{b_2}$ such that

$$a_1^2 - db_1^2 = a_2^2 - db_2^2 = N \text{ and } a_1 \equiv a_2, b_1 \equiv b_2 \text{ modulo } |N|.$$

Consider the numbers p, q defined by

$$\begin{aligned}
p + q\sqrt{d} &= \frac{a_1 + b_1\sqrt{d}}{a_2 + b_2\sqrt{d}} \\
&= \frac{(a_1 + b_1\sqrt{d})(a_2 - b_2\sqrt{d})}{(a_2 + b_2\sqrt{d})(a_2 - b_2\sqrt{d})} \\
&= \frac{(a_1a_2 - b_1b_2d) + (a_1b_2 - b_1a_2)\sqrt{d}}{a_2^2 - db_2^2} \\
&= \frac{(a_1a_2 - b_1b_2d)}{N} + \frac{(a_1b_2 - b_1a_2)}{N}\sqrt{d}
\end{aligned}$$

We claim that (p, q) is a non-trivial solution of (3).

The numerators are integral multiples of N because, using the congruences

$$a_1 \equiv a_2, b_1 \equiv b_2 \text{ modulo } |N|$$

we have $a_1a_2 - b_1b_2d \equiv a_1^2 - db_1^2 = N \equiv 0$ and $a_1b_2 - b_1a_2 \equiv a_2b_1 - b_2a_1 = 0$

Thus, $(p, q) \in \mathbb{Z}$.

Hence

$$\begin{aligned}
p^2 - dq^2 &= (p + q\sqrt{d})(p - q\sqrt{d}) \\
&= \frac{(a_1 + b_1\sqrt{d})(a_1 - b_1\sqrt{d})}{(a_2 + b_2\sqrt{d})(a_2 - b_2\sqrt{d})}
\end{aligned}$$

$$= \frac{\left(a_1^2 - b_1^2\sqrt{d}\right)}{\left(a_2^2 - b_2^2\sqrt{d}\right)} = \frac{N}{N}$$
$$= 1$$

Theorem (Niven *et al.*1991)

If the generalized Pell's equation $x^2 - dy^2 = N$ has an integral solution, then it has infinitely many integral solutions.

Proof

For $g, h, e, f \in \mathbb{Z}$, suppose (g, h) and (e, f) are solutions of equations (2) and (3).

Then $x + y\sqrt{d} = \left(g + h\sqrt{d}\right)\left(e + f\sqrt{d}\right)$ is a solution of equation (2).

Now,

$$\begin{aligned} x^2 - dy^2 &= \left(x + y\sqrt{d}\right)\left(x - y\sqrt{d}\right) \\ &= \left(g + h\sqrt{d}\right)\left(e + f\sqrt{d}\right)\left(g - h\sqrt{d}\right)\left(e - f\sqrt{d}\right) \\ &= (g^2 - dh^2)(e^2 - df^2) \\ &= N.1 \\ &= N \end{aligned}$$

Multiplying one solution of (2) by infinitely many solutions of (3). We get infinitely many solutions of (2).

Continued Fraction: If ξ is a real number, the simple continued fraction of ξ is given as

$$\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

for all $1 \le i \le n, a_i \in \mathbb{N}, a_0 \in \mathbb{Z}$. It is denoted by $\xi = [a_0; a_1, a_2, \cdots, a_n]$. As they are determined by repeated application of the division algorithm, each a_i are partial quotients. Every rational integer can be expressed as a simple continued fraction using the Euclidean algorithm. Hence, a real number is rational if and only if its continued fraction expansion is finite. The continued fraction is an infinite continued fraction if the number of terms in a simple continued fraction is infinite. As a result, every infinite continued fraction is irrational, and every irrational number can be written as an infinite continued fraction in exactly one way. The beginning parts of an infinite continued fraction representation for an irrational number provide rational approximations to the number. The convergent of the continued fraction refers to these rational numbers. The n^{th} convergents of the continued fraction $[a_0; a_1, a_2, \cdots]$ is $\frac{A_n}{B_n}$ and recurrence relation are

$$A_n B_{n-1} - A_{n-1} B_n = (-1)^{n-1}$$
$$A_{n+1} = a_{n+1} A_n + B_{n-1}, \ \ B_{n+1} = a_{n+1} B_n + B_{n-1}, \text{ for } n \ge 1.$$

Fundamental Solution: A solution $x + y\sqrt{d}$ to (2) is the fundamental solution in its class (Nagell 1951) if y is the minimal non-negative y among all solutions in the class. If two solutions in the class have the same minimal non-negative y, then the solution is the fundamental solution for $x > 0$.

If (x, y), where $x, y \in Z$, is a solution to (2), it is known as positive and (x, y) is called primitive if $gcd(x, y) = 1$. So, the primitive solutions of (3) if such solutions exist, there is one in which both x and y have their least values. A fundamental solution is one that is based on a continued fraction expansion of $\sqrt{d}$. Suppose there are positive integral solutions to (3).

Then the fundamental solution is least positive solution (x_1, y_1) such that $x_1 < u$ and $y_1 < v$ for all other positive solutions (u, v).

Nagell (1951) gives necessary conditions for a solution to (2) to be a fundamental solution, but does not give sufficient conditions.

METHODS

The methods of proof include inductive, deductive, contrapositive, and contradiction. In addition to theorems in general, Algebraic, Analytic, Topological, and number theoretic approaches may need new results. The main goal to is obtain the solutions of the generalized Pell's equation are relevant. It will be shown that how the solutions to the generalized Pell's equation are achieved using various strategies. So it is a descriptive study in which the proposition is proven using a number theoretic approach with theorems and examples. Based on a review and debate of previously available documents, the main result of the criterion for the solvability of generalized Pell's equation was reached.

RESULTS

A quadratic irrational is a positive discriminant number that solves a quadratic equation with integer coefficients but is not a perfect square. It will be concerned with the continued fraction expansions of quadratic irrationals $\xi_0 = \frac{b+\sqrt{d}}{c}$, where $b, c \neq 0$, d is positive integer, not perfect square have been studied extensively (Olds 1963). If $d > 0$, not perfect square, there is a positive integer r so that the continued fraction expansion of $\sqrt{d}$ in (Olds 1963) is

$$\sqrt{d} = [a_0; \overline{a_1, a_2, a_3, \cdots, a_r, 2a_0}]$$

where $a_0 = \lfloor\sqrt{d}\rfloor$ is a greatest integer and for all $1 \leq i \leq r$, each a_i is partial denominator of the continued fraction. The k^{th} convergent of $\xi_0$for $k \geq 0$ is given by

$$\frac{A_k}{B_k} = [a_0; \overline{a_1, a_2, a_3, \cdots, a_r, 2a_0}]$$

where

$$A_k = a_k A_{k-1} + A_{k-2},\, A_{-2} = 0, A_{-1} = 1$$

$$B_k = a_k B_{k-1} + B_{k-2},\, B_{-2} = 1, B_{-1} = 0$$

The complete quotients are given by

$$\frac{P_k+\sqrt{d}}{Q_k} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

where $P_0 = 0, Q_0 = 1$ and for $k \geq 1$

$$P_{k+1} = a_k Q_k - P_k\,,\ a_k = \left\lfloor \frac{P_k+\sqrt{d}}{Q_k} \right\rfloor,\quad d = P_{k+1}^2 + Q_k Q_{k+1}$$

A real number is periodic continued fraction expansion if and only if it is a quadratic irrational number in (Mollin 2001). Furthermore a quadratic irrational is a purely periodic continued fraction expansion if it has the form

$$\xi_0 = \overline{[a_0; a_1, a_2, \ldots a_{r-1}]}$$

for $a_n = a_{n+r}$, for all $n \geq 0$, where $r = r(\xi_0)$ is the period length of the simple continued fraction expansion. It is known that a quadratic irrational has such purely periodic expansion if and only if

$\xi_0 > 1$and $-1 < \xi_0' < 0$, where ξ_0' is the algebraic conjugate of ξ_0. So, quadratic irrational which satisfies these two conditions is reduced in (Mollin 2001).

The following theorem indicates a relation between generalized Pell's equation and simple continued fractions.

Theorem (Kumundury & Romero 1998)

If d be a positive non-perfect square integer, then $h_n^2 - dk_n^2 = (-1)^{n-1}\ q_{n+1}$ for every integers $n \geq -1$.

This theorem gives us solution to (2) for given value of N.

So, the following theorem establishes the connection between the convergence of $\sqrt{d}$ and the solutions of equation (2) for $0 < N < \sqrt{d}$.

Theorem (Niven *et al.*1991)

Let $0 < N < \sqrt{\text{d}}$ and (P, Q) be a solution of the equation $x^2 - dy^2 = N$. Then $\frac{P}{Q}$ is convergent in the expansion of $\sqrt{\text{d}}$.

Proof

Since (P, Q) is a solution of the equation (2) . Then

$$N = P^2 - dQ^2 = \left(P - Q\sqrt{d}\right)(P + Q\sqrt{d})$$

Since $0 < N < \sqrt{d} \Rightarrow \sqrt{d} > N > 0$

Then, we have

$$0 < \frac{P}{Q} - \sqrt{d} < \frac{N}{Q(P + Q\sqrt{d})} < \frac{\sqrt{d}}{Q(P + Q\sqrt{d})}$$

Since $Q\sqrt{d} < P \Rightarrow\ 2Q\sqrt{d} < P + Q\sqrt{d}$

Then, we have

$$0 < \frac{P}{Q} - \sqrt{d} < \frac{\sqrt{d}}{Q\left(P + Q\sqrt{d}\right)} < \frac{\sqrt{d}}{2Q^2\sqrt{d}} = \frac{1}{2Q^2}$$

$$\Rightarrow 0 < \frac{P}{Q} - \sqrt{d} < \frac{1}{2Q^2}$$

$$\Rightarrow \left|\frac{P}{Q} - \sqrt{d}\right| < \frac{1}{2Q^2}$$

It follows that $\frac{P}{Q}$ is a convergent of $\sqrt{d}$.

Let (p, q) be single solution to (2) and (r, s) be solution to (3).

Then

$$\begin{aligned}(p^2 - dq^2)(r^2 - ds^2) &= p^2r^2 - dp^2s^2 - dq^2r^2 + d^2q^2s^2 \\ &= (pr \pm dqs)^2 - d(ps \pm qr)^2 \\ &= N\end{aligned}$$

Thus, $(x, y) = (pr \pm dqs, ps \pm qr)$ is also solutions to (3) to be found by using incrementally larger values of (r, s), which can be easily computed using the standard technique for the Pell's equation.

DISCUSSION

The unique solution (x, y) with the least $x, y > 0$ is defined as the minimal positive solution of a class of solutions. Because all solutions to (3) may be created from its minimal positive answer, we only need to identify the minimal positive solution to (3) and a single solution from each class of equations to find all solutions to (2). This is demonstrated in the following theorem.

Theorem

Let (x, y) be a solution of $x^2 - dy^2 = N$ and (x_1, y_1) be a solution of $x^2 - dy^2 = 1$. Pair of linear recurrence relations be defined as

$$X_i = 2x_1 X_{i-1} - X_{i-2}, \qquad Y_i = 2y_1 Y_{i-1} - Y_{i-2}$$

with initial condition $(X_0, Y_0) = (x, y)$ and $(X_1, Y_1) = (xx_1 + yy_1 d, x\,y_1 + yx_1)$ all solutions to

$x^2 - dy^2 = N$ in (x, y) are given by (X_i, Y_i), for $i \in \mathbb{Z}$.

The Perceptual Quality Adaptation (PQa) technique can be used to find the smallest positive solution to (3). This technique is based on the continued fraction expansion of $\sqrt{d}$. Because general limitations on these solutions are known, the fundamental solutions to (2) may frequently be found via a Brute-Force Search technique. Although Euler demonstrated a more convenient method of solving Pellian equations by using the continuous fraction expansion of $\sqrt{d}$, Lagrange was the first to prove the existence of solutions. He presented a direct method of finding integral solutions of equations in his classic memoirs. Lagrange's reduction is another name for this procedure. The Lagrange-Matthews-Mollin (LMM) technique in is a modified version of this approach (Robertson 2004). The LMM technique is the main method for solving the generalized Pell's equation $x^2 - dy^2 = N$, is slightly more difficult to solve Pell's equation $x^2 - dy^2 = 1$, than the traditional continued fraction. While Lagrange (Matthews, 1961) knew about this procedure, it was mostly unknown until it was recently uncovered independently by another researcher. The following theorems characterize fundamental solutions to equation $x^2 - dy^2 = N$.

Theorem (Nagell 1951)

Let (a, b) be fundamental solution of equation $a^2 - db^2 = N$ and (a_1, b_1) be the least positive solution to the equation $a^2 - db^2 = 1$. Then

$$0 \le b \le \sqrt{\frac{N(a_1-1)}{2d}}, \text{ for } N > 0 \text{ and } \sqrt{\frac{|N|}{d}} \le b \le \sqrt{\frac{|N|(a_1+1)}{2d}}, \text{ for } N < 0$$

The following techniques were used to compute the solution of the generalized Pell's equation;

Continued Fractions Technique (Mollin *et al.* 1994)

For any positive solution to (3), there is a convergent $\frac{p}{q}$ to $\sqrt{d}$ such that $x = p$ and $y = q$, which is basis for Lagrange's proof that Pell's equation $x^2 - dy^2 = 1$ gives a non-trivial solution. Lagrange got a periodic continued fraction of $\sqrt{d}$ and explained where to get the positive solution to (3) among the convergent to $\sqrt{d}$. If d >0, not perfect square, then Pell's equation have infinitely many solutions. So, the continued fraction expansion of $\sqrt{d}$ plays most important role to get the solutions of such equation.

Theorem (Szuse & Rochett 1992)

Let p and q be two integers such that $p > q > 0$. Then $[a_0; a_1, a_2, \cdots, a_{n-1}, a_n]$ is the continued fraction of $\frac{p}{q}$ if and only if $\frac{p}{q}$ has $[0, a_0; a_1, a_2, \cdots, a_{n-1}, a_n]$ as its continued fraction.

Theorem (Olds 1963)

Let $d > 0$, not perfect square. Then the continued fraction expansion for

$$\sqrt{d} = [a_0; \overline{a_1, a_2, \cdots, a_{n-1}, a_n, 2a_0}], \text{ where } a_{n+1-j} = a_j \text{ for } j = 1,2, \cdots, n.$$

That is $\sqrt{d} = [a_0; \overline{a_1, a_2, \cdots, a_2, a_1, 2a_0}]$

Theorem (Olds 1963)

Let r be the length of the period of the expansion of $\sqrt{d}$. Then the equation

$$h_{nr-1}^2 - dk_{nr-1}^2 = (-1)^{nr} q_{nr} = (-1)^{nr}$$

with n even gives infinitely many solutions to (3).

In particular, it gives infinitely many solutions to equation (3) by the use of even number nr. Of course, if r is even implies that nr is even. If r is odd, this theorem obtains infinitely many solutions to equation $x^2 - dy^2 = -1$ using odd integers $n > 1$. It can be shown that solution of generalized Pell's equation (2) by using continued fraction.

Theorem (Kumundury & Romero 1998)

Let d be a positive integer, not a perfect square and suppose $|N| < \sqrt{d}$. If (u, v) is a positive solution in integers of $x^2 - dy^2 = N$, then there is a convergent (h_n, k_n) of the simple continued fraction expansion of $\sqrt{d}$ such that $\frac{u}{v} = \frac{h_n}{k_n}$

Theorem (Kumundury & Romero 1998)

Let d be a positive non- perfect square integer and $\frac{h_n}{k_n}$ be convergent to the continued fraction of $\sqrt{d}$. Let N be an integer such that $|N| < \sqrt{d}$. Then for some positive integer n, each positive solution $x = s, y = t$ of $x^2 - dy^2 = N$ with $gcd(s, t) = 1$ satisfies $s = h_n, t = k_n$.

All other positive solutions to equations $x^2 - dy^2 = \pm 1$ are to be found among $x_n = h_n, y_n = k_n$, where $\frac{x_n}{y_n}$ are the convergents of expansion of $\sqrt{d}$. If r is period of the expansion of $\sqrt{d}$ and r is even, then $x^2 - dy^2 = -1$ has no solution and all positive solutions to $x^2 - dy^2 = 1$ are given by

$$x = h_{nr-1}, y = k_{nr-1}, \text{ for } n = 1, 2, \cdots, n.$$

On the other, if r is odd, then x= $h_{nr-1}, y = k_{nr-1}$ are given all positive solutions to equation $x^2 - dy^2 = -1$ for $n = 1,3,5, \cdots$ and all positive solutions to $x^2 - dy^2 = 1$for $n = 2,4,6, \cdots$.

We conclude that all positive solutions are $(x_n, y_{n)}$, where $(x_n, y_{n)}$ integers are defined by

$$(x_n + y_n\sqrt{d}) = \left(x_1 + y_1\sqrt{d}\right)^n, \text{ for } n \epsilon \mathbb{N}$$

Then expanding by the Binomial Theorem, equating the rational and purely irrational parts of the result.

But if $|N| > \sqrt{d}$, the procedure is significantly more complicated (Dickson 1957) for solution of generalized Pell's equation. Although the continuous fraction strategy for solving Pell's equation is very nice for small values of d, the method's difficulty has been investigated to see if it is the most efficient for large d. In the length of the input d, a polynomial time technique would be an algorithm that took time limited by a fixed power of $log\, d$. The continuous fraction technique is not a polynomial time algorithm, and no polynomial time algorithm for solving Pell's equation is currently known.

PQa Technique (Olds 1963)

The technique for solving Pell's equations is using the PQa technique. It determines the quadratic irrationals continued fraction expansion of

$$\xi_0 = \frac{P_0 + \sqrt{d}}{Q_0}$$

for given P_0, Q_0, d and it finds some auxiliary variables.

Suppose P_0, Q_0, d are integers such that $Q_0 \neq 0, d > 0$, not a perfect square and $P_0^2 \equiv d(mod Q_0)$.

For $i \geq 1$, then we set

$$a_i = \left\lfloor \frac{P_i+\sqrt{d}}{Q_i} \right\rfloor,\ P_{i+1} = a_i Q_i + P_i,\ Q_{i+1} = \frac{d-P_{i+1}^2}{Q_i}$$

For $i \geq 0$, then we compute G_i and B_i by recurrence relation as follows;

$$G_i = a_i G_{i-1} + G_{i-2}, G_{-2} = -P_0, G_{-1} = Q_0$$

$$B_i = a_i B_{i-1} + B_{i-2}, \qquad B_{-2} = 1\ \ B_{-1} = 0$$

The sequence $a_0, a_1, a_2, \cdots$ is a key output of this algorithm which obtains the continued fraction expansion of $\xi_0 = \frac{\mathrm{P_0}+\sqrt{\mathrm{d}}}{\mathrm{Q_0}}$

So,

$$\xi_0 = \frac{P_0+\sqrt{d}}{Q_0} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}} = [a_0; a_1, a_2, \cdots]$$

where the a_i for $i \geq 0$ are the partial quotients of ξ_0.

Thus, we have a relation

$$G_i^2 - dB_i^2 = (-1)^{i+1} Q_{i+1} Q_0.$$

If we put $Q_0 = |N|$, then $(-1)^{i+1} Q_{i+1} = \frac{N}{|N|}$

So

$$G_i^2 - dB_i^2 = (-1)^{i+1} Q_{i+1} Q_0 = \frac{N|N|}{|N|} = N$$

Hence $(G_i,\ B_i)$ is a solution of generalized Pell's equation.

LMM Technique (Mollin 2000, Mollin & Goddard 2002)

The authors developed a nearly forgotten Lagrange approach for determining the solvability of general Pell's equation $x^2 - dy^2 = N$, where $gcd(x,y) = 1$ and and $d > 0$, is not a perfect square. The fundamental solutions are likewise created in the case of solvability. The main goal is to show a variant of Lagrange's algorithm that simply uses simple continuous fractions as a technique. Although (Niven 1942) provides an analogous approach, each of the instances $d = 2$ or $d = 3$ and N_0 requires its own treatment. In addition, unlike our procedure, the approach in (Niven 1942) necessitates the calculation of Pell's resolving's basic solution. The algorithm of Lagrange has been rediscovered (Mollin 1996).

The LMM technique is a fundamental solution to the generalized Pell's problem based on continued fraction. The goal of the LMM technique is to discover primitive solutions for each equivalence class in the solution set of equation $x^2 - dy^2 = N$. For $N_0, d > 0$, not a square, this technique discovers exactly one member from each family of solutions to the stated equation. Listing of $f > 0$, so that f^2 divides N. For each f in this list, then we set $m = \frac{N}{f^2}$. Finding all z so that

$$-\frac{|m|}{2} < z \leq \frac{|m|}{2} \text{ and } z^2 \equiv d(mod|m|).$$

For each such z, apply the PQa technique with $P_0 = z, Q_0 = |m|, d = d$. Continue the process either there is an $i \geq 1$ with $Q_i = \pm 1$ or without reaching an i with $Q_i = \pm 1$, we get the end of the first period for the sequence a_i.

In the next case, there will be no any i with $Q_i = \pm 1$. If we got an i with $Q_i = \pm 1$, then $r = G_{i-1}, s = B_{i-1}$. If $r^2 - ds^2 = m$, then add $x = f_r$, $y = f_s$ to the solution list.

Else, $r^2 - ds^2 = -m$. Test the next z, if the equation $t^2 - du^2 = -1$ does not have solution. Let the minimal positive solution be (t, u), and add $x = f(rt + sud), y = f(ru + st)$ to the list of solutions if the equation $t^2 - du^2 = -1$ has solutions. Alternatively, repeat the PQa technique for one more period, then add $r = G\ (i-1), s = B\ (i-1)$ and $x = fr, y = fs$ to the list of solutions for the next $Q_i = 1$. Because $gcd(r,s) = 1$, the solution to the equation $x^2 - dy^2 = m$ produced is primitive.

Brute-Force Search Technique (Mollin 1998, Leveque 1956)

The Brute-Force Search technique is a method of problem-solving. Pell's equation can be solved by brute-force search technique because the general conditions on these solutions are known. Suppose that (t,u) is the smallest solution to equation (2).

If $N > 0$, then, we search from $y_1 = 0$ to $y_2 = \sqrt{\frac{N(t-1)}{2d}}$.

If $N < 0$, then, we search from $y_1 = \sqrt{\frac{|N|}{2}}$ to $y_2 = \sqrt{\frac{|N|(t+1)}{2d}}$.

For $y_1 \leq y \leq y_2$ and if $N + dy^2$ is square, then $x = \sqrt{(N+dy^2)}$

If (x,y) and $(-x,y)$ are not equivalent, then add both to the list of answers; otherwise, just we add (x,y). This list, when completed, provides the fundamental solutions. If y_2 is not too huge, this technique works effectively. If (x,y) is not equivalent to $(-x,y)$, add both to the list of solutions, otherwise just we add (x,y) to the list. When finished, this list gives the fundamental solutions. This technique works well if y_2 is not too large, which means that $\sqrt{\frac{(t\pm1)|N|}{2d}}$ is not too large. Hence it suffices to search between the bounds y_1 and y_2.

CONCLUSIONS

There is a non-trivial solution to Pell's equation. Using a non-trivial solution of Pell's equation, we discussed some techniques for writing down all the solutions of a generalized Pell's equation. The generalized Pell's equation seems to have no universal techniques for finding all integer solutions. The generalized Pell's equation is described in this study, and different techniques are applied to investigate the general criterion for solvability of the generalized Pell's equation, with the objective of getting all positive integer solutions.

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