JETIR.ORG ISSN: 2349-5162 | ESTD Year : 2014 | Monthly Issue
JOURNAL OF EMERGING TECHNOLOGIES AND INNOVATIVE RESEARCH (JETIR)
An International Scholarly Open Access, Peer-reviewed, Refereed Journal

PARTITIONS IN PARTITION THEORY

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ABSTRACT

Here we try to study fundamental additive decomposition process i.e. the representation of positive integers by sums of other positive integers. If we consider order of term into account the sum is called composition otherwise consider it partition. Notation used here, $C\ (i)$ is no. of compositions of i and $P\ (i)$ for the no of partitions.

Thus, compositions (number three) are $3, 1 + 2, 2 + 1, 1 + 1 + 1$ which implies $C\ (3) = 4$ and $P(3) = 3$ since 3, $2 + 1, 1 + 1 + 1$ are partition of no. three here.

Compositions

If we consider order of term into account the sum is called compositions

In Particular,Take number four then no of partitions are : $(4), (13), (2^2), (1^22), (1^4)$; there are eight compositions of 4 are: $(4), (13), (31), (22), (112), (121), (211), (1111)$.

Definition . We let C(m, n) = the no. of compositions of n into m parts (exactly)

Definition . We let $C(N, M, n)$ total no.of of compositions of n with M parts exactly, each $\leq N$.

Obviously $C\ (N, M, n) = C\ (M, n)$ whenever $N \geq n$.

Interestingly enough, $C\ (N, M, n)$ possesses symmetry properties .

Definition A *partition* of $(\alpha_1, \alpha_2, \dots, \alpha_r)$ is a set of vectors $(\beta_1^{(i)}, \dots, \beta_r^{(i)})$, $1 \leq i \leq s$ (order disregarded), such that $\sum_{i=1}^{s}(\beta_1^{(i)}, \dots, \beta_r^{(j)}) = (\alpha_1, \alpha_2, \dots, \alpha_r)$ (here as explained earlier all vectors have non-negative integral coordinates not all zero). If we consider order of term into account , we call $(\beta_1^{(1)}, \ldots, \beta_r^{(1)})$, ... , $(\beta_1^{(s)}, \ldots, \beta_r^{(s)})$ a *composition* of $(\alpha_1 \ldots, \alpha_s)$.

Definition We let $P = (\alpha_1, \alpha_2, \ldots, \alpha_r : m)$ be *partition* of $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ with m parts, and $C\ (\alpha_1, \alpha_2, \ldots, \alpha_r; m)$ be the no. of compositions of $(\alpha_1, \alpha_2 \ldots., \alpha_r)$ with m parts.

Thus P = (2, 1, 1; 2) = 5, since there are five partitions of (2, 1, 1) into two parts: (2,1,0)(0,0, 1) , (2.0. 1)(0, 1,0) , (2,0,0)(0, 1,1) , (1,1,0)(1,0, 1) , (1, 1, 1)(1, 0 0) ; $C\ (2,1,1;\ 2) = 10$ since each of the rive partitions produces two compositions.

Definition We let $P(\alpha_1, \ldots, \alpha_r)$ (respectively $C\ (\alpha_1, \ldots . \alpha_r)$) is the total no. of partitions (respectively compositions) of $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ .For later convenience we define $C\ (0,0, \ldots, 0) = \frac{1}{2}$.

Observe that $\sum_{m \geq 1} P\ (\alpha_1, \ldots, \alpha_r : m) = P(\alpha_1, \ldots, \alpha_r)$,

$$and\ \ \sum_{m \geq 1} C\ (\alpha_1, \ldots, \alpha_r; m) = C\ (\alpha_1, \ldots, \alpha_r)\ .$$

Keywords- Compositions,Co-efficient, Partition, identities , decomposition.

Introduction:-

Partitions

Definition In Z^+ , $\sum_{\lambda=1}^{\lambda=r} a_\lambda = a_1 + a_2 + \cdots + a_r = i$ where a_λ are parts or summands of that partition .Also we can write the partition $a = (a_1, a_2, \ldots, a_r)$ or $a \vdash i$ read as a is partition of i . For understanding we can take four which can be written as a 1+1+1+1, 1+1+2, 1+3, 2+2 and 4, hence $P(4)$ = five

Also $a = (a_1, a_2, \ldots, a_r) \vdash i$, we sometimes write

$$a = (1^{f_1} 2^{f_2} 3^{f_3} \cdots)$$

where $\sum_{\beta \geq 1} f_\beta\, \beta = i$.

First we consider, In particular by an example if an integer partition of n

i = 25 =6+4+4+3+2+2+2+1+1 can be written as

i = $25 = 1(2) + 2(3) + 3(1) + 4(2) + 5(0) + 6(1)$.

i.e. $25\ = (1^2 2^3 3^1 4^2 5^0 6^1)$.

Note. Since empty seq. have partition of 0 i.e. zero

$\Rightarrow P(i) = 0$ *if* i *is negative and* $(0) = 1$. *For* $i = 1$ *to 6 the list of* $P(i)$ *is*

Since $1 = (1)$;we have $P(1) = 1$

$P(2) = 2$: $2 = (2)$, $1 + 1 = (1^2)$;

$P(3) = 3$: $3 = (3)$, $2 + 1 = (12)$, $1 + 1 + 1 = (1^3)$;

$P(4) = 5$: $4 = (4)$, $3 + 1 = (13)$, $2 + 2 = (2^2)$, $2 + 1 + 1 = (1^2 2)$, $1 + 1 + 1 + 1 = (1^4)$;

$P(5) = 7$: $5 = (5)$, $4 + 1 = (14)$, $3 + 2 = (23)$, $3 + 1 + 1 = (1^2 3)$, $2 + 2 + 1 = (12^2)$, $2 + 1 + 1 + 1 = (1^3 2)$, $1 + 1 + 1 + 1 + 1 = (1^5)$;

$P(6) = 11$: $6 = (6)$, $5 + 1 = (15)$, $4 + 2 = (24)$, $4 + 1 + 1 = (1^2 4)$, $3 + 3 = (3^2)$, $3 + 2 + 1 = (123)$, $3 + 1 + 1 + 1 = (1^3 3)$, $2 + 2 + 2 = (2^3)$, $2 + 2 + 1 + 1 = (1^2 2^2)$, $2 + 1 + 1 + 1 + 1 = (1^4 2)$, $1 + 1 + 1 + 1 + 1 + 1 = (1^6)$.

Also we have $P(10) = 42$, $P(20) = 627$, $P(50) = 204226$, $P(100) = 190569292$, *and* $P(200) =$ *3972999029388* i.e numbers of partitions increase fast.

Definition Let $P(S, \mathrm{i})$ = total no. of partitions of i from subset S $\in \mathcal{S}$ where $\mathcal{S}$ is collection of all partitions of i.

In Particular, Let $\mathcal{O}$ = Collection of partitions(odd parts) & $\mathcal{D}$ = Collection partitions (distinct parts). Below we tabulate partitions related to $\mathcal{O}$ and to $\mathcal{D}$.

$P(\mathcal{O}, 1) = 1$: $1 = (1)$,

$P(\mathcal{O}, 2) = 1$: $1 + 1 = (1^2)$,

$P(\mathcal{O}, 3) = 2$: $3 = (3)$, $1 + 1 + 1 = (1^3)$,

$P(\mathcal{O}, 4) = 2$: $3 + 1 = (13)$, $1 + 1 + 1 + 1 = (1^4)$,

$P(\mathcal{O}, 5) = 3$: $5 = (5)$, $3 + 1 + 1 = (1^2 3)$, $1 + 1 + 1 + 1 + 1 = (1^5)$,

$P(\mathcal{O}, 6) = 4$: $5 + 1 = (15)$, $3 + 3 = (3^2)$, $3 + 1 + 1 + 1 = (1^3 3)$, $1 + 1 + 1 + 1 + 1 + 1 = (1^6)$,

$P(\mathcal{O}, 7) = 5$: $7 = (7)$, $5 + 1 + 1 = (1^2 5)$, $3 + 3 + 1 = (13^2)$, $3 + 1 + 1 + 1 + 1 = (1^4 3)$, $1 + 1 + 1 + 1 + 1 + 1 + 1 = (1^7)$.

$P(\mathcal{D}, 1) = 1$: $1 = (1)$,

$P(\mathcal{D}, 2) = 1$: $2 = (2)$,

$P(\mathcal{D}, 3) = 2$: $3 = (3)$, $2 + 1 = (12)$,

$P(\mathcal{D},4) = 2: 4 = (4)\,, 3+1 = (13)\,,$

$P(\mathcal{D},5) = 3: 5 = (5)\,, 4+1 = (14)\,, 3+2 = (23)\,,$

$P(\mathcal{D},6) = 4: 6 = (6)\,, 5+1 = (15)\,, 4+2 = (24)\,, 3+2+1 = (123)\,,$

$P(\mathcal{D},7) = 5: 7 = (7)\,, 6+1 = (16)\,, 5+2 = (25)\,, 4+3 = (34)\,, 4+2+1 = (124)$

.

Clearly $P(\mathcal{O}.\,\text{n}) = P(\mathcal{D}.\,\text{n})$ for $\text{n} \le 7$.

Result

$C\,(m,n) = \binom{n-1}{m-1} = \frac{(n-1)!}{(m-1)!\,(n-m)!}$

Proof.

$\sum_{n=0}^{\infty} C\,(m,n) \cdot q^n = (q + q^2 + q^3 + q^4 + \cdots)^m$

$= \frac{q^m}{(I-q)^m}$

$= q^m \sum_{r=0}^{\infty} \binom{r+m-1}{r} q^r$ (Binomial series)

$= \sum_{n=m}^{\infty} \binom{n-1}{n-m} q^n = \sum_{n=0}^{\infty} \binom{n-1}{m-1} q^n.$

By comparing the co-efficient on both side , obtain the desired result.

Result *For all* $N, M, n \ge 1$

$C\,(N,M,n) = C\,(N,M,MN+M-n);$

$C\,(N,M.n) - C\,(N,M,n-1) \ge 0\ for\ 0 < n \le \frac{M(N+1)}{2}$

Proof. Proceeding same as in the first proof of Theorem,

We have

$$\sum_{n\ge 0} C\,(N,M,n) q^n = (q + q^2 + \cdots + q^N)^M$$

$$= (0 + q + q^2 + \cdots + q^N + 0 \cdot q^{N+1})^M.$$

We have the reqd. equations.

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