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CANCELLATION PROPERTIES OF PRODUCTS AND SMALL CANCELLATION LABELLINGS OF SOME INFINITE GRAPHS

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ABSTRACT

The major emphasis of this article is on finitely created groups that embed isometrically into their Cayley graphs, with the goal of demonstrating how these groups may be formed. This allows for the formation of groups that share some of the properties of the graphs in question. Creating a Hilbert space embeddable of groups missing Guoliang Yu's property A, as well as groups whose Cayley graphs certain expanders embed isometrically, is a viable option for many applications. Because these groups are not coarsely embeddable into Hilbert space, the Baum–Connes hypothesis does not hold true for them, and the hypothesis is therefore invalidated. Generally, we depend on small cancellation theory in graphs, and the most challenging technical difficulty is selecting which graphs should have suitable small cancellation labelings. From the discoveries of Fernandez, Leighton, and López-Presa on the uniqueness of r th roots for disconnected graphs in terms of the Cartesian product, it is possible to deduce new cancellation rules.

1. CANCELLATION PROPERTIES OF PRODUCTS OF GRAPHS

It has recently been shown that the isomorphism of the Cartesian powers Gr and Hr entails that the isomorphism of the Cartesian powers G and H is likewise isomorphic, as previously demonstrated. It is straight from Sabidussi and Vizing's unique prime factor decomposition theorem for connected graphs that they have reached their result.

In the case of connected graphs, the unique prime factorization entails the existence of a cancellation condition. We demonstrate the validity of the cancellation property, as well as the isomorphism of Gr and Hr, for certain infinite graphs when applied to Cartesian products without making use of the connectedness assumption. Beyond the direct and strong product cancellation features, we find that our proof style implies them for nonbipartite graphs, which is an important observation. Moreover, the implication Gr ∼= Hr =⇒ G ∼= H, Wherever authority is seized in relation to the powerful or the immediate result, this holds true. Interestingly, Lov'asz has previously shown the same findings, except for the cancelling characteristic, which is a new discovery.

$$A \times C \cong B \times C \implies A \cong B$$

It can be shown by Lov'asz in the situation when A and B to C homomorphisms are not fulfilled. Finally, we want to point out that the connotation is clear.

$$G^r \cong H^r \implies G \cong H$$

The lexicographic product has the same cancelling feature. As an epilogue, I'll provide some thoughts on infinite graphs.

2. RESULTS OF CANCELLATION PROPERTIES OF PRODUCTS OF GRAPHS

Using the fact that finite graphs form a commutative semiring with unit K1 when Cartesian multiplication and disjoint union are applied, the basic notion of the argument is established. A linked graph may only be described as a product of prime graphs. All (connected or unconnected) finite graphs may be included into a polynomial R with integer coefficients, which is consistent with Cartesian multiplication and disjoint union of the polynomial rings. It is only the graphs that cannot be decomposed with regard to the Cartesian product that are called indeterminants. Polynomial P(G) with positive coefficients represents every finite graph G in R.

Theorem 2.1 Let A, B, C be finite graphs such that A □ C ∼= B □ C. Then A ∼= B.

Proof. It is easy to see that P(A □ C) = P(A) · P(C). Clearly,

P(A□C) = P (B □ C)

Whence

P(A) · P(C) = P(B) · P(C),

and thus P(A) = P(B) by the cancellation property in R. Therefore A ∼= B□.

In addition, graphs that construct objects in a direct and firm manner may profit from this strategy. Remember that the vertex sets of graphs G and H's direct and strong products are identical, just in case you forgot. When the projections of two vertices of the direct product form edges in both component graphs, two vertices of the direct product are connected by an edge. Both the Cartesian and direct product edge sets are used to create the strong product edge set, which is comprised of the edges from both sets.

Cartesian and strong products have a unique prime factor decomposition feature in connected graphs that distinguishes them from other products. Units in the disjoint union are dispersed among their members. Similarly, to the manner that Cartesian products were added into R, commutative semirings may be inserted into R as well. We've discovered an easy strategy of proving the thesis by using this procedure.

Proposition 2.1 Let Gr and Hr be powers of G and H with respect to the strong product. If Gr ∼= Hr, then G ∼= H. Furthermore, if A, B, C are graphs such that A C ∼= B C, then A ∼= B.

Look at the direct result immediately. To simplify things, we'll designate this product as belonging to the Γ0 class of simple graphs with loops. Lov'asz's following theorem holds:

Theorem 2.2 Let G, H ∈ Γ0. If Gr ∼= Hr, where powers are taken with respect to the direct product, then G ∼= H. Furthermore, if A, B, C ∈ Γ0 and if there are homomorphisms from A and B to C, then A × C ∼= B × C, implies A ∼= B.

Take a look at how this relates to the results of our methodological analysis. Only if loops and nonbipartite graphs are permitted in R may a commutative semiring of graphs with unit be integrated in the language. K1 is not a direct product unit (DPU) because the one vertex graph K 1 with a loop is a direct product unit (DPU). It is true that this theorem holds for nonbipartite zero-graphs, despite the fact that it is not true for generic direct product graphs. A commutative semiring in the context of direct product and disjoint union is represented by the class of nonbipartite graphs in the number zero, and K1 is a unit in this context. Because each connected graph has a unique prime factorization in terms of the direct product, it is easy to embed R into this class. Because of this, the ramifications are as follows:

Proposition 2.2 Let Gr and Hr, the nonbipartite graphs, represent powers of G and H with regard to the direct product. If Gr ∼= Hr , then G ∼= H.

Theorem 2.3 If A, B, C are nonbipartite graphs such that A×C ∼= B×C, then A ∼= B.

The second section of Theorem 2.2, which is a special case of Lov'asz's result, is stronger than the second portion of Theorem 2.3. Proposition 2.2 is implied by Theorem 2.2, as shown in this section. Because loops are added to every vertex of G and H, the direct product is multiplied by the resulting graphs, and then the consequent graphs are erased, the strong product G H may be produced. The direct product, which is also known as the strong product, is a particular case of the direct product. We don't need this restriction for the strong product since any graph containing a loop is nonbipartite, so we don't have to worry about it. A component of Proposition 2.2 has previously been addressed, namely the first section. The existence of homomorphisms between A and B is always feasible as long as C contains at least one loop. It is possible to transfer A and B into a vertex and its loop in this manner.) Please keep in mind that, for the purpose of completeness, the lexicographic product is also subject to Proposition 2.2. It is the only graph product that is not a noncommutative subset of any other graph product. In addition to its definition and other properties, it has the following additional features.

3. 2 - CARTESIAN PRODUCT OF SPECIAL GRAPHS

Study of the cartesian product of two graphs was done. G1 r G2 has been defined as a generalised cartesian product employing the concept of distance. For generic G1 and G2, it is difficult to get G1 r G2. As a result, for r = 2, we examine this product in detail in this study. A finite, simple graph containing the vertex set V (G) and the edge set E (G) is defined as G = [V (G), E(G)] (G). If a route connects every pair of vertices in a graph G, then the graph is said to be linked. Assuming G is a linked graph, the shortest route between nodes u and u 0 in G has a length of dG(u, u0). There are two types of graph components: those that are maximally linked and those that are not. The only thing that makes up a linked graph, G, is G. This study focuses on a small, basic, and linked graph throughout. Pn, Cn, and Kn, respectively, represent the path, cycle, and complete graphs with n vertices. There are (m + n) vertices in the full bipartite graph, which is called Kmn. A graph with no edges is known as a null graph. R-similar components H are disjointly combined to form the graph G,

$$G = \bigcup_{\substack{\circ \\ i=1}}^{r} H^{(i)}$$

In order to understand the basics of graph theory's vocabulary, ideas, and findings For Pn, Cn, and Ks,t, we get G1 2 G2. We focus primarily on the product graph's connectivity.

Definition 3.1 The 2−cartesian product of graphs G1 = (V1, E1) and G2 = (V2, E2) is the graph G = (V, E) with the vertex set V = V1 × V2 and the edge set E defined as follows:

Two vertices (u, v) and (u’, v’) are adjacent in G if one of the following conditions is satisfied:’

(i) dG1 (u, u’) = 2 and dG2 (v, v’) = 0,

(ii) dG1 (u, u’) = 0 and dG2 (v, v’) = 2.

We denote this graph G by G1 ×2 G2.

In order to achieve the typical cartesian product G1 x G2 if we substitute 2 for 1, we must change the specification. Keep in mind that G1 x2 G2 is a null graph if the diameters of both G1 and G2 are less than two.

Definition 3.2. The grid graph G = Gm,n is defined as the graph with vertex set, V = {(ui , vj) : i = 1, 2, . . . , m and j = 1, 2, . . . , n} and edge set E = $\bigcup_{i=1}^{m}\{(u_i, v_j) \leftrightarrow (u_i, v_{j+1}) : 1 \le j \le n-1\} \cup$

$\bigcup_{j=1}^{n}\{(u_i, v_j) \leftrightarrow (u_{i+1}, v_j) : 1 \le i \le m-1\}$

(i) Each edge of Gm,n;

(ii) The edges (ui , v1) ↔ (ui , vn), for every i = 1, 2, . . . , m, In place of (ii), if we consider (ii)' then we get another semi tied grid graph denoted by G(m0),(n) ,

(iii) (ii)' The edges (u1, vj) ↔ (um, vj), for every j = 1, 2, . . . , n.

"Tied grid graph" is a term used to refer to any graph that has all of the aforementioned types of edgesG(m0), (n0).

3.1 2- Cartesian Product

The 2 - cartesian product of path graphs has been obtained. In this section we discuss G1×2G2 with G1 path graph and G2 cycle graph and G1 ×2 G2, if both G1 and G2 are cycle graphs.

We fix the following notations. The path graph Pm is the graph with, V (Pm) = {u1, u2, . . ., um} and E(Pm) = {(u1u2),(u2u3), . . . ,(um−1um)} and Cm(m ≥ 3) is a cycle graph with V (Cm) = V (Pm) and E(Cm) = E(Pm) ∪ {(umu1)}.

Path graphs may benefit from the following conclusion..

Proposition 3.1. [1] For m, n ≥ 3,

(a) If both m and n are even integers then, Pm ×2 Pn = $\left[\overset{4}{\underset{i=1}{\mathring{\bigcup}}}\left(G_{(\frac{m}{2}),(\frac{n}{2})}\right)^{(i)}\right]$

(b) If m is odd and n is even, then Pm ×2 Pn = $\left[\overset{2}{\underset{i=1}{\mathring{\bigcup}}}\left(G_{(\frac{m+1}{2}),(\frac{n}{2})}\right)^{(i)}\right] \mathring{\bigcup} \left[\overset{2}{\underset{j=1}{\mathring{\bigcup}}}\left(G_{(\frac{m-1}{2}),(\frac{n}{2})}\right)^{(j)}\right]$

(c) If m is even and n is odd, then Pm ×2 Pn = $\left[\overset{2}{\underset{i=1}{\mathring{\bigcup}}}\left(G_{(\frac{m}{2}),(\frac{n+1}{2})}\right)^{(i)}\right] \mathring{\bigcup} \left[\overset{2}{\underset{j=1}{\mathring{\bigcup}}}\left(G_{(\frac{m}{2}),(\frac{n-1}{2})}\right)^{(j)}\right]$

(d) If both m and n are odd integers, then

$$P_m \times_2 P_n = \left[G_{(\frac{m+1}{2}),(\frac{n+1}{2})}\right] \mathring{\bigcup} \left[G_{(\frac{m+1}{2}),(\frac{n-1}{2})}\right] \mathring{\bigcup} \left[G_{(\frac{m-1}{2}),(\frac{n+1}{2})}\right] \mathring{\bigcup} \left[G_{(\frac{m-1}{2}),(\frac{n-1}{2})}\right]$$

4. SMALL CANCELLATION LABELLINGS OF SOME GRAPHS

See 2 for further information on how to label a graph by assigning labels to the graph's directed edges. When no labelling of a long route (long in relation to the girth) occurs in two separate locations, a labelling fulfils a modest cancellation requirement; see 2.3. Graphs are infinite disjoint unions of finite graphs with degree limited uniformly, and a finite set of labels is of relevance to us. For a given degree D>2, examples include sequences of finite D-regular graphs. The only'small cancellation' labelling available before now was the Gromov labelling of select expanders [Gro] (cf. some explanations of this construction in [AD], [Cou]). It is impossible for Gromov's labelling to meet our tiny cancellation criteria since it is general in nature. Gromov's labelling defines a weak embedding in the sense of, but not a coarse embedding of the graphs (relators) into the associated group. (Recall that a map f: (X, dX)!(Y, dY) between metric spaces is a coarse embedding when dY (f(xn), f(yn))->∞ if and only if dX(xn, yn)->∞ for all sequences (xn)n∈N and (yn)n∈N.) Our research focuses on sequences of finite graphs of uniformly limited degree (n)nN, which have increasing girth and widths that are constrained by their girth. We create labels for them that adhere to far stricter guidelines than Gromov labels.

Theorem 4.1 For every λ>0 there exists a C' (λ)- small cancellation labelling of (Θn)n∈N over a finite set of labels.

It is reasonable to conclude that the graphs n are isometrically embedded into the Cayley graphs for groups created using this labelling in order to satisfy such a stringent tiny cancellation condition. To generate the requisite labelings, combinatorics approaches (graph colorings) and the Lov'asz local lemma [AGHR] are employed in conjunction with the AGHR (see e.g. [AS]). This is a first in terms of the subject matter. When comparing our technique to that of Gromov's, there is a significant difference in that we seek for any labelling that fulfils our criteria, but Gromov's method only analyses qualities of the generic labelling. As previously indicated in Section 2.4, this is required in order to have access to more powerful capabilities. Both strategies make use of a variety of distinct tools. In addition, our argument is shorter than Gromov's, as mentioned in [AD], when compared to his. Following that, we will describe how the little cancellation labels that we create are employed in real-world situations. While the design of this article and the general combinatorial approach described in it are important tools, we anticipate that they will be used in a variety of circumstances beyond the scope of this article.

4.1 non-exact groups with the Haagerup property

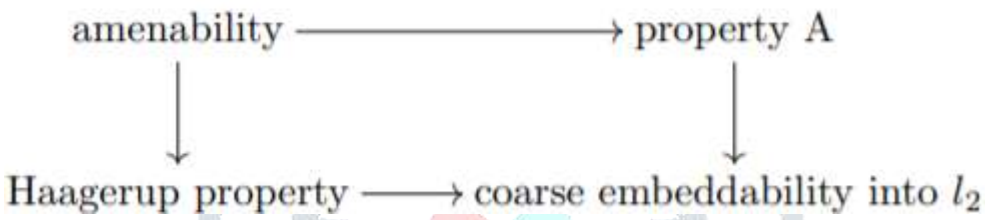
Property A, or coarse amenability, was introduced by Guoliang Yu [Y] for his studies on the Baum–Connes conjecture. A uniformly discrete metric space (X, d) has property A if, for every ε>0 and R>0, there exist a collection of finite subsets {Ax}x∈X, Ax⊆X×N for every x∈X, and a constant S>0 such that

(1) $|A_x \triangle A_y|/|A_x \cap A_y| \leqslant \varepsilon$ when $d(x, y) \leqslant R$;

(2) Ax⊆B(x, S)×N.

With respect to certain limited generator sets, groups that are finitely created are said to have property A if they are approximately amenable to the word metric in a general sense.

[Wil1], [NY] are only a few instances of its various alternative formulations and key applications; for more information, see [Wil1], [NY]. Property A is similar to amenability in that it is a weak (non-equivariant) version of the latter notion. In the case of countable discrete groups, properties such as the presence of a compact Hausdorff space [HR] and the exactness [GK1] of the reduced C-algebra, as well as the nuclearity of the uniform Roe algebra [R] and numerous other geometric and analytic properties, are all equivalent to property A. Property A implies coarse embeddability in a Hilbert space [Y] when the space is a Hilbert space. Amenability is similar to the Haagerup characteristic in that it implies (that is, a-T-menability in the sense of Gromov). As an illustration, see for example [NY, p. 124], the following diagram, which depicts the correlations (with arrows denoting implications) between those attributes for different groups. There are non-equivariant alternatives to the notions on the left and right, as you can see in the diagram:



Do groups which are coarsely embeddable into Hilbert space have the property A, which had previously been an open question? For example, see [A-D, Remark 3.8 (2)], [GK2, pp. 257 and 261], [NSW, p. 6], [AD, footnote p. 27], [Wil1, p. 251], or [NY, Open question 5.3.3]. Answering this question in the affirmative has sparked a great deal of fresh study and thinking in the field. Using the same strategy we used in [AO2] to arrive at a negative response, we can now make an even stronger case.

Theorem 4.2 There exist finitely created groups for CAT(0) cubical complexes that act appropriately but do not contain property A.

Possessing property PW, which is what it means to properly act in a space with walls [HP], [Ni], and [CN], is synonymous with correctly acting in a CAT(0) cubical complex (in a language of [Cor]). The Haagerup condition implies that there is equivariant coarse embeddability in a Hilbert space in a Hilbert space. As shown by Theorem 2, there are no additional implications between the qualities depicted in the image. To my knowledge, the only finitely created groups that do not have attribute A are those made by Gromov monsters [Gro], as of this writing. [No], [NSW, pages 6 and 28], [AD, pages 251 and 7.5], and [NY, Open question 4.5.4] are all good places to go for further information. It should be noted that embeddable into l2 coarsely non-amenable spaces were constructed in [No] (locally finite case) and [AGS] (all cases) (bounded geometry case). The examples from [Os1] serve as the foundation for our design.

As various experts pointed out, several Baum–Connes theories were deemed to be doomed due to the lack of property A for a group in a number of them. As shown in Theorem 2, there are groups that do not possess property A, but do fulfil the Haagerup property in some other way. For such groups, the Baum–Connes hypothesis [HK] holds true as well. Infinite graphical small cancellation presentations result in coarsely non-amenable groups that can be embedded in a Hilbert space that was established in this work, according to the authors. Graphs with infinity-growing girths are members of the indefinitely large family of related graphs, which has an endlessly large number of members. In order to accommodate the presence of realtors, which are graphs with walls, the group has its own walling. As a consequence, the group's activities are contained inside a room divided by walls. It is okay to take this course of action if certain other prerequisites are satisfied as well. In Section 5, the appropriate lacunary walling condition is investigated. Despite the fact that this idea was informed by the author's past collaboration with Goulnara Arzhantseva [AO2], it stands on its own merits. Particularly useful is the equivalent of [AO2, Main theorem, and Theorem 1], which is found.

Theorem 4.3 A complex with the necessary lacunary walling condition is called X. In this case, the wall pseudo-metric is correct. To put it another way: A group operating correctly with respect to one of these complexes performs appropriately with respect to another one of them.

Theorem 2's lacunary walling requirement is met for a space that is adequately acted upon by the group. A CAT(0) cubical complex may be handled by the group. The infinite family of realtors, on the other hand, embeds isometrically into the Cayley graph through the small cancellation condition. For this reason, we infer that the whole family is coarsely non-amenable as a consequence of Willett [Wil2].

5. CONCLUSION

Every connected component of a Cartesian product of infinitely many nontrivial graphs is referred to as a weak Cartesian product. For example, the weak Cartesian product of every linked (finite or infinite) network has a unique prime factorization. A linked graph with regard to that product's rth root is unique if it does exist, according to this rule. Assuming that G and H have only a limited number of components, the Cartesian product of Gr and Hr is equivalent to the Cartesian product of G and H in the situation of infinite graphs. Infinite graphs, even those that are linked, do not follow the cancellation rule. To learn more about infinite graphs, go here.

REFERENCES

1. W. D¨orfler and W. Imrich, Uber das starke Produkt von endlichen ¨ Graphen (German), Osterreich. Akad. Wiss. Math.-Natur. Kl. S.-B. II ¨ 178 (1970) 247–262.

2. A. Fern´andez, T. Leighton, and J. L. L´opez-Presa, Containment propeties of product and power graphs, Discrete Appl. Math., in press: doi:10.1016/j.dam.2006.06.001.

3. W. Imrich and S. Klavˇzar, Product Graphs: Structure and Recognition, John Wiley & Sons, New York, 2000.

4. L. Lov´asz, On the cancellation law among finite relational structures, Period. Math. Hungar. 1 (1971) 145–156.

5. R. McKenzie, Cardinal multiplication of structures with a reflexive relation, Fund. Math. 70 (1971) 59–101.

6. Arzhantseva, G., Guentner, E. & Spakula, J. ˇ , Coarse non-amenability and coarse embeddings. Geom. Funct. Anal., 22 (2012), 22–36.

7. Arzhantseva, G. & Osajda, D., Infinitely presented small cancellation groups have the Haagerup property. J. Topol. Anal., 7 (2015), 389–406.

8. Baum, P., Guentner, E. & Willett, R., Expanders, exact crossed products, and the Baum-Connes conjecture. Ann. K-Theory, 1 (2016), 155–208.

9. Chatterji, I. & Niblo, G., From wall spaces to CAT(0) cube complexes. Internat. J. Algebra Comput., 15 (2005), 875–885.

10. Gromov, M., Random walk in random groups. Geom. Funct. Anal., 13 (2003), 73–146; Addendum, 147–177.