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A fixed point theorem in b- metric space.

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Abstract: In this paper we have established a fixed point in b - metric space .Our results generalize many previous results of the literature.

1. **Introduction:** One of the most popular generalization of metric space is b-metric space. It was introduced by Bakhtin 1989 [1] as quasi- metric space and proved a contraction principle for such spaces. In 1993 Czerwik introduced them under the name “b-metric space” first for S=2 [7] and then for an artribary S≥1 in [8] with application to fixed points. Later an many researcher Boriceanu [2,3] Bota [4] Chung [6], Du [9], Kir [5] and many others established fixed point theorem in b-metric space.

In this paper. We have extended some known fixed point results in b-metrix space. We have studied contractive type mappings and showed the validity of the result in b-metric space. We have established results for single map and two maps.

2. Preliminaries

Definition (2.1) [8]: Let X be a nonempty set and s ≥ 1 be a given real number. A function d : XxX → R+ is said to be a b-metric on X if the following conditions hold:

(i) d(x, y) = 0 if and only if x = y;

(ii) d(x, y) = d(y; x) for all x; y 2 X;

(iii) $d(x, y) \le s\,(d(x; z) + d(z; y))$ for all x; y; z ϵ X.

The pair (X; d) is called a b-metric space.

We observe that if s = 1, then the ordinary triangle inequality in a metric space is satisfied, however it does not hold true when s > 1. Thus the class of b-metric spaces is effectively larger than that of the ordinary metric spaces. That is, every metric space is a b-metric space, but the converse need not be true. The following examples illustrate the above remarks.

Example (2.2) [8]: Let X = {1, 0, - 1}. Define $d : X \times X \to R^+$ by d(x, y) = d(y, x) for all x, y ϵ X; d(x, x) = 0; x ϵ X and d(-1, 0) = 3; d(-1, 1) = d(0,1) = 1. Then (X, d) is a b-metric space, but not a metric space since the triangle inequality is not satisfied. Indeed, we have that

d(-1, 1) + d(1, 0) = 1 + 1 = 2 < 3 = d(-1, 0):

It is easy to verify that s = $3/2$.

Example (2.3)[8] : Let X = R and $d : X \times X \to R^+$ be such that d(x; y) =j x- y j^2 for any x, y ϵ X. Then (X; d) is a b-metric space with s = 2, but not a metric space.

Definition (2.4) [8]: Let (X, d) be a b-metric space, x ϵ X and (x_n) be a sequence in X. Then

(i) (x_n) converges to x if and only if $d(x_n, x) = 0$. We denote this by $\lim_{n\to\infty} x_n = x \; as \; x_n \to x(n \to \infty)$ x_n .

(ii) (x_n) is Cauchy if and only if $\lim_{m,n\to\infty} d(x_n, x_m) = 0$

(iii) (X; d) is complete if and only if every Cauchy sequence in X is convergent.

Proposition (2.5) [8] : In a b−metric space (X, d) the following assertions hold:

(i) a convergent sequence has a unique limit,

(ii) each convergent sequence is Cauchy,

(iii)in general, a b−metric is not continuous.

In general a b−metric function d for k > 1 is not jointly continuous in all of its two variables.

Definition (2.6) [8] : Let (X, d) be a b−metric space. If Y is a nonempty subset of X, then the closure Y of Y is the set of limits of all convergent sequences of points in Y , i.e.,

Y = {x ϵ X : there exists a sequence $\{x_n\}$ in Y such that $\lim_{n\to\infty} x_n = x$}

Definition. (2.7) [8]: Let (X, d) be a b−metric space. Then a subset Y ⊂ X is called closed if and only if for each sequence $\{x_n\}$ in Y which converges to an element x, we have x ϵ Y (i.e., Y = Ȳ).

Definition (2.8) [8]: Let (X, d) be a b-metric space and let T : X →X be a given mapping. We say that T is continuous at x_0 ϵ X if for every sequence (x_n) in X, we have $x_n \to x_0$ $as\ n \to \infty \Rightarrow T(x_n) \longrightarrow T(x_0)\ as\ n \to \infty$.

If T is continuous at each point x0 2 X, then we say that T is continuous on X.

3. Main Results:

Theorem 3.1: Let 0 ≤ α ≤ 1, p and q be non-negative numbers such that p + q < 1 and

(i) $\alpha|p - q| < 1-(p+q)$

and T : X→X be mapping of a complete b-metric space (X, d) such that whenever x, y are distinct elements in X.

(ii) $$d(Tx,Ty) \le \alpha \max.\{d(x,y), d(y,Tx), d(y,Tx)\} + (1-\alpha)[pd(x,Ty)+qd(y,Tx)]. \quad \_\_\_\_\_\_(1)$$

such that $\max.\left\{\frac{\alpha+(1-\alpha)ps}{1-(1-\alpha)ps}.\frac{ps}{1-ps}\right\}<1$ and s ≥ 1.

Then T has a unique fixed point.

Proof: Let $x_0 \in X$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X defined as

$$x_n = Tx_{n-1} = T^n x_0, n = 1,2,3.............(2)$$

By (1) and (2) we obtain

$$d(x_n, x_{n+1}) = d(Tx_{n-1}Tx_n)$$
$$\le \alpha \max\{d(x_{n-1},x_n), d(x_{n-1},Tx_{n-1}), d(x_n,Tx_n)\}$$
$$+(1-\alpha)\{pd(x_{n-1},Tx_n)+qd(x_n,Tx_{n-1})\}.$$
$$= \alpha \max\{d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1})\}$$
$$+(1-\alpha)\{pd(x_{n-1},x_{n+1}), qd(x_n,x_n)\}.$$

Or, $d(x_n,x_{n+1}) \le \alpha \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}$

If. $\max.\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} = d(x_{n-1},x)$, then

$$d(x_n x_{n+1}) \le \alpha d(x_{n-1}, x_n) + (1-\alpha) ps[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$

Or, $1-[(1-\alpha)ps]d(x_n, x_{n+1}) \le \alpha + (1-\alpha)ps \ d(x_{n-1}, x_n)$

Or, $d(x_n, x_{n+1}) \le \dfrac{\alpha + (1-\alpha)ps}{1-(1-\alpha)ps}$

If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$. Then,

$$d(x_n, x_{n+1}) \le \alpha d(x_n, x_{n+1}) + (1-\alpha)ps[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$

Or, $\left(\dfrac{1}{1-\alpha}\right) d(x_n, x_{n+1}) \le \dfrac{\alpha}{(1-\alpha)} + ps[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$

Or, $\left(\dfrac{1}{1-\alpha}\right) - \left(\dfrac{\alpha}{1-\alpha}\right) d(x_n, x_{n+1}) \le ps[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$

Or, $d(x_n, x_{n+1}) \dfrac{ps}{1-ps} d(x_{n-1}, x_n)$

Thus, $d(x_n, x_{n+1}) \le \max\left\{\dfrac{\alpha + (1-\alpha)ps}{1+(1-\alpha)ps}, \dfrac{ps}{1-ps}\right\} d(x_{n-1}, x_n)$

$$\le \beta \ d(x_{n-1}, x_n) \text{ when } \beta = \max.\left[\frac{\alpha + (1+\alpha)ps}{1-(1-\alpha)ps}, \frac{ps}{1-ps}\right].$$

Again,

$$
\begin{aligned}
d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\
&\preceq \alpha \ \max.\ \{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1})\} \\
&\quad + (1-\alpha)[pd(x_n, Tx_{n+1}) + qd(x_{n+1}, Tx_n)] \\
&= \alpha \ \max.\ \{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} \\
&\quad + (1-\alpha)[pd(x_n, x_{n+2}) + qd(x_{n+1}, x_{n+1})] \\
&\le \alpha \ \max.\ \{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} \\
&\quad + (1-\alpha)[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]
\end{aligned}
$$

If $d(x_n, x_{n+2})$ is maximum, Then

$$d(x_{n+1}, x_{n+2}) \le \alpha d(x_{n+1}, x_{n+2}) + (1-\alpha)ps[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]$$

Or, $\left(\dfrac{1}{1-\alpha}\right) d(x_{n+1}, x_{n+2}) \le \left(\dfrac{\alpha}{1-\alpha}\right) d(x_{n+1}, x_{n+2}) + ps[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]$

Or, $d(x_{n+1}, x_{n+2}) \le \dfrac{ps}{1-ps} d(x_n, x_{n+1})$

Thus, $d(x_{n+1}, x_{n+2}) \le \max.\left\{\dfrac{\alpha + (1-\alpha)ps}{1-(1-\alpha)ps}, \dfrac{ps}{1-ps}\right\} \dfrac{d(x_n, x_{n+1})}{d(x_n, x_{n+1})} \quad \le \beta d(x_n, x_{n+1})$

Proceeding it this way, we have :

$$d(x_{n+1}, x_{n+2}) \le \beta d(x_n, x_{n+1})$$

$$\leq \beta^2 d(x_{n-1}, x_n)$$

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$$\leq \beta^n d(x_0, x_1)$$

However if $p, q \in \left[0, \tfrac{1}{2}\right)$, then β < 1.

If max. {p, q} ≥ ½. Then .

$\dfrac{\alpha + (1-\alpha)x}{1-(1-\alpha)x} \leq \dfrac{x}{1-x} \Rightarrow 1/2 \leq x \quad \forall x \in [0,1),$ it follow from (i) that 0≤β≤1.

Now we claim that $\{x_n\}_{n=j}^{\infty}$ is a Cauchy sequence in X.

Let m, n > 0 with m>n.

$$d(x_m, x_n) \leq s d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_{n+3}) + \ldots.$$

$$\leq s\beta^n d(x_0, x_1) + s^2 \beta^{n+1} d(x_0, x_1) + s^3 \beta^{n+2}(x_0, x_1) + \ldots. + s^m \beta^{n+m+1} d(x_0, x_1)$$

$$\leq s\beta^n d(x_0, x_1)\left[1 + s\beta + (s\beta)^2 + \ldots.. + (s\beta)^{m-1}\right]$$

$$\leq s\beta^n d(x_0, x_1)\left[\frac{1-(s\beta)^{n-(m-1)}}{1-s\beta}\right]$$

When we take m, n→∞ , $\lim_{n\to\infty} d(x_n, x_m) = 0$

Hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence is X. Since $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence $\{x_n\}$ converges to u in X (say) as X is a complete b-metric space.

Now we shall show that u is the fixed point of T. For this we consider :

$$d(Tu, Tx_n) \leq \alpha \max.\{d(u, x_n), d(u, Tu), d(x_n, Tx_n)\}$$

$$+ (1-\alpha)\left[pd(u, Tx_n) + qd(x_n, Tu)\right]$$

Taking n→∞ we have.

$$d(u, Tu) \leq \alpha \max.\{d(u,u), d(u,Tu), d(u,u)\} + (1-\alpha)\left[pd(u,u) + qd(u,Tu)\right]$$

Or , $d(u,Tu) \leq \alpha d(u,Tu) + (1-\alpha) q d(u,Tu)$

Or, $(1-q)(u,Tu) \leq 0,$ this possible only if $d(u,Tu) = 0 \quad \Rightarrow$ Tu=u

Hence u is the fixed point of T. Now we claim that u is the unique fixed point of T. For this Let u≠v and Tv=v. New we have:

$$d(u,v) = d(Tu,Tv) \leq \alpha \max.\{d(u,v), d(u,Tu), d(v,Tv)\} + (1-\alpha)\left[pd(u,Tv) + qd(v,Tv)\right]$$

$$\leq \alpha \max.\{d(u,v), d(u,u), d(v,v) + (1-\alpha)\}\left[pd(u,v) + qd(u,v)\right]$$

$$= \alpha d(u,v) + (1-\alpha)(p+q) d(u,v).$$

Or, $\left(\dfrac{1}{1-\alpha}\right) d(u,v) \leq \left(\dfrac{\alpha}{1-\alpha}\right) d(u,v) + (p+q) d(u,v)$

Or, d(u,v),≤ (p+q) d(u,v)

Or, $1-(p+q) d(u,v) \leq 0$ which is a contradiction Hence d(u,v)=0 i.e. u =v. is unique fixed point of T.

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