JETIR.ORG ISSN: 2349-5162 | ESTD Year : 2014 | Monthly Issue
JOURNAL OF EMERGING TECHNOLOGIES AND INNOVATIVE RESEARCH (JETIR)
An International Scholarly Open Access, Peer-reviewed, Refereed Journal

NUMERICAL INVESTIGATION OF UNCONDITIONALLY STABLE SPLINE FUNCTION FOR THREE-DIMENSIONAL TIME-FRACTIONAL TELEGRAPH EQUATIONS

Uday Singh

Department of Mathematics and Computer Science

Rani Durgavati University, Jabalpur, MP ,India.

Email Id- udayrdvv@gmail.com

Abstract

Telegraph equations are hyperbolic partial differential equations that may be used to represent reaction-diffusion processes in a variety of engineering and biological disciplines. The development of numerical techniques for telegraph type equations has received a lot of interest in the literature in recent years. The primary goal of this article is to introduce and evaluate a new approach for approximating the time-fractional telegraph equation using spline functions. Initially, an operational matrix technique based on the consolidation of Fibonacci wavelets and block pulse functions is presented to derive the solutions to Time-Fractional Telegraph Equations (TFTs). The suggested approach converts the fractional model into an algebraic equation system that can be solved using the Newton iteration method. The Crank Nicolson approach is also offered for the solution of three-dimensional time-fractional telegraph equations using the Trigonometric Quintic B-spline (TQBS). The rationale behind using the collocation method is to select specific collocation spots where the differential equation is fulfilled exactly. The suggested technique combats nonlinearity by employing a quasilinearization linearization procedure. The discretization of the time-fractional derivative is done using the Caputo fractional derivative formula. The calculated solutions are obtained using a combination of the Caputo fractional derivative and a trigonometric Quintic B-spline. The main objective is to verify the well-posedness and produce a numerical solution for an initial-boundary value issue for a hyperbolic equation using finite-difference methods. Accordingly, the research developed the exponentially fitted approach for solving initial boundary value problems using finite difference formulae and temporal frequencies. The scheme’s convergence is demonstrated using normal analytical approaches, demonstrating that the method is unconditionally stable and has an order of convergence. MATLAB software is used to run the numerical simulations. Two model examples with boundary layer behaviour are investigated to support the theoretical conclusion. Furthermore, the graphs show that numerical and exact solutions are near together, demonstrating the method's precision.

Keywords: Time Fractional Telegraph Equation, Three-Dimensional, Quintic B-spline, Fibonacci Wavelets, Crank Nicholson

1. Introduction:

Fractional Differential Equations (FDEs) have piqued the interest of scientists and engineers around the world in recent decades due to their potential value and applications in quantum mechanics, astrophysics, hadron spectroscopy, engineering, classical mechanics and other fields, and other fields [1]. The fractional telegraph equation, which belongs to the group of FDEs, has been studied by a significant number of scholars during the last two decades [2]. One of the primary mathematical models developing in the study of electrical signals in transmission lines and wave phenomena is the fractional telegraph equation. It is a member of the hyperbolic partial differential equations family [3]. Cascaval examined the Fractional Telegraph Equation (FTE) for the first time in 2002, discussing several features of the time-fractional telegraph equation, such as the solution's asymptotic behaviour [4]. Many authors have lately investigated the time-fractional telegraph equations.

Time Fractional Telegraph Equations (TFTE)

To solve TFTE, several numerical and analytical algorithms have been developed. Cascaval [5] studied the well-posedness and asymptotical examination of TFTE using the Riemann–Liouville technique. Chen has established an analytical solution for the TFTE with three different nonhomogeneous border conditions utilising variable separation. Momani [6] explored approximate solutions of space and TFTE using the Adomian decomposition approach. Huang [7] found analytical solutions for three main TFTE issues: Cauchy and signalling difficulties using Laplace and Fourier transforms, and the boundary problem using spatial Sine transform. Dehghan and Shokri proposed a numerical approach for solving hyperbolic telegraph equations using collocation points, which they approximated with thin-plate spline radial fundamental functions [8]. Yousefi employed the Legendre multi-wavelet Galerkin technique to solve the hyperbolic telegraph equation. For the numerical solution of TFTE, Wang discussed and examined the Galerkin mixed finite element technique [9]. The use of suitable B-splines for the numerical solution of TFTE is motivated by the success of B-splines in the numerical solution of differential equations. There is no research on the usage of splines for the fractional telegraph differential equation [10], as far as researchers know. There is various research in the literature that uses splines to solve fractional partial differential equations. Tasbozan used the cubic B-spline collocation method to obtain a numerical solution for the fractional diffusion problem [11]. For the solution of fractional boundary value issues, Akram and Tariq proposed a numerical methodology based on the quintic spline collocation method [12]. For the solution of the fractional diffusion problem, the cubic B-spline collocation method was applied. Similarly, the quintic B-spline collocation approach was used to solve a time-fractional super-diffusion fourth-order differential equation. The quadratic B-spline Galerkin technique was used to solve numerical TFTE solutions [13].

Dimensional Problem in Time Fractional Telegraph Equation

Several numerical methods for multidimensional hyperbolic partial differential equations have been developed in recent years. For instance, Gao and Chi incorporated the unconditionally stable difference techniques to solve a one-space-dimensional linear hyperbolic model. Mohanty and Jain [14] developed unconditionally stable alternating dimension implicit techniques for two-dimensional and three-dimensional hyperbolic equations. There are few published articles on the numerical solution of three-dimensional fractional equations, as far as the authors are aware. Chen devised an ADI finite difference approach for the fractional sub-diffusion equation for specific three-dimensional situations. For space-fractional diffusion equations, Wang and Du considered rapid ADI finite difference algorithms [15]. Other researchers developed a Backward Euler (BE) ADI difference system for solving the integro-differential equation as a result. The work focuses on the three-dimensional time-fractional telegraph equation using the unique spline function as a result of the aforementioned concerns.

2. Literature Review:

To solve the three-dimensional time-fractional telegraph problem, Xuehua Yang *et al* [16] devised an efficient alternating direction implicit (ADI) finite difference approach. In the temporal direction, the completely discrete scheme is constructed using the L1 discrete formula, and in the spatial direction, the finite difference approach. An ADI method is created and implemented to lower the computing cost of addressing three-dimensional issues. The research next verifies the scheme's stability and convergence in L 2 and H1 norms, respectively, using the energy technique. Finally, various numerical examples are provided to support the theoretical conclusions.

The Cauchy issue for the time-fractional telegraph equation of dispersed order was introduced by N.Vieira*et al* [17]. A representation of the fundamental solution of this equation in terms of convolutions using the Fox H-function is derived using the Fourier, Laplace, and Mellin transform approach. Some specific density function options in the form of basic functions are investigated. In the Laplace domain, fractional moments of the basic solution are obtained. Finally, the asymptotic behaviour of the second-order moment (variance) in the time domain is investigated using Tauberian theorems.

The one-dimensional time-fractional telegraph equation, a family of explicit-implicit (E-I) difference techniques and implicit-explicit (I-E) different methods, was proposed by Xiaozhong Yang *et al* [18]. The two approaches are based on a hybrid of the conventional implicit and explicit difference methods. The E-I and I-E difference schemes are unconditionally stable, with 2nd order spatial precision, 2nd order time accuracy, and considerable time savings, and their computation efficiency is superior to the traditional implicit scheme, according to theoretical analysis and numerical testing. The E-I and I-E difference approaches developed in this article are successful in solving the time-fractional telegraph equation, according to the research.

To solve the two-dimensional time-fractional telegraph problem, N. Abdi et al [19] suggested the Compact Finite Difference (CFD) and rotating four-point compact explicit decoupled group (CEDG) techniques. The CEDG approach is formed from a rotational CFD approximation formula combined with the grid points being arranged in a group. When compared to the CFD approach on the conventional grid, this method outperforms it in terms of CPU timings and iteration while maintaining the same order of accuracy. Using Fourier analysis, this verified the stability and convergence of the suggested systems. To show the efficacy of the suggested methodologies, certain numerical experiments are carried out.

From the 2h-spaced standard and rotational Crank–Nicolson FD approximations, Ajmal Ali et al [20] proposed the modified group iterative approach for solving the two-dimensional (2D) fractional hyperbolic telegraph differential equation with Dirichlet boundary conditions. The results of the novel four-point modified explicit group relaxation approach show that the suggested method has a faster rate of convergence than current systems. The competence of the group iterative scheme is compared to that of its point iterative equivalents using numerical testing. The matrix norm technique proves the stability of the resulting modified group approach. The acquired findings are tabulated, and it is determined that precise and approximate solutions are symmetric.

Using the Laguerre wavelet collocation approach, Kumbinarasaiah Srinivasa et al [21] established an effective numerical methodology for solving the fractional-order (1 + 1) dimensional telegraph problem. Convergence analysis is described in terms of a theorem, and several instances are shown to evaluate the effectiveness of the suggested approach. The fractional-order telegraph problem is turned into a system of algebraic equations using the Laguerre wavelet characteristics, and the suggested scheme's solutions are more accurate when compared to the analytical solution and other approaches in the literature.

For the efficient and accurate numerical solution of a time-fractional diffusion equation in two spatial dimensions, Fouad Mohammad Salama et al [22] proposed the modified hybrid explicit group (MHEG) iterative approach. In the Caputo interpretation, the time-fractional derivative is defined. In the suggested technique, a Laplace transformation is applied in the temporal domain, and a new finite difference scheme based on the grouping strategy is examined for spatial discretization. The matrix analysis approach proves the unique solvability, unconditional stability, and convergence. The suggested algorithm's feasibility and efficiency are demonstrated by a comparison of numerical results with analytical and other approximate solutions.

The Elzaki decomposition approach was suggested by Nehad Ali Shah et al [23] for evaluating the solution of fractional-order telegraph equations. Within the Caputo derivative operator, the approximate analytical solution is achieved. The examples are offered as a solution to show that the proposed technique is feasible. With the assistance of the figure, the outcome of the proposed approach and the precise solution is illustrated and examined. With minimal computing labour and a quick convergence rate to accurate solutions, the analytical technique yields the series form solution. The findings revealed a simple and effective method for analysing challenges in connected fields of science and technology.

For the analysis of one space dimensional time-dependent partial differential equations, Brajesh Kumar Singh et al [24] presented the Hybrid Cubic B-spline differential quadrature technique (in short, HCB-DQM). The proposed approach was used in particular for the 1D telegraph equation (1D TE). Because the HCB-DQM uses DQM with hybrid Cubic-B-splines as basic functions, the 1D TE is reduced to a system of first-order ordinary differential equations (ODEs) that can be solved using the SSP-RK43 technique. The suggested HCB-DQM is shown to yield stable solutions for the 1D telegraph problem. The precise outcomes are compared to the assessed solutions. Furthermore, comparing the appraised results to outcomes that have just been published. The provided findings appear to be in good agreement with the exact solutions.

Abdelkebir Saad *et al* [25] devised a fast technique for solving one-dimensional time-space fractional telegraph equations. The conformable sense is used to characterise fractional derivatives. This algorithm is based on fourth-order shifted Chebyshev polynomials. The fractional telegraph equations in time and space are reduced to a linear system of second-order differential equations, which is solved using Newmark's approach. Finally, several numerical examples are shown to demonstrate the algorithm's dependability and efficacy.

3. Research Problem Definition and Motivation:

Fractional Differential Equations (FDEs) have become a lot of interest in recent years since they can simulate a lot of things in science and engineering. Meanwhile,no method offers an accurate solution for fractional differential equations in general, only approximate solutions may be determined using linearization or perturbation methods. Despite the fact that earlier researchers have done a lot of effort, studies on FTEs are still in the works. B-splines have been used by some academics to solve FDEs, but only a few studies have been done on FTEs. Furthermore, there is no study has been done on using B-spline to solve fractional telegraph equations. According to this survey, there are just a few numerical solution techniques for such equations, and the majority of these numerical schemes are distributed with ordinary differential equations and one-dimensional differential equations of the distributed order. Parabolic partial differential equations have typically been used to simulate suspension flows. Hyperbolic equations with parabolic asymptotic properties, such as the telegraph equation, can sometimes be used to better model them. The solution of the fractional telegraph equation has become an essential research topic due to the continual application of the fractional telegraph equation. As a common fractional diffusion wave equation, fractional Telegraph Equations (FTE) are frequently utilised in wave propagation, random walk theory, signal analysis, and other engineering domains. Physically, the time-fractional derivative explains the physical phenomena connected to the process in the fractional telegraph equation model, which is known as historical dependency. The development of effective simulation methods, particularly for multi-dimensional situations, is hampered by this. Only a few individuals have looked at the three-dimensional fractional situation, particularly numerical equation solutions. With this scheme's encouraging findings, attempts are now being undertaken to adapt the formulation to solve the more difficult telegraph equation with fractional-order derivatives. This prompts the presentation of a novel approach for solving the time-fractional telegraph equation based on the use of spline functions to solve the dimensional problem.

4. Proposed Research Methodology:

The primary objective of this article is to introduce and evaluate a new approach for approximating the time-fractional telegraph equation using the spline function. A competent numerical approach for the Time-Fractional Telegraph Equation (TFTE) is provided in this research study. The suggested approach makes use of a relatively new form of B-spline known as the Trigonometric Quintic B-spline (TQBS). Finite difference approaches (explicit and implicit schemes), a prominent numerical method, are often used to successfully solve equations. For three-dimensional equations with Dirichlet initial-boundary conditions, implicit finite difference techniques are used in this work. The flow diagram for the proposed work is shown in Figure 1.

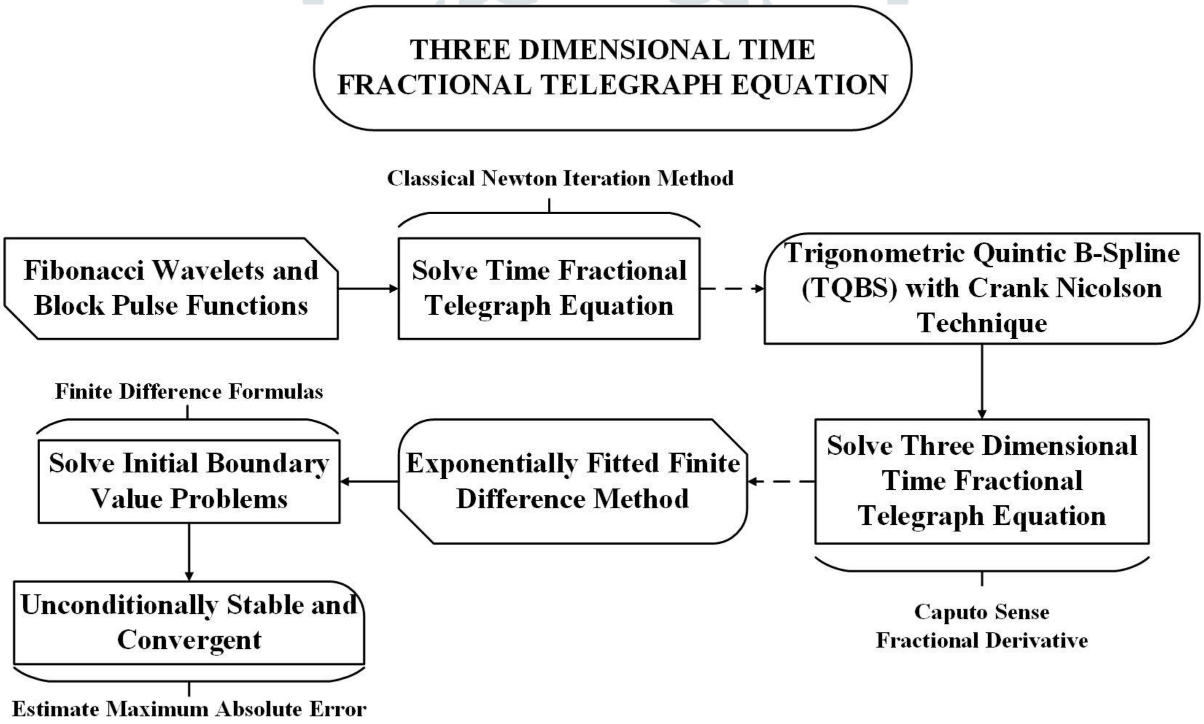


Figure 1: Flow Diagram of the Proposed Work

The time-fractional derivative is the primary impediment to the creation of effective simulation algorithms in the Fractional Telegraph Equation (FTE) model, particularly for dimensional issues. A competent numerical approach for the time-fractional telegraph equation (TFTE) is provided in this research study. For the solution of Time-Fractional Telegraph Equations, an operational matrix technique based on the combination of Fibonacci wavelets and block pulse functions is presented (TFTs). The suggested approach converts the fractional model into an algebraic equation system that can be solved using the Newton iteration method. For the solution of three-dimensional time-fractional telegraph equations subject to particular starting and Dirichlet boundary conditions, the trigonometric Quintic B-spline (TQBS) using the Crank

Nicolson approach is presented. The discretization of the time-fractional derivative is done using the Caputo fractional formula. The calculated solutions are obtained using a combination of the Caputo fractional derivative and a trigonometric Quintic B-spline. Boundary conditions cannot be met in differential equations, but the number of viable solutions is reduced in algebraic equations. The exponentially fitted technique employs temporal frequencies in finite-difference formulae to solve initial boundary value issues. The suggested algorithm's maximum absolute error and rate of convergence are estimated, revealing that it is unconditionally stable and convergent.

a. Time Fractional Telegraph Equation:

Using the wavelet-based approach, this research investigates analytic and approximate solutions to the time-fractional telegraph problem. For the solution of Time-Fractional Telegraph Equations (TFTs), a new and efficient operational matrix approach based on the combination of Fibonacci wavelets and block pulse functions is given. The suggested approach converts the fractional model into an algebraic equation system that can be solved using the Newton iteration method. The proposed method's error analysis is also looked at. The objective of this article is to employ block pulse functions to develop fractional-order operational matrices of integration for Fibonacci wavelets. On the interval [0, 1], the block-pulse functions are defined as

$$b_l(\eta)=\begin{cases}1, & lr \le \eta < (l+1)r \\ 0, & Otherwise\end{cases} \qquad (1)$$

From the above equation, $l=0,1,2,\ldots,N-1, N\in Z^{+}$ and r represents the $1/N$. Afterwards, any function $f(\eta)\in L^2[0,1)$ may be distended using the block-pulse function as follows:

$$f(\eta)\simeq f_N(\eta)=\sum_{l=0}^{N-1} a_l b_l(\eta)=A^T B_N \qquad (2)$$

Where, $B_N(\eta)$ indicates the $[b_0(\eta),b_1(\eta),\ldots,b_{N-1}(\eta)]^T$ together with A denotes the $[a_0,a_1,\ldots,a_{N-1}]^T$. The model obtains by integrating the vector $B_N(\eta)$ as

$$\int_0^{\eta} B_N(s)ds \simeq \Delta B_N(\eta) \qquad (3)$$

The operational integration matrix for block-pulse functions is denoted by

$$\Delta=\frac{r}{2}\begin{pmatrix}1 & 2 & 2 & \ldots & 2\\ 0 & 1 & 2 & \ldots & 2\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & \ldots & 1\end{pmatrix} \qquad (4)$$

Following that, the fractional-order operational matrix F^{α} for block pulse functions is provided by

$$(I^{\alpha}B_N)(\eta)\simeq F^{\alpha}B_N(\eta) \qquad (5)$$

Where,

$$F^{\alpha}=\frac{1}{N^{\alpha}\Gamma(\alpha+2)}\begin{pmatrix} 1 & \eta_1 & \eta_2 & \eta_3 & \dots & \eta_{N-1} \\ 0 & 1 & \eta_1 & \eta_2 & \cdots & \eta_{N-2} \\ 0 & 0 & 1 & \eta_1 & \dots & \eta_{N-3} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & 1 & \eta_1 \\ 0 & 0 & 0 & & 0 & 1 \end{pmatrix} \tag{6}$$

The formula for the η_l in equation (6) is as follows:

$$\eta_l=(l+1)^{\alpha+1}-2l^{\alpha+1}+(l+1)^{\alpha+1} \tag{7}$$

The succeeding step is to use block pulse functions to create matrices of fractional order integration associated with Fibonacci wavelets. The model obtains the result by integrating the vector (24)

$$\int_0^{\eta}\Phi(s)ds\approx Q\Phi(\eta) \tag{8}$$

Where Q denotes the Fibonacci wavelet of $2^{k-1}M\times 2^{k-1}M$ order's operational integration matrix. The matrices $\Phi(\eta)$ in (8) are the Fibonacci wavelet matrices of $1\times 2^{k-1}M$ order, which are given by

$$\Phi(\eta)=\left[\phi_{1,0},\phi_{1,1},\dots,\phi_{1,M-1},\phi_{2,0},\phi_{2,1},\dots,\phi_{2,M-1},\dots\phi_{2^{k-1},0},\phi_{2^{k-1},1},\dots\phi_{2^{k-1},M-1}\right]^T \tag{9}$$

The Fibonacci wavelets (9) can alternatively be described using block-pulse functions (1).

$$\Phi(\eta)=\Phi_{n,m}B_N(\eta) \tag{10}$$

Assume that $D^{\alpha}\Phi(\eta)$ value to get the Fibonacci wavelet operational matrix of the integration general order.

$$D^{\alpha}\Phi(\eta)=Q^{\alpha}_{n,m}\Phi(\eta) \tag{11}$$

The fractional-order operational matrix of integration for the Fibonacci wavelet is represented by the matrix. After evaluating the relationships (5), (10) and (11) the model gets to the following,

$$(D^{\alpha}\Phi)(\eta)\approx(D^{\alpha}\Phi_{n,m}B_N)(\eta)=\Phi_{n,m}(D^{\alpha}B_N)(\eta)\approx\Phi_{n,m}F^{\alpha}B_N(\eta) \tag{12}$$

Correspondingly, the system derives the following relationship from (11) and (12).

$$Q^{\alpha}_{n,m}\Phi(\eta)=Q^{\alpha}_{n,m}\Phi_{n,m}B_N(\eta)=\Phi_{n,m}F^{\alpha}B_N(\eta) \tag{13}$$

This gives the Fibonacci wavelets the needed operational matrix of general order integration:

$$Q^{\alpha}_{n,m}=\Phi_{n,m}F^{\alpha}\left[\Phi_{n,m}\right]^{-1} \tag{14}$$

For example, if the model is $k=2$, $M=3$ and $\alpha=1.25$, the fractional-order operational matrix $Q^{1.25}_{6\times6}$ associated with the Fibonacci wavelets, further the equivalent fractional-order operational matrix reads.

$$Q_{6\times6}^{1.25} = \begin{bmatrix} -0.1147 & 0.1579 & 0.1475 & 0.4426 & 0.1130 & -0.0885 \\ -0.3194 & -0.0278 & 0.4514 & 0.3732 & 0.1169 & -0.1006 \\ -0.1950 & 0.0864 & 0.2679 & 0.4248 & 0.1180 & -0.0965 \\ 0 & 0 & 0 & -0.1147 & 0.1579 & 0.1475 \\ 0 & 0 & 0 & -0.3194 & -0.0278 & 0.4514 \\ 0 & 0 & 0 & -0.1950 & 0.0864 & 0.2679 \end{bmatrix} \quad (15)$$

Similarly, for $k = 2$, $M = 3$ and $\alpha = 1$ the matrix have

$$Q_{6\times6} = \begin{bmatrix} 0 & 0.2887 & 0 & 0.5000 & 0 & 0 \\ -0.4330 & 0 & 0.5916 & 0.4330 & 0 & 0 \\ -0.1745 & 0.1663 & 0.2500 & 0.4880 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2887 & 0 \\ 0 & 0 & 0 & -0.4330 & 0 & 0.5916 \\ 0 & 0 & 0 & -0.1745 & 0.1663 & 0.2500 \end{bmatrix} \quad (16)$$

i. Description of the Fibonacci wavelet method

This section demonstrates the validity of the general order integration operational matrices based on Fibonacci wavelets built in the previous section for solving the time-fractional telegraph equations with both starting and boundary conditions. Consider the fractional-order telegraph equation below.

$$\frac{\partial^{\alpha} f(\eta,\zeta)}{\partial \zeta^{a}} + \frac{\partial^{\alpha-1} f(\eta,\zeta)}{\partial \zeta^{\alpha-1}} + f(\eta,\zeta) = \frac{\partial^{2} f(\eta,\zeta)}{\partial \eta^{2}} + g(\eta,\zeta), \quad 1 < \alpha \le 2 \quad (17)$$

with initial conditions

$$f(\eta,0) = y_1(\eta), \quad \frac{\partial f(\eta,0)}{\partial \zeta} = y_2(\eta), \quad 0 \le \eta \le 1 \quad (18)$$

and Dirichlet boundary conditions

$$f(0,\zeta) = u_0(\zeta), \quad f(1,\zeta) = u_1(\zeta), \quad 0 < \zeta \le 1 \quad (19)$$

With second-order continuous derivatives, Where, y_1, y_2, u_1 and u_2 be appropriate to $L^2(0,1]$. To solve the telegraph equation (17), the model uses a Fibonacci wavelet basis to approximate the highest partial derivative.

$$\frac{\partial^4 f(\eta,\zeta)}{\partial \eta^2 \partial \zeta^2} \approx \Phi^T(\eta) H \Phi(\zeta) \quad (20)$$

The unknown Fibonacci wavelet coefficient vector is provided by (12). This might be obtained by integrating two times with regard to ζ

$$\frac{\partial^2 f(\eta,\zeta)}{\partial \eta^2} \approx \Phi^T(\eta) H \left(Q^2 \Phi(\zeta)\right) + \zeta\, y_2''(\eta) + y_1''(\eta) \quad (21)$$

Now, two-times integrate (21) concerning η, and thus obtain the following equation.

$$f(\eta,\zeta)\approx\left(Q^2\Phi(1)\right)^T H\left(Q^2\Phi(\zeta)\right)+\zeta\left(y_2(1)-y_2(0)-y_2'(0)\right)+ \left(y_1(1)-y_1(0)-y_1'(0)\right)+\frac{\partial f(\eta,\zeta)}{\partial\eta}\Big|_{\eta=0}+f(0,\zeta) \tag{22}$$

In addition, the equation stated that

$$\frac{\partial f(\eta,\zeta)}{\partial\eta}\Big|_{\eta=0}\approx-\left(Q^2\Phi(1)\right)^T H\left(Q^2\Phi(\zeta)\right) +\zeta\left(y_2(1)-y_2(0)-y_2'(0)\right)+\left(y_1(1)-y_1(0)-y_1'(0)\right)+\eta L(\zeta)+u_0(\zeta) \tag{23}$$

By substituting (23) for (22) the model get

$$f(\eta,\zeta)\approx\left(Q^2\Phi(\eta)\right)^T H\left(Q^2\Phi(\zeta)\right)+\zeta\left(y_2(\eta)-y_2(0)-\eta y_2'(0)\right)+ \left(y_1(1)-y_1(0)-\eta y_1'(0)\right)+\eta L(\zeta)+u_0(\zeta) \tag{24}$$

The model is obtained by taking the fractional derivative on both sides of (24) with regard ζ .

$$\frac{\partial f(\eta,\zeta)}{\partial\eta}\Big|_{\eta=0}\approx-\left(Q^2\Phi(1)\right)^T H\left(Q^2\Phi(\zeta)\right)+\eta D^{\alpha}L(\zeta)+D^{\alpha}u_0(\zeta),$$

$$\frac{\partial f(\eta,\zeta)}{\partial\eta}\Big|_{\eta=0}\approx-\left(Q^2\Phi(1)\right)^T H\left(Q^2\Phi(\zeta)\right)+ \left(y_2(1)-y_2(0)-y_2'(0)\right)\frac{\Gamma(2)}{\Gamma(3-\alpha)}\zeta^{2-\alpha}+\eta D^{\alpha-1}L(\zeta)+D^{\alpha-1}u_0(\zeta) \tag{25}$$

Where

$$D^{\alpha-1}L(\zeta)=-\left(Q^2\Phi(1)\right)^T H\left(Q^2\Phi(\zeta)\right)-D^{\alpha}u_0(\zeta)+D^{\alpha}u_1(\zeta) \tag{26}$$

$$D^{\alpha-1}L(\zeta)=-\left(Q^2\Phi(1)\right)^T H\left(Q^2\Phi(\zeta)\right) -\left(y_2(1)-y_2(0)-y_2'(0)\right)\frac{\Gamma(2)}{\Gamma(3-\alpha)}\zeta^{2-\alpha}-D^{\alpha-1}u_0(\zeta)+D^{\alpha-1}u_1(\zeta) \tag{27}$$

Obtain a system of algebraic equations of the type by substituting the estimates (21), (24), (25) and (26) into the supplied model (17) at the predefined collocation locations.

$$\begin{aligned}&\left(Q^2\Phi(\eta)\right)^T H\left(Q^2\Phi(\zeta)\right)+\eta D^{\alpha-1}L(\zeta)+D^{\alpha-1}u_0(\zeta)+\left(Q^2\Phi(\eta)\right)^T H\left(Q^{3-\alpha}\Phi(\zeta)\right)+\\&\left(y_2(\eta)-y_2(0)-\eta y_2'(0)\right)\frac{\Gamma(2)}{\Gamma(3-\alpha)}\zeta^{2-\alpha}+\eta D^{\alpha-1}L(\zeta)+D^{\alpha-1}u_0(\zeta)+\\&\left(Q^2\Phi(\eta)\right)^T H\left(Q^2\Phi(\zeta)\right)+\zeta\left(y_2(\eta)-y_2(0)-\eta y_2'(0)\right)+\left(y_1(\eta)-y_1(0)-\eta y_1'(0)\right)+\\&\eta L(\zeta)+u_0(\zeta)-\Phi^T(\eta)H\left(Q^2\Phi(\zeta)\right)-\zeta y_2''(\eta)-y_1''(\eta)=g(\eta,\zeta)\end{aligned} \tag{28}$$

The unknown Fibonacci wavelet coefficient vector may be obtained by solving the system of algebraic Equation (28), and then substituting the values of in (24), the system can yield an approximate solution of the supplied time-fractional telegraph equation (17).

b. Solving Three-Dimensional Equation Problem

For the solution of three-dimensional time-fractional telegraph equations, the Trigonometric Quintic B-spline (TQBS) using the Crank Nicolson approach is presented, subject to specified starting boundary conditions and Dirichlet boundary requirements. The suggested technique combats nonlinearity by employing a quasilinearization linearization procedure. The discretization of the time-fractional derivative is done using the Caputo fractional derivative formula. The considered solution is obtained using a combination of the Caputo fractional derivative and a trigonometric Quintic B-spline.

In three dimensions, consider the hyperbolic equation

$$\frac{\partial^2 v}{\partial t^2}+2\alpha\frac{\partial v}{\partial t}+\beta^2 v=A\frac{\partial^2 v}{\partial x^2}+B\frac{\partial^2 v}{\partial y^2}+C\frac{\partial^2 v}{\partial z^2}+f(x,y,z,t) \qquad (29)$$

Where, A,B,C,α,β are Positive functions of x,y and z these are in the domain $\Omega=\{(x,y,z,t)\,|\,0<x,y,z<1,t>0\}$. The telegraph equation is the name given to the equation for. $A=B=C=1$. Initially,

$$v(x,y,z)=v_1(x,y,z);\quad \frac{\partial v}{\partial t}(x,y,z,0)=v_2(x,y,z) \qquad (30)$$

The Dirichlet boundary conditions are defined as follows:

$$\begin{aligned} &v(0,y,z,t)=a_1(y,z,t) && v(1,y,z,t)=a_2(y,z,t)\\ &v(x,0,z,t)=b_1(x,z,t) && v(x,1,z,t)=b_2(x,z,t)\\ &v(x,y,0,t)=c_1(x,y,t) && v(x,y,1,t)=c_2(x,y,t)\end{aligned} \qquad (31)$$

i. Quintic Trigonometric B-Spline Collocation Method

Let divide the domain $[a,b]$ into equal-length $[x_i,x_{i+1}]$ intervals n, which include $x_i=a+(i\times h), i=0,1,\ldots,n, a=x_0, b=x_n$ and $h=\frac{1}{n}(b-a)$. The degree q, order $q+1$, polynomial B-spline is defined as

$$B_{q,r}(x)=L_{q,r}B_{q-1,r}(x)+(1-L_{q,r+1})B_{q-1,r+1}(x),\quad x\in[x_r,x_{r+1+q}] \qquad (32)$$

where $q>0$, $L_{q,r}=\frac{(x-x_r)}{(x_{r+q}-x_r)}$ and $B_{0,r}(x)=\begin{cases}1 & if\ \ x\in[x_r,x_{r+1}]\\ 0 & Otherwise\end{cases}$. The model obtains fifth-degree $q=5$ basis spline functions by combining (29) -(31) with

$$B_{5,r}(x)=\frac{1}{120h^5}\times\begin{cases}(x-x_{r-3})^5 & x\in[x_{r-3},x_{r-2}]\\ h^5+5h^4(x-x_{r-2})+10h^3(x-x_{r-2})+10h^2(x-x_{r-2})^3 & \\ +5h(x-x_{r-2})^4-5(x-x_{r-2})^5 & x\in[x_{r-2},x_{r-1}]\\ 26h^5+50h^5(x-x_{r-1})+20h^3(x-x_{r-1})^2-20h^2(x-x_{r-1})^3 & \\ -20h(x-x_{r-1})^4+10(x-x_{r-1})^5 & x\in[x_{r-1},x_r]\\ 26h^5+50h^5(x_{r+1}-x)+20h^3(x_{r+1}-x)^2-20h^2(x_{r+1}-x)^3 & \\ -20h(x_{r+1}-x)^4+10(x_{r+1}-x)^5 & x\in[x_r,x_{r+1}]\\ h^5+5h^4(x_{r+2}-x)+10h^3(x_{r+2}-x)+10h^2(x_{r+2}-x)^3 & \\ +5h(x_{r+2}-x)^4-5(x_{r+2}-x)^5 & x\in[x_{r+1},x_{r+2}]\\ (x_{r+3}-x)^5 & x\in[x_{r+2},x_{r+3}]\\ 0 & Otherwise\end{cases} \tag{33}$$

The spatial domain $[a,b]$ is divided evenly into N subintervals of the following lengths h as follows: $a=x_0<x_1<x_2,\ldots,<x_N=b$. In terms of the value of the spline function at the nodes, a quintic trigonometric spline approximation $u(x,t)$ to the analytic solution $U(x,t)$ may be written as:

$$u(x,t)=\sum_{m=-2}^{N+2}\delta_m(t)TB_m(x) \tag{34}$$

Where, $\delta_m(t), m=-2,\ldots,(N+2)$ is the quintic trigonometric B-spline basis function defined at nodes and $TB_m(x)$ is the time-dependent parameters to be acquired from boundary conditions and B-spline formulation. Outside $[x_0,x_N]$, the extra nodes x_{-2},x_{-1},x_{N+1}, and x_{N+2}, are positioned in a way of $x_0-x_{-1}=x_{-1}-x_{-2}=x_{N+1}-x_N=x_{N+2}-x_{N+1}=h$. Each quintic trigonometric B-spline basis function takes nonzero values at a maximum of five consecutive intervals, from x_{m-2} to x_{m+2}. Let's define the numerical solution $u(x,t)$ at $x=x_m$ and $t=t_n$, with and represent the mesh size t in both directions. As a result, the numerical values of u_m^n and their derivatives determined using the B-spline representation Equation (34) are:

$$\begin{aligned}u_m^n&=\alpha_1\delta_{m-2}^n+\alpha_2\delta_{m-1}^n+\alpha_3\delta_m^n+\alpha_2\delta_{m+1}^n+\alpha_1\delta_{m+2}^n\\ (u_x)_m^n&=-\beta_1\delta_{m-2}^n-\beta_2\delta_{m-2}^n+\beta_2\delta_{m+1}^n+\beta_1\delta_{m+2}^n\\ (u_{xx})_m^n&=\gamma_1\delta_{m-2}^n+\gamma_2\delta_{m-2}^n+\gamma_3\delta_m^n+\gamma_2\delta_{m+1}^n+\gamma_1\delta_{m+2}^n\end{aligned} \tag{35}$$

Where,

$\alpha_1=\csc(h)\csc(3h/2)\csc(2h)\csc(5h/2)\sin^4(h/2)$, $\alpha_2=\dfrac{(5+8\cos(h))\sec(h/2)\sec(h)}{4(1+2\cos(h))(1+2\cos(h))+2\cos(2h)}$,

$\alpha_3=\dfrac{(5+6\cos(h))\sec^2(h/2)\sec(h)}{(4+8\cos(h))+8\cos(2h)}$, $\beta_1=(5/4)\csc(3h/2)\csc(2h)\csc(5h/2)\sin^2(h/2)$,

$$\beta_2 = \frac{5(1+4\cos(h))\csc(h/2)\sec(h)}{8(1+2\cos(h))(1+2\cos(h))+2\cos(2h)},\ \gamma_1 = \frac{5(3+5\cos(h))(\csc^2(h/2)\sec(h/2)\sec(h))}{16(1+2\cos(h))(1+2\cos(h))+2\cos(2h)},$$

$$\gamma_2 = \frac{(3+\cos(h)+4\cos(2h))(\csc^2(h)\sec(h))}{32(1+2\cos(h))(1+2\cos(h))+2\cos(2h)},\ \gamma_3 = \frac{-5(2+5\cos(h)+\cos(2h))(\csc^2(h)\sec(h))}{8(1+2\cos(h)+2\cos(2h))}$$

The time derivative is discretized using the Crank-Nicolson technique, while the space derivatives are approximated using a quintic trigonometric B-spline. When the generalised θ scheme is applied to the telegraph equation (29), the model produces

$$(u_t)_m^n + (1-\theta) f_m^{n+1} = 0,\ 0 \le \theta \le 1 \qquad (36)$$

Where, $f_m^n = (u^\eta u_x)_m^n - v(u_{xx})_m^n$, $m = -2,-1,\ldots,(N+1),(N+2)$ and $n = 0,1,2,\ldots,T$. The time derivative involved in the above equation is discretized using finite difference $(u_t)_m^n = \frac{u_m^{n+1} - u_m^n}{k}$, resulting in the equation

$$u_m^{n+1} + \theta k\left((u^\eta u_x)_m^{n+1} - v(u_{xx})_m^{n+1}\right) = F_m^n - (1-\theta)k\left((F_m^n)^\eta G_m^n - vH_m^n\right),\ 0 \le \theta \le 1 \quad (37)$$

Where, $F_m^n = u_m^n$, $G_m^n = (u_x)_m^n$, $H_m^n = (u_{xx})_m^n$. To discretize u and its spatial derivatives by quintic trigonometric splines, replace $(u)_m^n, (u_x)_m^n$, and $(u_{xx})_m^n$ in the equation by their respective spline representations from Equations (35) to arrive at the final scheme. consider the Crank-Nicolson scheme for time integration, $\theta = 0.5$.

$$\begin{aligned}&\left(\alpha_1 + 2\theta k F_m^n G_m^n \alpha_1 - \theta k (F_m^n)^2 \beta_1 - \theta v k \gamma_1\right)\delta_{m-2}^{n+1} \\ &+ \left(\alpha_2 + 2\theta k F_m^n G_m^n \alpha_2 - \theta k (F_m^n)^2 \beta_2 - \theta v k \gamma_2\right)\delta_{m-2}^{n+1} + \left(\alpha_3 + 2\theta k F_m^n G_m^n \alpha_3 - \theta k (F_m^n)^2 \beta_3 - \theta v k \gamma_3\right)\delta_{m-2}^{n+1} \quad (38) \\ &= F_m^n - (1-\theta)k\left((F_m^n)^2 G - vH_m^n\right) + 2\theta k (F_m^n)^2 G_m^n\end{aligned}$$

For $m = 0,1,2,\ldots,N$ and $n = 0,1,2,\ldots,T$. The above system Equation (38) consists of $(N+1)$ equations in $(N+5)$ unknowns, namely $\delta_{-2}, \delta_{-1}, \delta_0, \ldots, \delta_N, \delta_{N+1}, \delta_{N+2}$. Only by reducing this system to a system of $(N+1)$ equations in $(N+1)$ unknowns can a unique solution be found. This is accomplished by removing the parameters from the system using the value of Equation (35) u_m^n and the boundary conditions. As a result, construct the following system of equations.

$$AX = B \qquad (39)$$

Where, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & \ldots & 0 \\ a_{21} & a_{22} & a_{23} & e & 0 & 0 & \ldots & 0 \\ a & b & c & d & e & 0 & \ldots & 0 \\ 0 & a & b & c & d & e & \ldots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & a & b & c & d & e \\ 0 & \ldots & 0 & 0 & a & a_{N,N-1} & a_{N,N} & a_{N,N+1} \\ 0 & \ldots & 0 & 0 & 0 & a_{N+1,N-1} & a_{N+1,N} & a_{N+1,N+1} \end{bmatrix}$, and $X = [\delta_0\ \delta_1 \ldots \delta_N]^T$

$$B = F_m^n - \frac{\Delta t}{2}\left(\left(F_m^n\right)^\eta G_m^n - vH_m^n\right) + \frac{\Delta t}{2}\eta\left(F_m^n\right)^\eta G_m^n = \left\{u - \frac{\Delta t}{2}\left(u^\eta u_x - vu_{xx}\right) + \frac{\Delta t}{2}\eta u^\eta u_x\right\}\Big|_{x=x_m}^{t=t_n} \quad (40)$$

Here, $a_{11} = c + \frac{(\beta_2\gamma_3)a - (\beta_1\gamma_3)b}{\beta_1\gamma_2 - \beta_2\gamma_1}, a_{12} = d + \frac{(2\beta_2\gamma_2)a - (\beta_1\gamma_2 + \beta_2\gamma_1)b}{\beta_1\gamma_2 - \beta_2\gamma_1}$,

$$a_{13} = e + \frac{(\beta_1\gamma_2 + \beta_2\gamma_1)a - (2\beta_1\gamma_1)b}{\beta_1\gamma_2 - \beta_2\gamma_1},\ a_{21} = b - \frac{(\beta_1\gamma_3)a}{\beta_1\gamma_2 - \beta_2\gamma_1},\ a_{22} = c - \frac{(\beta_1\gamma_2 + \beta_2\gamma_1)a}{\beta_1\gamma_2 - \beta_2\gamma_1},$$

$$a_{23} = d - \frac{(2\beta_1\gamma_1)a}{\beta_1\gamma_2 - \beta_2\gamma_1},\ a_{N,N-1} = -\frac{2\beta_1\gamma_1 e}{\beta_1\gamma_2 - \beta_2\gamma_1} + b,\ a_{N,N} = c - e\frac{\beta_1\gamma_2 + \beta_1\gamma_2}{\beta_1\gamma_2 - \beta_2\gamma_1},\ a_{N,N+1} = d - \frac{\beta_1\gamma_3 e}{\beta_1\gamma_2 - \beta_2\gamma_1},$$

$$a_{N+1,N-1} = a - \frac{(2\beta_1\gamma_1)d - (\beta_1\gamma_2 + \beta_2\gamma_1)e}{\beta_1\gamma_2 - \beta_2\gamma_1},\ a_{N+1,N} = b - \frac{(\beta_1\gamma_2 + \beta_2\gamma_1)d - 2(\beta_2\gamma_2 e)}{\beta_1\gamma_2 - \beta_2\gamma_1},$$

$$a_{N+1,N+1} = c - \frac{(\beta_1\gamma_3 + \beta_2\gamma_1)d - 2(\beta_2\gamma_3)e}{\beta_1\gamma_2 - \beta_2\gamma_1}$$

The values of a, b, c, d and e are as follows:

$$a = \alpha_1 + \frac{\Delta t}{2}\eta\left(F_m^n\right)^{\eta-1}G_m^n\alpha_1 - \frac{\Delta t}{2}\left(F_m^n\right)^\eta\beta_1 - \frac{\Delta t}{2}v\gamma_1,$$

$$b = \alpha_2 + \frac{\Delta t}{2}\eta\left(F_m^n\right)^{\eta-1}G_m^n\alpha_2 - \frac{\Delta t}{2}\left(F_m^n\right)^\eta\beta_2 - \frac{\Delta t}{2}v\gamma_2$$

$$c = \alpha_3 + \frac{\Delta t}{2}\eta\left(F_m^n\right)^{\eta-1}G_m^n\alpha_3 - \frac{\Delta t}{2}v\gamma_3$$

$$d = \alpha_2 + \frac{\Delta t}{2}\eta\left(F_m^n\right)^{\eta-1}G_m^n\alpha_2 - \frac{\Delta t}{2}\left(F_m^n\right)^\eta\beta_2 - \frac{\Delta t}{2}v\gamma_2$$

$$e = \alpha_1 + \frac{\Delta t}{2}\eta\left(F_m^n\right)^{\eta-1}G_m^n\alpha_2 - \frac{\Delta t}{2}\left(F_m^n\right)^\eta\beta_1 - \frac{\Delta t}{2}v\gamma_1$$

Consequently, finding the solution at the first-time level, δ^n, require an initial vector $\delta^0 = \left(\delta_0^0, \delta_1^0 \ldots, \delta_N^0\right)^T$. This initial vector is obtained from the given initial conditions.

c. Initial Boundary Value Problems

Boundary conditions cannot be met in differential equations, but the number of viable solutions is reduced in algebraic equations. By utilising temporal frequencies in finite-difference formulae, the exponentially fitted technique is meant to solve initial boundary value issues. With the order of convergence one, the parameter uniform convergence analysis was conducted. Three model examples with boundary layer behaviour are investigated to validate the theoretical finding. The scheme's maximum absolute error, as well as its rate of convergence, are calculated. The suggested approach is both convergent and unconditionally stable.

i. Exponentially Fitted Method for Boundary Value Problem

In this section, numerical solutions for problems (29) - (30) will be determined in the stated ranges using finite-difference methods. The integral constraint $\int_0^L u(x,t)dx = 0$ makes computations complicated. To overcome this difficulty, reduce the problem to an equivalent problem with classical conditions. Let u be a solution of (29)-(30). Integration of Equation (29) over $[0, L]$ gives

$$\frac{\partial v}{\partial t}\left[v_1(L,t) - u_x(0,t)\right] = 0\,,\ \forall t \in [0,T] \qquad (41)$$

Relation (41) and condition (30) yields

$$v_1(0,t) = 0 \qquad (42)$$

Now assume that v satisfies (29)-(30), the boundary condition $v_1(0,t) = 0$ and the compatibility conditions

$$\int_0^L f(x,y,z)dx = 0 \ \int_0^L v_2(x,y,z)dt = 0 \qquad (43)$$

then, the integration of Equation (29) with respect to v over $[0, L]$ gives

$$\frac{\partial^2}{\partial t^2}\int_0^L v(x,y,z,t)dx = 0 \qquad (44)$$

Hence $$\int_0^L v(x,y,z,t)dx = d_1 t + d_2 t \qquad (45)$$

Compatibility conditions (43) and (45) imply that $\int_0^L v(x,y,z,t)dx = 0$, and conclude that problems (29)-(30) and (42) are equivalent. The boundary value problem and (42) are solved by finite-difference methods by dividing the space interval $0 \le x \le L$ into M+1 subintervals each of width h, so that $(M+1)h$ the time interval t is discretized in steps each of length $l > 0$. The finite-difference method may be applied for $m = 1,2,\ldots,M$ and $n = 0,1,2,\ldots$

$$\frac{S_m^{n-1} - 2S_m^n + S_m^{n+1}}{L^2} - \frac{a(t)}{2h^2 L}\left[S_{m+1}^{n+1} + S_{m+1}^{n-1} - S_{m-1}^{n-1} + 2S_m^{n-1} - 2S_m^{n+1}\right] - H(x_m, t_n) = 0 \qquad (46)$$

Which after re-arranging becomes

$$-PaS_{m-1}^{n-1} + (2Pa+1)S_{m-1}^{n-1} - PaS_{m-1}^{n-1} = 2S_m^n - S_{m-1}^{n-1} + (2Pa+1)S_m^{n-1} - PaS_{m-1}^{n-1} + L^2 H(x_m, t_n) \quad (47)$$

Where, $P = L/2h^2$. The local truncation error $T_u = T_u[u(x,t); h, L]$ associated with (47) at the point $(x,t) = (x_m, t_n)$ may be written down from (46) and (29)

$$T_u = \frac{u(x,t-l)-2u(x,t)+u(x,t+L)}{L^2} -$$
$$\frac{a(t)}{2h^2L}[(u(x+h,t+L)+u(x-h,t+l)-u(x+h,t+l)-u(x-h,t-l)+2u(x,t-L)-2u(x.t+L))]$$
$$-H(x,t)-[u_{tt}-a(t)u_{xxt}-H(x,t)] \qquad (48)$$

Expanding $u(x,t\pm L)$, $u(x\pm h,t\pm L)$ in (48) as Taylor's series about (x,t) yields

$$T_u = \left[\frac{1}{12}u_{ttt} - \frac{a(t)}{6}u_{xxttt}\right]L^2 - \frac{a(t)}{12}h^2 u_{xxxxt} + \ldots, \qquad (49)$$

The convergence matrix is $O(h^2+L^2)$ as $h, L \to 0$. The finite-difference method (47) may be applied for $m = 1,2,\ldots,M$ and $n = 0,1,2,\ldots$. In these cases, $m = 0$, it requires some modifications and maybe simplified a little when $m = M+2$. Let A^{n+1} be a vector whose elements are ordered in rows parallel to the t-axis which is of an order $(M+2)$ that takes the form.

ii. Implementation and Stability Analysis

The Von Neumann method is employed to analyse the stability of the finite-difference approximations presented in (47). This approach looks for the circumstances in which minor form mistakes occur.

$$Z_m^n = A_m^n - \tilde{A}_m^n = e^{i\beta mh}e^{\alpha nl} \qquad (50)$$

Where, β is real and α is complex, $i = \sqrt{-1}$ and $\tilde{A}_m^n$ is perturbed numerical solution necessary conditions for the error to grow as $n \to \infty$.

$$\left|e^{\alpha l}\right| \le 1 + M_x \qquad (51)$$

From the equation, M_x is a non-negative constant independent of h, l. The condition in (51) makes no allowance for growing solutions if $M_x = 0$. Substituting Equation (50) into (47) leads to the local stability Equation.

$$2pa(t)e^{al}(1-\cos(\beta h)) - 2pa(t)e^{-\alpha l}(1-\cos(\beta h)) + e^{\alpha l} + e^{-\alpha l} - 2 = 0 \quad (52)$$

Which implies

$$e^{\alpha l} = \frac{1 \pm 4pa(t)\sin^2\frac{\beta h}{2}}{1 \pm 4pa(t)\sin^2\frac{\beta h}{2}} \qquad (53)$$

The von-Neumann necessary condition for stability is $\left|e^{\alpha l}\right| \le 1$, that is the stability restriction is $4pa(t) > 0$, which implies $a(t) > 0$ and $p > 0$ which agrees with the conditions.

5. Experimentation and Results Discussion:

A trigonometric quintic B-spline based on the Fibonacci wavelet method for the solution of the time-fractional telegraph equation is presented and discussed. Two test examples are used in this section to demonstrate the scheme's efficiency, accuracy, and computing complexity. The exact solution, approximate solution, and error values were computed for α = 1.95 and different M α values. The results were demonstrated in Tables 2-3. All tests are run on a Windows 10 computer with MATLAB R2021b (64 bit, CPU 2.20 GHz, 8.0 GB of RAM). To demonstrate the nature of the solution for varied times t, the numerical solution is also visually shown in relation to the analytic solution.

Table 1: System Configuration

MATLAB	Version R2021a
Operating System	Windows 10 Home
Memory Capacity	6GB DDR3
Processor	Intel Core i3 @ 3.5GHz

Table 1 shows the Matlab simulation machine configuration for solving stochastic differential equations using an appropriate numerical integration approach. The absolute error is used to assess the accuracy of the suggested technique in this study.

$$E_{abs} = \left| f(\eta,\zeta) - f_{n,m}(\eta,\zeta) \right| \tag{54}$$

Numerical Examples:

In this section, several illustrated cases have been provided to demonstrate the applicability and efficiency of the proposed wavelet technique. In both linear and nonlinear scenarios, numerical examples are addressed.

Example 1. Consider the following αth order telegraph equations

$$\frac{\partial^{\alpha} f(\eta,\zeta)}{\partial \zeta^{\alpha}} + \frac{\partial^{\alpha-1} f(\eta,\zeta)}{\partial \zeta^{\alpha-1}} + f(\eta,\zeta) = \frac{\partial^{2} f(\eta,\zeta)}{\partial \eta^{2}} + g(\eta,\zeta) \tag{55}$$

subject to the initial conditions $f(\eta,0)=0$, $\frac{\partial f(\eta,0)}{\partial \zeta} = \eta(\eta-1)$, $0 \le \eta \le 1$

and Dirichlet boundary conditions $f(0,\zeta) = f(1,\zeta) = 0$, $0 < \zeta \le 1$

Where, $g(\eta,\zeta) = \left(\frac{\Gamma(2)}{\Gamma(3-\alpha)} \zeta^{2-\alpha} + \zeta \right) (\eta^2 - \eta) - 2\zeta$. The exact solution to this problem is $f(\eta,\zeta) = (\eta^2 - \eta)\zeta$.

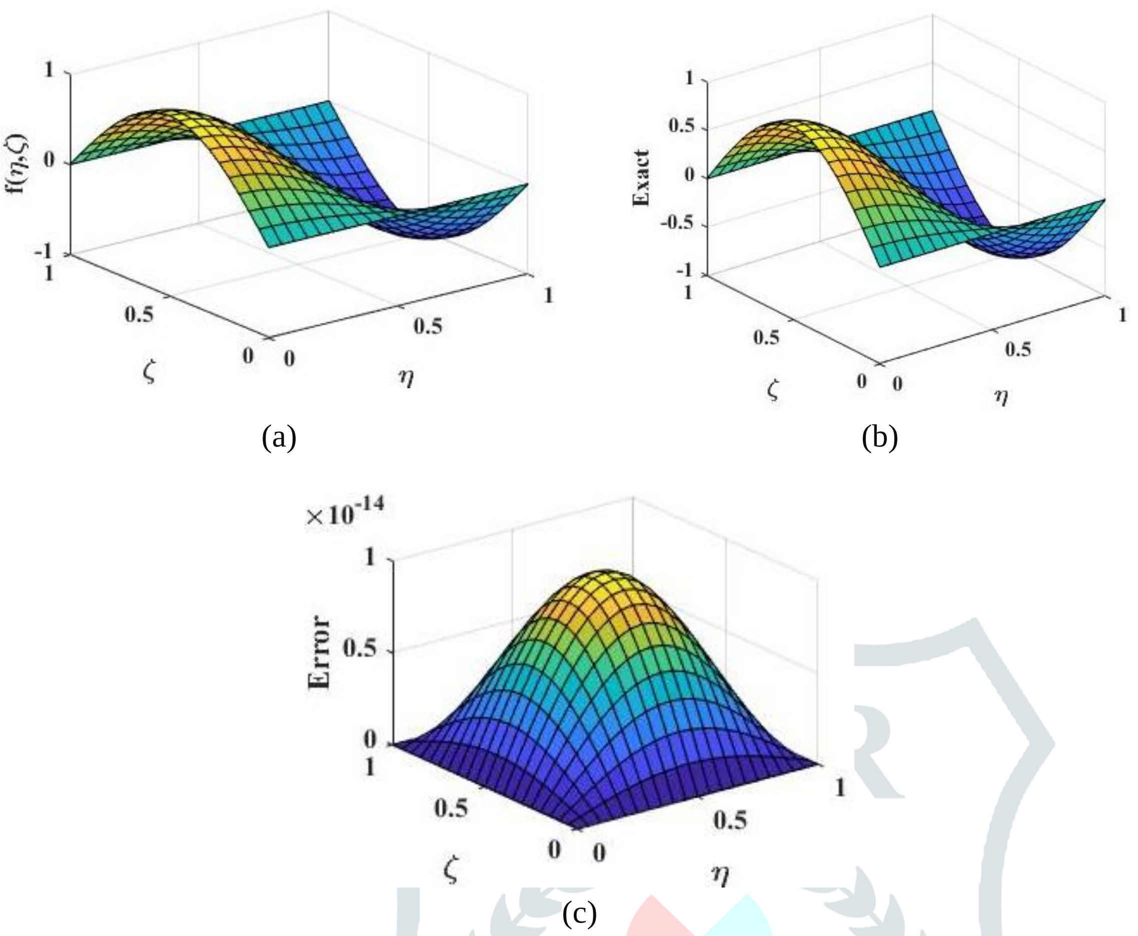


Figure 2: Approximate Solution of Example 1

Figure 2 depicts the approximate solutions to Example 1 visually. The approximate solution is shown in Figure 2(a), the precise solution is shown in Figure 2(b), and the graphical representation of absolute error is shown in Figure 2(c). The acquired absolute errors of (55) for various values are given in Table 2 to show the effectiveness and accuracy of the suggested approach. The approximation results achieved with Fibonacci wavelets are better and more accurate, as seen in these tables.

Table 2: Absolute Error of Proposed Method for Example 1

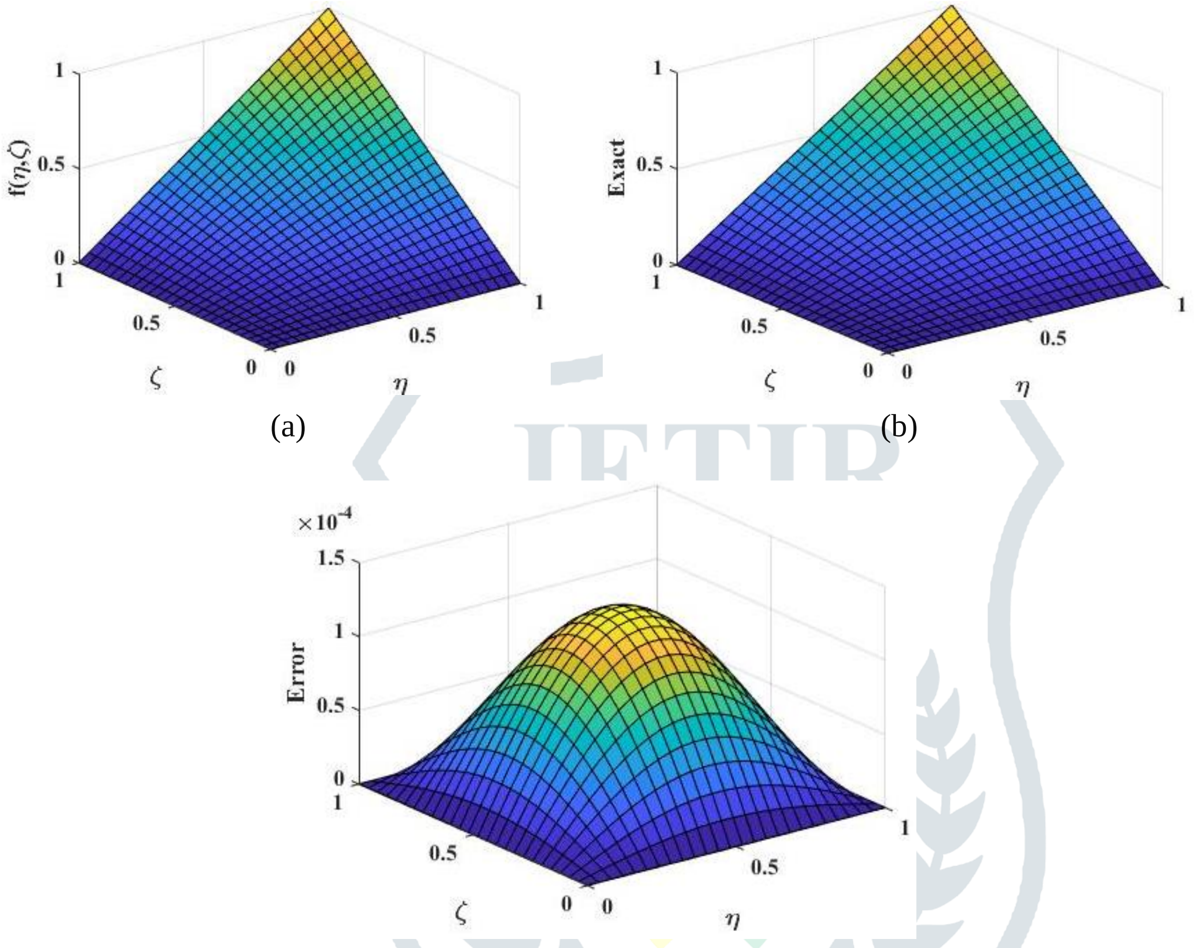
(η,ζ)	**Proposed Method**		
	$\alpha = 1.1$	$\alpha = 1.2$	$\alpha = 1.5$
(0.1,0.1)	1.5276 × 10−18	6.6353 × 10−18	1.9741 × 10−18
(0.2,0.2)	0	4.5797 × 10−18	4.3715 × 10−18
(0.3,0.3)	0	4.6491 × 10−18	8.6736 × 10−18
(0.4,0.4)	1.5821 × 10−18	0	6.6475 × 10−1
(0.5,0.5)	2.2413 × 10−1	7.6328 × 10−18	6.7446 × 10−18
(0.6,0.6)	0	7.9103 × 10−18	6.2172 × 10−18
(0.7,0.7)	4.7462 × 10−18	2.6645 × 10−18	1.4433 × 10−18
(0.8,0.8)	4.7462 × 10−18	1.1142 × 10−18	3.3307 × 10−18
(0.9,0.9)	6.9389 × 10−18	1.2490 × 10−18	2.3453 × 10−18

Example 2: Consider the following αth order telegraph equations

$$\frac{\partial^{\alpha} f(\eta,\zeta)}{\partial \zeta^{a}}+\frac{\partial^{\alpha-1} f(\eta,\zeta)}{\partial \zeta^{a-1}}+f(\eta,\zeta)=\pi\frac{\partial^{2} f(\eta,\zeta)}{\partial n^{2}}+g(\eta,\zeta), \quad 1<\alpha\leq 2 \, (56)$$

with initial conditions $f(\eta,0)=\dfrac{\partial f(\eta,0)}{\partial\zeta}=0$, $0\le\eta\le1$ and Dirichlet boundary conditions $f(0,\zeta)=0$, $f(1,\zeta)=\zeta^3\sin(1)$, $0<\zeta\le1$

Where, $g(\eta,\zeta)=\left(\dfrac{3\Gamma(3)}{\Gamma(4-\alpha)}\zeta^{(3-\alpha)}+\dfrac{3\Gamma(3)}{\Gamma(5-\alpha)}\zeta^{(4-\alpha)}+\zeta^3\right)\sin^2(\eta)-2\pi\zeta^3\cos(2\eta)$. The exact solution of (64) is $f(\eta,\zeta)=\zeta^3\sin^2(\eta)$.



(c)

Figure 3: Numerical Results for Approximate and Exact Solution

Figure 3 shows the approximate solution, absolute error, and precise solution of Example 2 in 3D at α = 1.95. Figure 3(a) denotes the approximate solution and figure 3(b) stated the exact solution. The absolute error implementation was explicated in figure 3(c). The approximate solutions are in good agreement with the actual solution of the issue, as shown in Figure 3. The numerical solutions derived using the Fibonacci wavelet matrix approach agree with the exact solutions sufficiently.

Table 3: Comparison of Maximum Absolute Error for Example 2

α	**Fibonacci Wavelet**		**Legendre Wavelet**	
	M=3	M=4	M=3	M=4
1.15	4.1184×10^{-5}	6.3146×10^{-6}	4.6439×10^{-5}	7.5593×10^{-6}
1.35	3.0138×10^{-5}	6.4611×10^{-6}	4.5474×10^{-5}	7.3935×10^{-6}
1.55	3.0138×10^{-5}	5.8519×10^{-6}	4.4293×10^{-5}	7.1980×10^{-6}

1.75	4.1642×10^{-5}	6.8947×10^{-6}	4.2909×10^{-5}	6.9749×10^{-6}
1.95	4.2956×10^{-5}	6.2791×10^{-6}	4.1016×10^{-5}	6.6688×10^{-6}

Table 3 presents a comparison table for the greatest absolute error for various values M and α demonstrates the validity of the suggested strategy. Table 3 shows that the approximate solution achieved using Fibonacci wavelets is more precise than the approximate solution found using Legendre wavelets.

8. Conclusion

Fractional differential equations have sparked a lot of curiosity in recent years. The FD technique, on the other hand, has several inherent drawbacks, particularly for problems with high dimensions, strong gradients, and complicated geometry. At the moment, there is relatively little study on the subject of fractional telegraph equations. B-splines have been used by several academics to solve FDEs, but only a few studies have been done on FTEs. As a result, for the solution of three-dimensional time-fractional telegraph equations, research integrated Trigonometric Quintic B-spline (TQBS) with the Crank Nicolson approach is presented. The calculated solutions are obtained using a combination of the Caputo fractional derivative and a trigonometric Quintic B-spline. Fibonacci wavelets and the conventional newton iteration approach are used to generate solutions to Time-Fractional Telegraph Equations (TFTs). The proposed scheme engages the usual finite forward difference formulation for initial-boundary problems respectively.

Numerical solutions derived using the Matlab software are found to be superior to those previously published in the literature. The scheme's key advantages are ease of implementation, reduced complexity, and low computing costs. Higher-dimensional FDEs can be handled well using this method. The scheme's applicability, simplicity, and strength in solving the time-fractional telegraph problem with accuracies extremely near to the actual solutions are demonstrated numerically. The computational order of convergence $O\left(h^2 + L^2\right)$is conformable with the theoretical estimations. Furthermore, the suggested scheme's convergence is investigated, and the scheme is shown to be unconditionally stable. The numerical simulation has been run for two test examples, which show that the proposed scheme can efficiently be employed for the numerical treatment of time-fractional problems. The simulation results show a superior agreement with the exact solution as compared to those found in the literature.

9. Acknowledgment: Author is thankful to the Editor and Referee for their valuable guidance and suggestions.

References:

[1] Akram, T., Abbas, M., Iqbal, A., Baleanu, D. and Asad, J.H., 2020. Novel numerical approach based on modified extended cubic B-spline functions for solving non-linear time-fractional telegraph equation. Symmetry, 12(7), p.1154.

[2] Kumar, A., Bhardwaj, A. and Dubey, S., 2021. A local meshless method to approximate the time-fractional telegraph equation. Engineering with Computers, 37(4), pp.3473- 3488.

[3] Ali, A. and Ali, N.H.M., 2019, August. Explicit group iterative methods for the solution of two-dimensional time-fractional telegraph equation. In AIP Conference Proceedings (Vol. 2138, No. 1, p. 030006). AIP Publishing LLC.

[4] Hafez, R.M. and Youssri, Y.H., 2020. Shifted Jacobi collocation scheme for multidimensional time-fractional order telegraph equation. Iranian Journal of Numerical Analysis and Optimization, 10(1), pp.195-223.

[5] Majeed, A., Kamran, M. and Asghar, N., 2021. Solution of non-linear time fractional telegraph equation with source term using B-spline and Caputo derivative. International Journal of Nonlinear Sciences and Numerical Simulation.

[6] Hafez, R.M., 2018. Numerical solution of linear and nonlinear hyperbolic telegraph type equations with variable coefficients using shifted Jacobi collocation method. Computational and Applied Mathematics, 37(4), pp.5253-5273.

[7] Hassani, H., Avazzadeh, Z. and Machado, J.T., 2020. Numerical approach for solving variable-order space–time fractional telegraph equation using transcendental Bernstein series Engineering with Computers, 36(3), pp.867-878.

[8] Sweilam, N.H., Nagy, A.M. and El-Sayed, A.A., 2020. Sinc-Chebyshev collocation method for time-fractional order telegraph equation. Appl. Comput. Math, 19(2), pp.162-174.

[9] Ibrahim, W. and Bijiga, L.K., 2021. Neural Network Method for Solving Time- Fractional Telegraph Equation. Mathematical Problems in Engineering, 2021.

[10] Modanli, M. and Akgül, A., 2020. On Solutions of Fractional order Telegraph partial differential equation by Crank-Nicholson finite difference method. Applied Mathematics and Nonlinear Sciences, 5(1), pp.163-170.

[11] Tapaswini, S. and Behera, D., 2020. Analysis of imprecisely defined fuzzy space- fractional telegraph equations. Pramana, 94(1), pp.1-10.

[12] Ezz-Eldien, S.S., Doha, E.H., Wang, Y. and Cai, W., 2020. A numerical treatment of the two-dimensional multi-term time-fractional mixed sub-diffusion and diffusion- wave equation. Communications in Nonlinear Science and Numerical Simulation, 91, p.105445.

[13] Wang, F. and Hou, E., 2020. A direct meshless method for solving two-dimensional second-order hyperbolic telegraph equations. Journal of Mathematics, 2020.

[14] Abbaszadeh, M. and Dehghan, M., 2021. A finite-difference procedure to solve weakly singular integro partial differential equation with space-time fractional derivatives. Engineering with computers, 37(3), pp.2173-2182.

[15] Xu, X. and Xu, D., 2018. Legendre wavelets direct method for the numerical solution of time-fractional order telegraph equations. Mediterranean Journal of Mathematics, 15(1), pp.1-33.

[16] Yang, X., Qiu, W., Zhang, H. and Tang, L., 2021. An efficient alternating direction implicit finite difference scheme for the three-dimensional time-fractional telegraph equation. Computers & Mathematics with Applications, 102, pp.233-247.

[17] Vieira, N., Rodrigues, M.M. and Ferreira, M., 2021. Time-fractional telegraph equation of distributed order in higher dimensions. Communications in Nonlinear Science and Numerical Simulation, p.105925.

[18] Yang, X. and Liu, X., 2021. Numerical analysis of two new finite difference methods for time-fractional telegraph equation. Discrete & Continuous Dynamical Systems- B, 26(7), p.3921.

[19] Abdi, N., Aminikhah, H. and Sheikhani, A.R., 2021. High-order rotated grid point iterative method for solving 2D time fractional telegraph equation and its convergence analysis. Computational and Applied Mathematics, 40(2), pp.1-26.

[20] Ali, A., Abdeljawad, T., Iqbal, A., Akram, T. and Abbas, M., 2021. On Unconditionally Stable New Modified Fractional Group Iterative Scheme for the Solution of 2D Time- Fractional Telegraph Model. Symmetry, 13(11), p.2078.

[21] Srinivasa, K. and Rezazadeh, H., 2021. Numerical solution for the fractional-order one-dimensional telegraph equation via wavelet technique. International Journal of Nonlinear Sciences and Numerical Simulation, 22(6), pp.767-780.

[22] Salama, F.M., Abd Hamid, N.N., Ali, N.H.M. and Ali, U., 2022. An efficient modified hybrid explicit group iterative method for the time-fractional diffusion equation in two space dimensions. AIMS Mathematics, 7(2), pp.2370-2392.

[23] Shah, N.A., Dassios, I. and Chung, J.D., 2021. A Decomposition Method for a Fractional-Order Multi-Dimensional Telegraph Equation via the Elzaki Transform. Symmetry, 13(1), p.8.

[24] Singh, B.K., Shukla, J.P. and Gupta, M., 2021. Study of One Dimensional Hyperbolic Telegraph Equation Via a Hybrid Cubic B-Spline Differential Quadrature Method. International Journal of Applied and Computational Mathematics, 7(1), pp.1- 17.

[25] Saad, A. and Brahim, N., 2021. An efficient algorithm for solving the conformable time-space fractional telegraph equations. Moroccan Journal of Pure and Applied Analysis, 7(3), pp.413-429.