GENERALIZED FINITE ELEMENT METHOD-ISOPARAMETRIC DOMAIN TRANSFORMATION (MINIMUM ELEMENTS APPROACH)

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Abstract: In this paper, I have presented a new approach for solving general 2D partial differential equations using the finite element method with the help of isoparametric quadrilateral element for mapping arbitrary shape domains. The concept behind this approach is to partition the whole geometrical domain into the minimum quadrilateral elements. This approach focuses on using isoparametric elements (minimum) to cover the whole complex domain. This is achieved by partitioning the whole domain into 1 to 4 isoparametric elements (irregular, curved edge type elements) then it is mapped into regular rectangular elements in the natural coordinate system using the Lagrangian interpolation basis function and Jacobin matrix. After this, the **method of residual** and **Galerkin method** is used to solve the weak form of the governing differential equation.

Keywords: finite element method, fem, isoparametric formulation, Jacobian transformation, minimum element domain partition, Galerkin method, Laplace equation.

1. INTRODUCTION:

The finite element method (FEM) has been widely used to solve the partial differential equation in the complex domain. But the major problem in working with FEM is dealing with the generation of mesh and connectivity of elements. In this paper, a simple approach for eliminating the overuse of mesh generation is presented. Note that the FEM procedure will be the same as standard FEM only difference is that in this approach very less elements are needed. The idea behind this is that the whole arbitrary shaped geometrical domain can be partitioned into 1 to 4 isoparametric quadrilateral elements in other words, by displacing the vertices of the quadrilateral element we can exactly map the given domain. The isoparametric element is nonregular and has curved sides. The term isoparametric is derived from the use of the same shape functions(interpolation basis functions) to define the element geometry and nodal displacement in the element [2]. At least I can say that there is no need for complex meshing or any mesh generation software in this approach just coordinates of domain points are needed and these vertices are easily generated using any vector graphics editor. I used Inkscape for lower order and Blander for higher order. both are open sources.

The concept for mapping of circular and annular domain are given in section 2. Section 3 and 4 highlight the mathematical background of shape functions, transformation, and jacobian for 1D and 2D geometry.. To check the versatility of this approach the general differential equation is solved for the Dirichlet boundary condition in section 5.1 and in section 5.2. the Poisson equation is solved on the L-Shape domain.

2. **Partition of the circular domain using a single second-order quadrilateral element.**

It is well known that in FEM, Lagrangian or Hermite basis functions are used for two purposes, one for approximating the given differential equation as trial and test function in weak form and second, for mapping of irregularly shaped mesh element into the standard rectangular element (in natural coordinates system s,t) [1]. Now the given circular domain is mapped by displacing vertices of quadrilateral (second-order, 9 nodes) on circumference as shown in fig 2.1. after transformation (theory given in section 2) this circular domain is exactly mapped into standard 9 nodes rectangular element in the natural coordinate system as shown in Fg.2.2. once the domain is mapped into a standard element, we can further divide this standard element into any order here I have taken the 6th order so that there will be a total of 49 domain points. For display purposes, I have interpolated intermediate points and plotted them as shown in Fig. 2.3. similarly, the annular domain is mapped from 4 elements, second-order each as shown in Fig.
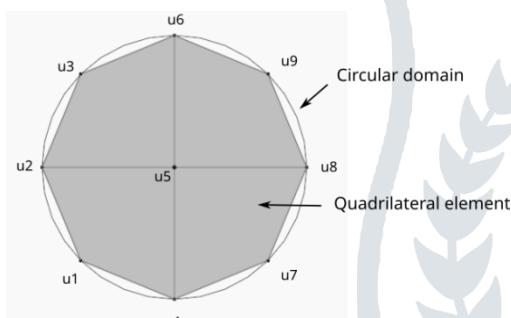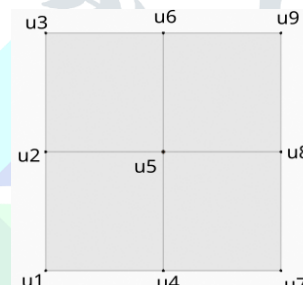

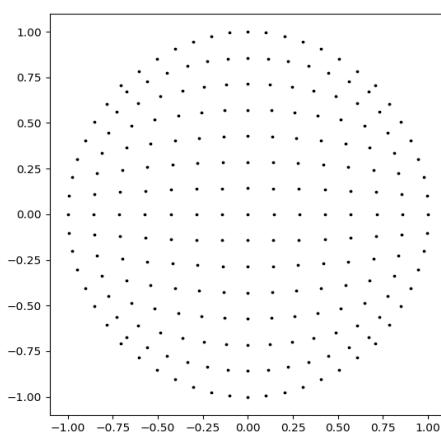Fig.2.1. Nine nodes **single** quadrilateral element   Fig. 2.2. Nine nodes standard rectangular element.
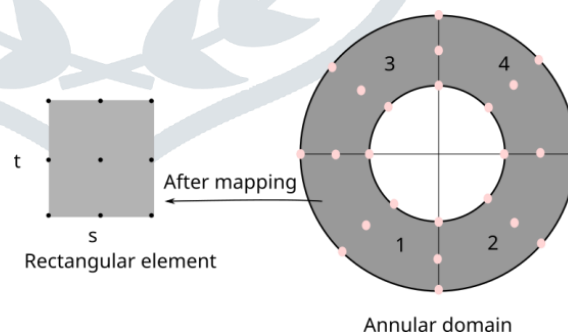

Fig.2.4 Annular domain mapped from four elements, second-order each.

Fig. 2.3 Result of the mapped circular domain from 9 nodes single quadrilateral element.
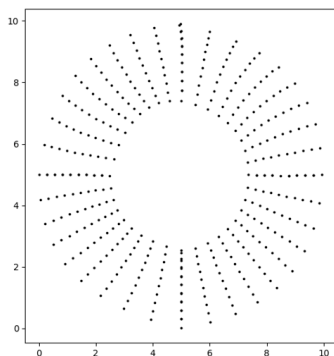
Fig. 2.5. Result of the mapped annular domain from 4 quadrilateral elements.

To find the approximate solution of a given differential equation it is necessary to have good enough interior points in the domain. To increase the internal points we can increase the order of approximation. It is not necessary to have a large order for approximating the domain. It is observed that 1 to 4 order of shape functions are sufficient to map the domain.

3. Shape functions

The Lagrange linear polynomial shape functions along X-axis are given by

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

And along Y-axis

$$l_i(y) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(y - y_j)}{(y_i - y_j)}$$

Taking the tensor product of the above two-liner shape function, we can obtain the 2D basis shape functions $\phi(s, t)$.

4. Transformations and Jacobian

Jacobin is the matrix that is useful in the transformation of one coordinate system to another; it may be Cartesian to Cartesian or Cartesian to polar etc.

1D Case:

This transformation is used in the calculation of line integral on the boundary of the domain. For line integration $f = f(x, y)$ and $x = x(s), y = y(s)$ these are the parametric equation used for the transformation of the simple horizontal line called a endpoints distances master element, and its are -1 and 1.
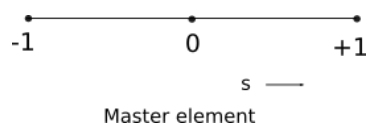

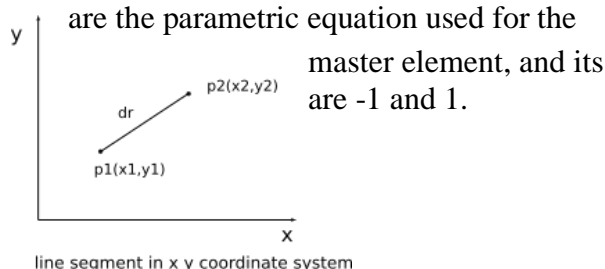Fig. 4.1   Fig. 4.2

To map an inclined line in the XY coordinate system to the master element in a natural coordinate system, the transformation equation in terms of Lagrange linear basis function $\phi(s)$ are

$x = x_1\phi_1 + x_2\phi_2$ And $y = y_1\phi_1 + y_2\phi_2$

Here $\phi_1(s)$ and $\phi_2(s)$ are the linear shape functions or linear basis functions in the natural coordinate system, that is $\phi = \phi(s)$.

The line integration on the segment dr is written as

$$\int_a^b f(x,y)dr = \int_{-1}^1 f(x(s), y(s))\sqrt{(dx)^2 + (dy)^2}$$

$$\int_a^b f(x,y)dr = \int_{-1}^1 f(x(s), y(s))\sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2}\ ds$$

Where $\sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2}$ is the Jacobin J.

For linear shape function on boundary only x or y variable present therefore

$$\int_a^b f(x)dx = \int_{-1}^1 f(x(s))J\ ds$$

Where $J = \frac{dx}{ds}$

Numerically, line integration is solved by using Gauss Quadrature integration [9]. which is

$$\int_a^b f(x)dx = \frac{b-a}{2}\sum_{i=1}^n w_i f(x(s))J$$

Where, w_i are the Gauss-Quadratures weights. And note that Jacobian is evaluated at Gauss sampling points. To find the derivative of shape function ϕ, as we know that shape function ϕ is the function of s only and we have to calculate the derivative with the respective x. then

$$\frac{d\phi}{dx} = \frac{d\phi}{ds}\frac{ds}{dx}$$

$$\therefore \frac{d\phi}{dx} = \frac{d\phi}{ds}\frac{1}{|J|}$$

Where, $J = \frac{dx}{ds}$ is the Jacobin.

2D Case:

In this case, transformation equations are [1]

$x = \sum_{i=1}^N x_i\phi_i(s,t)$　　　　　...1

$$y = \sum_{i=1}^{N} y_i \phi_i(s,t) \qquad \ldots 2$$

Where (x_i, y_i) the coordinates of domain points, (x, y) are the new points in the domain corresponding to natural coordinates (s, t) in the master element. Where N is the number of points in the domain.

Suppose we have a function $f(x, y)$ and in the Cartesian coordinate system and by transformation technique we got the values of x and y such that $x = x(s,t)$ and $y = y(s,t)$. Where, s and t are horizontal and vertical axes in the natural coordinate system. Suppose we want to integrate $f(x, y)$ function, then the surface integration with transformation is given by [2]

$$\int \int f(x,y)dxdy = \int \int f(x(s,t), y(s,t))J\, dsdt \qquad \ldots 3$$

Calculation of Jacobian in 2D:

Let $f = f(x, y)$ and $x = x(s,t), y = y(s,t)$ According to the chain rule [2]

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Consider the above equation as a system of two linear equations with two unknown $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. In matrix form

$$\begin{pmatrix} \dfrac{dx}{ds} & \dfrac{dy}{ds} \\ \dfrac{dx}{dt} & \dfrac{dy}{dt} \end{pmatrix} \begin{pmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f}{\partial s} \\ \dfrac{\partial f}{\partial t} \end{pmatrix}$$

$$D = \begin{vmatrix} \dfrac{dx}{ds} & \dfrac{dy}{ds} \\ \dfrac{dx}{dt} & \dfrac{dy}{dt} \end{vmatrix}$$

$$D_x = \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{dy}{ds} \\ \dfrac{\partial f}{\partial y} & \dfrac{dy}{dt} \end{vmatrix}$$

$$D_y = \begin{vmatrix} \dfrac{dx}{ds} & \dfrac{\partial f}{\partial x} \\ \dfrac{dx}{dt} & \dfrac{\partial f}{\partial y} \end{vmatrix}$$

According to the crammers rule [2]

$\frac{\partial f}{\partial x} = \frac{D_x}{D}$ and $\frac{\partial f}{\partial y} = \frac{D_y}{D}$. here $D = |J|$ is the 2D Jacobin

Once we know the Transformation and Jacobian then we can integrate and differentiate the shape functions. Using Gauss-Quadrature integration (ref. 9), equation 2 is written as

$$\int_a^b \int_c^d f(x,y)dxdy = \left(\frac{b-a}{2}\right)\left(\frac{d-c}{2}\right)\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j f\big(x(s,t), y(s,t)\big)J$$

Note that, here jacobian is evaluated at Gauss sampling points.

Where w_i or w_j are Gauss-Legender weight points.

4.1 complete steps involved in FEM

1. Map the given domain by displacing the vertices of the quadrilateral elements. The order of this element depends on the corners and the curve boundary appears in the domain.
2. Transform these quadrilateral elements into standard rectangular elements.
3. Increase the order of the standard rectangular element (subdivide) so that domain contains optimum nodes for a solution.
4. Make the week form of the given differential equation by using Galerkin and Residual method.
5. Obtain the element level stiffness, and load matrices. (for other problems, Mass and Robinson matrices)
6. Apply the Neumann and Robin (if present) boundary condition at the element level.
7. Assemble the matrices (if elements are more than 1) and obtain global level matrices
8. Apply Dirichlet boundary condition.
9. Solve global level matrix equation.

In this paper, problems are solved using Python language.

5. NUMERICAL EXAMPLES:

5.1. Solving Elliptical equation on a square domain

Week formulation:

Consider the 2d second-order elliptical equation [2]

$$-\nabla.(c\nabla u) = f, in\ \Omega \quad \ldots 1$$

$$u = g, on\ \partial\Omega$$

Where Ω is the 2D domain, $f(x,y)$ and $c(x,y)$ are given functions on Ω, $g(x,y)$ is given function on $\partial\Omega$, and $u(x,y)$ is an unknown function.

First multiply equation 1 by test function $v(x,y)$ on both sides and integrate over the whole domain Ω (as per the rule of the **Residual method**)

$$-\nabla.(c\nabla u) = f, in\ \Omega$$

$$\Rightarrow -\nabla.(c\nabla u)v = fv\ \ in\ \Omega$$

$$\Rightarrow -\int_\Omega \nabla.(c\nabla u)v\ dxdy = \int_\Omega fv\ dxdy$$

Where $u(x,y)$ is trial function and $v(x,y)$ is the test function

According to integration by pats above equation become

$$\int_{\Omega}(c\nabla u).\nabla v \, dxdy - \int_{d\Omega}(c\nabla u.\hat{n})v \, ds = \int_{\Omega} fv \, dxdy \qquad \ldots 2$$

The highlighted term plays a very important role in applying the boundary condition. Here
$d\Omega$ is the total boundary of given surface and $\hat{n}$ is the unit normal to the given boundry

Dirichlet boundary condition: If the solution on the boundary is specified that is $u = u_0$ or 0 or $g(x,y)$, then we can choose the test function $v(x,y) = 0$ on $d\Omega$.

Hence above equation become

$$\int_{\Omega}(c\nabla u).\nabla v \, dxdy = \int_{\Omega} fv \, dxdy \qquad \ldots 3$$

Let $trial \; solution \; u = \sum_{i=1}^{N} u_i \phi_i(x,y)$

Where N is the number of basis functions used for interpolation.

And according to Galerkin, the test basis function is the same as the trial function that is, $v = \phi_j$

Putting this function in equation 3 we get

$$\sum_{\{i,j=1\}}^{N} \int_{\Omega} u_i (c\nabla\phi_i).\nabla\phi_j \, dxdy = \sum_{i}^{N} \int_{\Omega} f\phi_j \, dxdy$$

Here both $\phi_i \; and \; \phi_j$ are same, it is just tensor product of each other up to total shape's functions.

The above equation in matrix form will become

$$[k][u] = [f]$$

Where,

$$k = a_i \sum_{i,j=1}^{N} \int_{\Omega} \left(\frac{\partial\phi_i}{\partial x}\frac{\partial\phi_j}{\partial x} + \frac{\partial\phi_i}{\partial y}\frac{\partial\phi_j}{\partial y}\right)dxdy \; , N \times N \; matrix$$

$$f = \sum_{i,j=1}^{N} \int_{\Omega} \phi_i dxdy \; , \qquad N \times 1 \; matrix \; or \; column \; vector$$

And $X = N \times 1 \; unknown \; column \; vector$ that we want to calculate.

All these equations are element-wise. after this we have to assemble these matrices into single global matrices (if more than one element is present in the problem) so that we will get global stiffness matrix K, and global load vector L. this matrix equation is nothing but a set of algebraic equations in matrix form

$$KU = F$$

solve the following equation on the domain $\Omega = [-1,1] \times [-1,1]$:

$$-\nabla.(\nabla u) = -y(1-y)\left(1-x-\frac{x^2}{2}\right)e^{x+y} - x(1-x)(-3y-y^2)e^{x+y}$$

$u = -1.5y(1-y)e^{-1+y}$ on $x = -1$,

$u = 0.5y(1-y)e^{1+y}$ on $x = -1$,

$u = -2x\left(1-\frac{x}{2}\right)e^{x-1}$ on $y = -1$,

$u = 0$ on $y = 1$.

The analytical solution is $u = xy\left(1-\frac{x}{2}\right)(1-y)e^{x+y}$
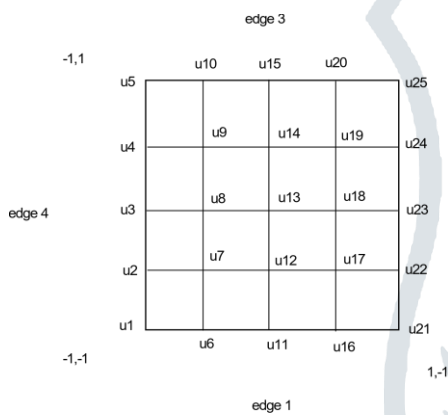
Problem domain:

Fig. 5.1

Here the **single element** of order 4 is used. The exact and FEM solutions are given in Table 5.1

Table 5.1

Node numbers	FEM Solution	Exact Solution	Absolute Error
1	0.40600585	0.40600585	0
2	0.2789127	0.2789127	0
3	0	0	0
4	-0.454897995	-0.454897995	0
5	-1	-1	0
6	0.25102143	0.25102143	0
7	0.175103454	0.172443488	0.002659966
8	0.000199869	0	0.000199869
9	-0.285383061	-0.28125	0.004133061
10	-0.618270477	-0.618270477	0
11	0	0	0
12	0.000628707	0	0.000628707
13	-6.03E-05	0	6.03417E-05
14	-0.001128325	0	0.001128325
15	0	0	0
16	-0.227448997	-0.227448997	0
17	-0.159966556	-0.15625	0.003716556

18	-0.000189973	0	0.000189973
19	0.260513906	0.254838921	0.005674984
20	0.560211134	0.560211134	0
21	0	0	0
22	0	0	0
23	0	0	0
24	0	0	0
25	0	0	0

5.2. **Solving Poisson's equation on L-shaped domain [3]**

Equation 1 in the above example becomes the Poisson equation when c=1.

Solve, $-\nabla.(\nabla u) = 1$ on L shaped domain given in Fig. 5.2.1
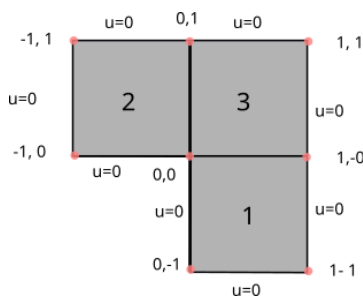

Fig.5.2.1. L-Shaped problem domain (homogeneous Dirichlet boundary on all edges)

The given geometrical domain is mapped into **3 elements** of order 1 and the problem is solved using 3th order Lagrange shape functions so that total there are total 16+16+16=48 nodes in whole domain. The intermediate node solutions are also interpolated using lagrange shape functions.

This problem represents a simple diffusion model for the temperature distribution u(x, y) in an L-Plate. The specific source term in this example models uniform heating of the plate, and the boundary condition models the edge of the plate being kept at an ice-cold temperature. Fig 5.2.2 and 5.2.3 show the contour plot and 3d plot respectively on an L-shaped domain.
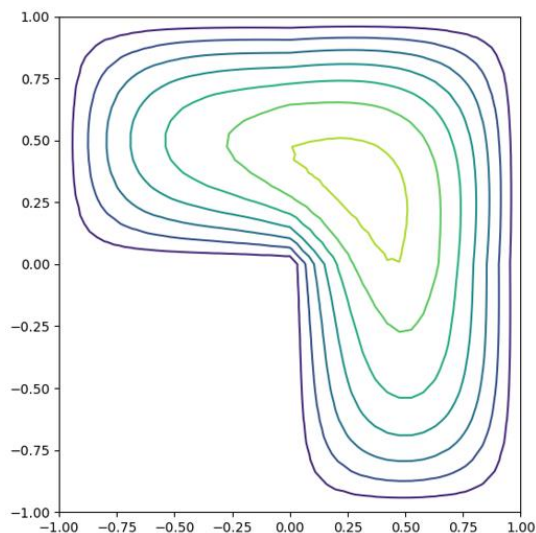

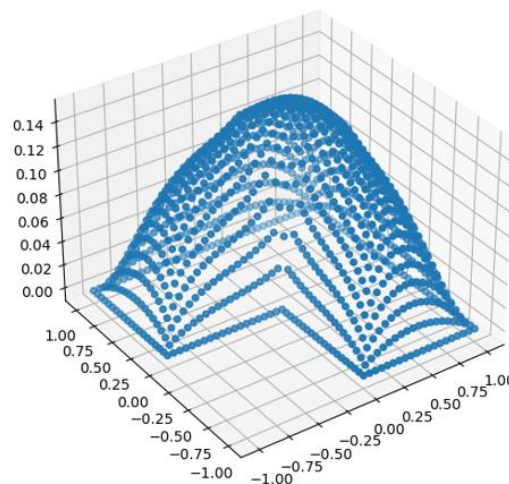Fig 5.2.2. Contour plot of fem solution

Fig. 5.2.3. Three-dimensional plot of fem solution.

4. CONCLUSION:

The technique of Iosparmetic formulation in partitioning the complex domain into minimum isoparametric elements is found to be very accurate and simple. As evidence that the approach has substantial potential for applications, we note that it has been fairly easy to partition the domain into isoparametric elements just by providing the coordinates of domain points. It is observed that we can generate many complex shapes using minimum elements. This approach can easily be extended into 3D geometry also.

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