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On Some Difference Sequence Classes of Interval Numbers

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Abstract: In this article our aim to introduce some difference sequence classes of interval numbers with associated sequence of real numbers and studies some topological and algebraic properties. Also we give some inclusion relations.

Keywords: Difference sequence, paranorm space, Young functions, Completeness.
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1. Introduction

The concept of interval arithmetic was first suggested by Dwyer [1] in 1951. After developed by Moore [11], Moore and Yang [14]. Furthermore several authors have studied various aspects of the theory and applications of interval numbers in differential equations [14], [15], [16]. The sequence of interval numbers was first introduced by Chiao [21] and defined usual convergence. Bounded and convergence sequences spaces of interval numbers were introduced by Sengonul and Eryilmaz [ ] and showed that these spaces are complete metric space.

A set consisting of closed interval of real numbers x such that $a \le x \le b$ is called an interval number. A real interval can also be considered as a set. Denote the set of all real valued closed intervals by $\mathbb{R}$. Any member of $\mathbb{R}$ is called closed interval and denoted by $\bar{x}$. Thus $\bar{x} = \{x \in \mathbb{R} : a \le x \le b\}$. In [20], an interval number is closed subset of real line $\mathbb{R}$.

Let x_l and x_r be the first and last points of the interval number $\bar{x}$ respectively. For $\bar{x}_1$, $\bar{x}_2 \in \mathbb{R}$, we have

$$\bar{x}_1 = \bar{x}_2 \Leftrightarrow x_{1_l} = x_{2_l}, x_{1r} = x_{2_r} .$$

$$\bar{x}_1 + \bar{x}_2 = \{x \in \mathbb{R} : x_{1_l} + x_{2_l} \le x \le x_{1r} + x_{2_r}\}$$

$$\alpha \bar{x} = \{x \in \mathbb{R} : \alpha x_{1_l} \le x \le \alpha x_{1_r}\} \text{ if } \alpha \ge 0.$$

$$= \{x \in \mathbb{R} : \alpha x_{1_r} \le x \le \alpha x_{1_l}\} \text{ if } \alpha < 0.$$

and

$$\bar{x}_1 . \bar{x}_2 = \{x \in \mathbb{R} : \min(x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r}) \le x \le \max(x_{1_l}.x_{2_l}, x_{1_l}.x_{2_r}, x_{1_r}.x_{2_l}, x_{1_r}.x_{2_r})\}$$

The set of all interval numbers $\mathbb{R}$ is complete metric space under the metric defined by –

$$d(\bar{x}, \bar{y}) = \max\{|x_{1_l} - x_{2_l}|, |x_{1_r} - x_{2_r}|\} \text{ (see [18])} .$$

Let us consider the transformation $f:\mathbb{N}\to\mathbb{R}$ by $k\to f(k)=\bar{x}$ where $\bar{x}=(\bar{x}_k)$ which is known as sequence of interval numbers. $\bar{x}_k$ denotes the k^{th} term of the sequence $\bar{x}=(\bar{x}_k)$. The set of all sequences of interval numbers is denoted by w^i can be found in [18].

2. Definitions and Main Results

Let $\phi=(\phi_n)_n$ be a sequence of Young functions i.e. $\phi_n:\mathbb{R}^+\to\mathbb{R}^+$ is an increasing and convex function such that $\phi_n(x)=0$ for $x>0$ and $\phi_n(0)=0$. The Musielak-Orlicz sequence space ℓ^ϕ is given by –

$$\ell^\phi=\left\{x=(x_n)_n:\sum_n\phi_n(\lambda|x_n|)<\infty,\lambda>0\right\}$$

.This becomes Banach space under the norm(Luxemburg)

$$|x|_\phi=\inf\left\{\eta>0:\sum_n\phi_n\left(\frac{|x_n|}{\eta}\right)\le 1,\eta>0\right\}$$

Let $\phi=(\phi_k)$ be the sequence of Young functions. The space consisting of all those sequences $\bar{x}=(\bar{x}_k)$ in w^i such that

$\phi\left(\frac{|\bar{x}_k|^{\frac{1}{k}}}{\eta}\right)\to 0$ as $k\to\infty$ for some $\eta>0$ is known as class of entire sequences of interval numbers defined by sequence of Young functions and is denoted by $\bar{\Gamma}_\phi$. The space consisting of all those sequences $\bar{x}=(\bar{x}_k)$ in w^i such that $\sup_k\left(\phi\left(\frac{|\bar{x}_k|^{\frac{1}{k}}}{\eta}\right)\right)<\infty$ for some $\eta>0$ is known as class of analytic sequences of interval numbers defined by sequence of Young functions and is denoted by $\bar{\Lambda}_\phi$.

3. Main Results

Let $\bar{x}=(\bar{x}_k)$ be sequence of interval numbers, p be positive integer, $A=(a_{nk})$ be non negative regular matrix and $\phi=(\phi_k)$ be a sequence of Young functions, we define the following classes of sequences of interval numbers as follows :

$$\bar{\Gamma}_\phi\left(A,p,\Delta^s_{(v,r)}\right)=\left\{\bar{x}=(\bar{x}_k):\lim_{k\to\infty}\sum_k a_{nk}\left[d\left(\phi\left(\frac{\left|\Delta^s_{(v,r)}\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}=0\right\}$$

$$\bar{\Lambda}_\phi(A,p,\Delta^s_{(v,r)})=\left\{\bar{x}=(\bar{x}_k):\sup_n\left(\sum_k a_{nk}\left[d\left(\phi\left(\frac{\left|\Delta^s_{(v,r)}\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}\right)<\infty\right\}$$

for some $\eta>0$. Where r and s be two non-negative integers and $v=(v_k)$ be a sequence of non-zero reals and $\left(\Delta^s_{(v,r)}x_k\right)=\left(\Delta^{s-1}_{(v,r)}x_k-\Delta^{s-1}_{(v,r)}x_{k+r}\right)$ and $\Delta^0_{(v,r)}x_k=v_kx_k$ for all $k\in N$, which is equivalent to the following binomial representation:

$$\Delta^s_{(v,r)}x_k=\sum_{i=0}^{s}(-1)^i\binom{s}{i}v_{k+ri}x_{k+ri}.$$

We can specialize these classes as follows:

(a) If $A=I$, the unit matrix then –

$$\overline{\Gamma}_{\phi}\left(I,p,\Delta^{s}_{(v,r)}\right)=\left\{\overline{x}=\left(\overline{x}_k\right):\lim_{k\to\infty}\left[d\left(\phi\left(\frac{\left|\Delta^{s}_{(v,r)}\overline{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}=0\right\}$$

$$\overline{\Lambda}_{\phi}\left(I,p,\Delta^{s}_{(v,r)}\right)=\left\{\overline{x}=\left(\overline{x}_k\right):\sup_{k}\left(\left[d\left(\phi\left(\frac{\left|\Delta^{s}_{(v,r)}\overline{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}\right)<\infty\right\}$$

(b) If we take $\phi(x)=x$ then we get –

$$\overline{\Gamma}\left(A,p,\Delta^{s}_{(v,r)}\right)=\left\{\overline{x}=\left(\overline{x}_k\right):\lim_{k\to\infty}\sum_{k}a_{nk}\left[d\left(\frac{\left|\Delta^{s}_{(v,r)}\overline{x}_k\right|^{1/k}}{\eta},0\right)\right]^{p_k}=0\right\}$$

$$\overline{\Lambda}\left(A,p,\Delta^{s}_{(v,r)}\right)=\left\{\overline{x}=\left(\overline{x}_k\right):\sup_{n}\left(\sum_{k}a_{nk}\left[d\left(\frac{\left|\Delta^{s}_{(v,r)}\overline{x}_k\right|^{1/k}}{\eta},0\right)\right]^{p_k}\right)<\infty\right\}$$

(c) If $A=\left(a_{nk}\right)$ is Cesaro matrix of order 1 and $p_k=p$ then we have -

$$\overline{\Gamma}_{\phi}\left(p,\Delta^{s}_{(v,r)}\right)=\left\{\overline{x}=\left(\overline{x}_k\right):\lim_{k\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left[d\left(\phi\left(\frac{\left|\Delta^{s}_{(v,r)}\overline{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p}\right\}=0$$

$$\overline{\Lambda}_{\phi}\left(p,\Delta^{s}_{(v,r)}\right)=\left\{\overline{x}=\left(\overline{x}_k\right):\sup_{n}\left(\frac{1}{n}\sum_{k=1}^{n}\left[d\left(\phi\left(\frac{\left|\Delta^{s}_{(v,r)}\overline{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p}\right)<\infty\right\}$$

The space $\overline{\Gamma}(\Delta^{s}_{(v,r)})$ is defined as follows ;

$$\overline{\Gamma}(\Delta^{s}_{(v,r)})=\left\{\overline{x}=\left(\overline{x}_k\right):\lim_{k\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\left|\Delta^{s}_{(v,r)}\overline{x}_k\right|^{1/k}}{\eta}=0\right\}\text{ for some }\eta>0.$$

Theorem 3.1: If d is translation invariant then the class of sequence $\overline{\Gamma}_{\phi}\left(p,\Delta^{s}_{(v,r)}\right)$ is closed under addition and scalar multiplication of interval numbers.

Proof: Let $\overline{x}=\left(\overline{x}_k\right)\in\overline{\Gamma}_{\phi}\left(p,\Delta^{s}_{(v,r)}\right)$ and $\overline{y}=\left(\overline{y}_k\right)\in\overline{\Gamma}_{\phi}\left(p,\Delta^{s}_{(v,r)}\right)$

In order to prove the result, we need to find some $\eta_3>0$ such that

$$\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\left|\Delta^{s}_{(v,r)}\left(a\overline{x}_k+b\overline{x}_k\right)\right|^{1/k}}{\eta_3},0\right)\right)\right]^{p}\to 0\quad\text{as}\quad k\to\infty$$

Since $\overline{x}=\left(\overline{x}_k\right)\in\overline{\Gamma}_{\phi}\left(p\right)$ and $\overline{y}=\left(\overline{y}_k\right)\in\overline{\Gamma}_{\phi}\left(p\right)$, there exists some $\eta_1>0$ and $\eta_2>0$ such that –

$$\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\left|\Delta^{s}_{(v,r)}\overline{x}_k\right|^{1/k}}{\eta_1},0\right)\right)\right]^{p}\to 0\quad\text{as}\quad k\to\infty\quad\text{and}$$

$$\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\left|\Delta_{(v,r)}^{s}\bar{y}_k\right|^{1/k}}{\eta_2},0\right)\right)\right]^p \to 0 \text{ as } k\to\infty.$$

Since ϕ is non-decreasing, we have

$$\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\left|\Delta_{(v,r)}^{s}\left(a\bar{x}_k+b\bar{y}_k\right)\right|^{1/k}}{\eta_3},0\right)\right)\right]^p \le \sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\left|\Delta_{(v,r)}^{s}\left(a\bar{x}_k\right)\right|^{1/k}}{\eta_3}+\frac{\left|\Delta_{(v,r)}^{s}\left(b\bar{y}_k\right)\right|^{1/k}}{\eta_3},0\right)\right)\right]^p$$

$$\le \sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{|a|^{1/k}\left|\Delta_{(v,r)}^{s}\bar{x}_k\right|^{1/k}}{\eta_3}+\frac{|b|^{1/k}\left|\Delta_{(v,r)}^{s}\bar{y}_k\right|^{1/k}}{\eta_3},0\right)\right)\right]^p$$

$$\le \sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{|a|\left|\Delta_{(v,r)}^{s}\bar{x}_k\right|^{1/k}}{\eta_3}+\frac{|b|\left|\Delta_{(v,r)}^{s}\bar{y}_k\right|^{1/k}}{\eta_3},0\right)\right)\right]^p$$

Take η_3 such that

$$\frac{1}{\eta_3}=\min\left\{\frac{1}{|a|^p}\frac{1}{\eta_1},\frac{1}{|b|^p}\frac{1}{\eta_2}\right\}$$

Then,

$$\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\left|\left(a\Delta_{(v,r)}^{s}\bar{x}_k+b\Delta_{(v,r)}^{s}\bar{y}_k\right)\right|^{1/k}}{\eta_3},0\right)\right)\right]^p \le \sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\left|\Delta_{(v,r)}^{s}\bar{x}_k\right|^{1/k}}{\eta_1}+\frac{\left|\Delta_{(v,r)}^{s}\bar{y}_k\right|^{1/k}}{\eta_2},0\right)\right)\right]^p$$

$$\le \sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\left|\Delta_{(v,r)}^{s}\bar{x}_k\right|^{1/k}}{\eta_1},0\right)\right)\right]^p+\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\left|\Delta_{(v,r)}^{s}\bar{y}_k\right|^{1/k}}{\eta_2},0\right)\right)\right]^p$$

Hence $$\sum_{k=1}^{n}\frac{1}{n}\left[d\left(\phi\left(\frac{\left|\left(a\Delta_{(v,r)}^{s}\bar{x}_k+b\Delta_{(v,r)}^{s}\bar{y}_k\right)\right|^{1/k}}{\eta_3},0\right)\right)\right]^p \to 0 \text{ as } k\to\infty.$$

This completes the proof.

Theorem 3.2. The class of sequence $\bar{\Gamma}_\phi\left(p,\Delta_{(v,r)}^{s}\right)$ is a complete metric space under the metric $'h'$ defined by –

$$h\left(\bar{x},\bar{y}\right)=\sup_n\left[\frac{1}{n}\sum_{k=1}^{n}d\left(\phi\left(\frac{\left|\Delta_{(v,r)}^{s}(\bar{x}_k-\bar{y}_k)\right|^{1/k}}{\eta},0\right)\right)\right]^p$$

Proof. Let $\left\{\bar{x}^{(i)}\right\}$ be Cauchy sequence in $\bar{\Gamma}_\phi\left(p,\Delta_{(v,r)}^{s}\right)$

Then for any given $\varepsilon>0$ there exists a positive integer n_1 such that

$$h\left(\bar{x}^{(i)},\bar{y}^{(j)}\right)<\varepsilon \qquad \text{for all } \quad i,j\ge n_1.$$

Therefore

$$\sup_n\left[\frac{1}{n}\sum_{k=1}^{n} d\left(\phi\left(\frac{\left|\Delta^s_{(v,r)}\bar{x}_k^{(i)}-\Delta^s_{(v,r)}\bar{y}_k^{(j)}\right|^{1/k}}{\eta},0\right)\right)\right]^p<\varepsilon$$ for all $i, j \ge n_1$. Consequently $\left\{\bar{x}_k^{(i)}\right\}$ is a Cauchy sequence in the metric space of interval numbers which is complete and so $\bar{x}_k^{(i)} \to \bar{x}_k$ as $i \to \infty$.

Once can find that -

$$\left[\frac{1}{n}\sum_{k=1}^{n} d\left(\phi\left(\frac{\left|\Delta^s_{(v,r)}\bar{x}_k^{(i)}-\Delta^s_{(v,r)}\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^p<\varepsilon,\quad i\ge n_1.$$

Now

$$\left[\frac{1}{n}\sum_{k=1}^{n} d\left(\phi\left(\frac{\left|\Delta^s_{(v,r)}\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^p \le \left[\frac{1}{n}\sum_{k=1}^{n} d\left(\phi\left(\frac{\left|\Delta^s_{(v,r)}\bar{x}_k-\Delta^s_{(v,r)}\bar{x}_k^{(n_1)}\right|^{1/k}}{\eta},0\right)\right)\right]^p$$

$$+\left[\frac{1}{n}\sum_{k=1}^{n} d\left(\phi\left(\frac{\left|\Delta^s_{(v,r)}\bar{x}_k^{n_1}\right|^{1/k}}{\eta},0\right)\right)\right]^p$$

$<\varepsilon+0$ as $n\to\infty$.

Thus $$\left[\frac{1}{n}\sum_{k=1}^{n} d\left(\phi\left(\frac{\left|\Delta^s_{(v,r)}\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^p<\varepsilon$$

and so $\left(\bar{x}_k\right)\in\bar{\Gamma}_\phi\left(p,\Delta^s_{(v,r)}\right)$.

Hence $\bar{\Gamma}_\phi\left(p,\Delta^s_{(v,r)}\right)$ is a complete metric space. This completes the proof.

Theorem 3.3. Let $\bar{x}=\left(\bar{x}_k\right)$ be sequence of interval numbers. The sequence class $\bar{\Gamma}_\phi\left(A, p,\Delta^s_{(v,r)}\right)$ is complete w.r.t the topology generated by the paranorm h defined by –

$$h\left(\bar{x}\right)=\sup_k\left(\sum_{k=1}^{n} a_{nk}\left[d\left(\phi\left(\frac{\left|\Delta^s_{(v,r)}\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}}$$

Where $M=\max\left\{1,\sup_k\left(\frac{p_k}{M}\right)\right\}$.

Proof. Obviously $h(\theta)=0$ and $h\left(-\bar{x}\right)=h\left(\bar{x}\right)$. It can also be easily seen that $h\left(\bar{x}+\bar{y}\right)\le h\left(\bar{x}\right)+h\left(\bar{y}\right)$ as d is translation invariant.

Now for any scalar λ, we have $|\lambda|^{p_k/M}<\max\left(1,\sup|\lambda|\right)$, so that $h\left(\lambda\bar{x}\right)<\max\left(1,\sup|\lambda|\right)$, λ fixed implies $\lambda\bar{x}\to\theta$. Now let $\lambda\to\theta$, $\bar{x}$ fixed for $\sup|\lambda|<1$, we have

$$\left(\sum_{k=1}^{n} a_{nk}\left[d\left(\phi\left(\frac{\left|\Delta_{(v,r)}^{s}\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}} < \varepsilon \text{ for some } N > N(\varepsilon).$$

Also for $1 \le n \le N$ and $\left(\sum_{k=1}^{n} a_{nk}\left[d\left(\phi\left(\frac{\left|\Delta_{(v,r)}^{s}\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}} < \varepsilon$ there exists m such that

$$\left(\sum_{k=m}^{n} a_{nk}\left[d\left(\phi\left(\frac{\left|\lambda\Delta_{(v,r)}^{s}\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}} < \varepsilon.$$

Taking λ small enough, we then find

$$\left(\sum_{k=m}^{n} a_{nk}\left[d\left(\phi\left(\frac{\left|\lambda\Delta_{(v,r)}^{s}\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}} < 2\varepsilon \text{ for all } k.$$

Hence $h(\lambda\bar{x}) \to 0$ as $\lambda \to 0$. So h is a paranorm on $\bar{\Gamma}_\phi(A,p)$.

To show the completeness, let $\left\{\Delta_{(v,r)}^{s}\bar{x}^{(i)}\right\}$ be Cauchy sequence in $\bar{\Gamma}_\phi\left(A,p,\Delta_{(v,r)}^{s}\right)$.

Then for given $\varepsilon > 0$ there exists positive integer r such that –

$$\left(\sum a_{nk}\left[d\left(\phi\left(\frac{\left|\Delta_{(v,r)}^{s}\bar{x}_k^{i} - \Delta_{(v,r)}^{s}\bar{x}_k^{j}\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}} < \varepsilon \text{ for all } j \to \infty\ i,j \ge r.$$

Since d is translation invariant, so

$$\left(\sum a_{nk}\left[d\left(\phi\left(\frac{\left|\Delta_{(v,r)}^{s}\bar{x}_k^{i} - \Delta_{(v,r)}^{s}\bar{x}_k^{j}\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}\right)^{\frac{1}{M}} < \varepsilon \text{ for all } i,j \ge r. \text{ and each } n.$$

Hence

$$\left[d\left(\phi\left(\frac{\left|\Delta_{(v,r)}^{s}\bar{x}_k^{i} - \Delta_{(v,r)}^{s}\bar{x}_k^{j}\right|^{1/k}}{\eta},0\right)\right)\right] < \varepsilon \text{ for all } i,j \ge r.$$

Therefore $\{\Delta_{(v,r)}^{s}\bar{x}_k^{i}\}$ is a Cauchy sequence, consequently $\left\{\bar{x}^{(i)}\right\}$ is a Cauchy sequence in the metric space of interval numbers which is complete and hence $\bar{x}^{(j)} \to \bar{x}$ as $j \to \infty$

Keeping $r_0 \ge r$ and letting $j \to \infty$, once can find that –

$$\left(\sum a_{nk}\left[d\left(\phi\left(\frac{\left|\Delta_{(v,r)}^{s}\bar{x}_k^{i} - \Delta_{(v,r)}^{s}\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]\right) < \varepsilon \text{ for all } r_0 \ge r.$$

Since d is translation invariant, therefore

$$\left(\sum a_{nk}\left[d\left(\phi\left(\frac{\left|\Delta^s_{(v,r)}\bar{x}^i_k-\Delta^s_{(v,r)}\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}\right)^{1/M}<\varepsilon$$

i.e $\bar{x}^{(i)}\to\bar{x}$ in $\bar{\Gamma}_\phi(A,p)$. It can be easily seen that $\bar{x}\in\bar{\Gamma}_\phi\left(A,p,\Delta^s_{(v,r)}\right)$.

Thus $\bar{\Gamma}_\phi\left(A,p,\Delta^s_{(v,r)}\right)$ is complete. This completes the proof.

Theorem 3.4. If $0<\inf p_k\le p_k\le 1$, then $\bar{\Gamma}_\phi\left(A,p,\Delta^s_{(v,r)}\right)\subset\bar{\Gamma}_\phi\left(A,\Delta^s_{(v,r)}\right)$.

Proof. Let $\bar{x}=(\bar{x}_k)\in\bar{\Gamma}_\phi\left(A,p,\Delta^s_{(v,r)}\right)$. Since $0<\inf p_k\le p_k\le 1$, the result follows from the following inequality

$$\sum_k a_{nk}\left[d\left(\phi\left(\frac{\left|\Delta^s_{(v,r)}\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]\le\sum_k a_{nk}\left[d\left(\phi\left(\frac{\left|\Delta^s_{(v,r)}\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}$$

Theorem 3.5. If $1\le p_k\le\sup p_k<\infty$, then $\bar{\Gamma}_\phi\left(A,\Delta^s_{(v,r)}\right)\subset\bar{\Gamma}_\phi\left(A,p,\Delta^s_{(v,r)}\right)$..

Proof. $\bar{x}=(\bar{x}_k)\in\bar{\Gamma}_\phi\left(A,\Delta^s_{(v,r)}\right)$. Since $1\le p_k\le\sup p_k<\infty$ then for each $0<\varepsilon<1$ there exist a positive integer n_0 such that

$$\sum_k a_{nk}\left[d\left(\phi\left(\frac{\left|\Delta^s_{(v,r)}\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]\le\varepsilon<1 \text{ for some } n\ge n_0.$$

The result follows from the following inequality

$$\sum_k a_{nk}\left[d\left(\phi\left(\frac{\left|\Delta^s_{(v,r)}\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right]^{p_k}\le\sum_k a_{nk}\left[d\left(\phi\left(\frac{\left|\Delta^s_{(v,r)}\bar{x}_k\right|^{1/k}}{\eta},0\right)\right)\right].$$

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