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Fixed Point Theorems in Generalized weakly contractive maps

Dr. Preeti Sengar

SGSITS Indore (M.P.)

Abstract: In this paper, we prove a fixed point theorem for mappings satisfying an implicit relation in a complete fuzzy Metric Space.

Keywords: Complete fuzzy metric space, weakly compatible, generalized weakly compatible

1. Introduction: The concept of a fuzzy set was introduced by Zadeh [4] in 1965.This concept was used in topology and analysis by many authors. Sessa (1986) extended the concept of commuting mapping of Jungck (1976) by introducing the concept of weakly commuting mapping. Pant (1994) modified of concept of weakly commuting mappings and introduced the concept of R-weakly commuting mappings.

Amari and Moutawakil (2002) defined the (E. A) Property for self-mappings for both non compatible as well as compatible mappings. Mihet (2010) defined the (E. A) Property in fuzzy metric space. Common (E. A) Property is introduced by Ali et al. (2010) which relaxes the continuity of mappings and replaces the completeness condition of the space with closeness of range which is more wide condition. It is to be noted that (E. A) Property requires either completeness of the whole space or any of the range space or continuity of mappings.

In this chapter, we first establish some common fixed point theorems in fuzzy metric spaces for the sequence of self-mappings using an implicit relation and the property (E. A).

Definition 1[4]: A binary operation $*:[0,1]^2 \to [0,1]$ is called a continuous t-norm if $([0,1],*)$ is an abelian topological monoid, that is,

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $p * 1 = p$ for all $p \in [0,1]$,
(4) $p * q \le r * s$ whenever $p \le r$ and $q \le s$, for each $p, q, r, s \in [0,1]$.

Four typical examples of a continuous t-norms are $p *_1 q = \min\{p, q\}$, $p *_2 q = pq/\max\{p, q, \lambda\}$ for $0 < \lambda < 1$ and $p *_3 q = pq, p *_4 q = \max\{p + q - 1, 0\}$.

Definition 2[4]: A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy sets on $X^2 \times [0,\infty)$ satisfying following conditions, for each $x, y, z \in X$ and $s, t > 0$:

(i) $M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1$ if and only if $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,

(v) $M(x, y, \cdot) : [0,\infty) \to [0, 1]$ is left continuous;

Definition 3[2] : Let A and S be the mapping from a metric space X into itself, then the mapping is said to weakly compatible if they are commute at their coincidence points, that is,

$Ax = Sx$ implies that $ASx = SAx$.

Definition 4[3]: A self-map $T: X \to X$ is said to be generalized weakly contractive map if there exists a $\varphi \in \emptyset$ such that $M(Tx, Ty, t) \leq M(x, y, t) - \varphi(M(x, y, t))$ with $\lim_{t\to\infty} \varphi(t) = 0$ for all $x, y \in X$.

We denote $R^+ = [0, \infty)$ is positive real number, N the set of natural number and R the set of real number. We write $\emptyset = \{\varphi: R^+ \to R^+\}$ where φ satisfies following conditions:

- φ is continuous
- φ is non decreasing
- $\varphi(t) > 0 \; for \; t > 0$
- $\varphi(t) = 0$

Definition 5[3]: Two self-mappings A and B of a fuzzy metric space (X, M, $*$) are said to be R-weakly commuting if there exist a positive real number R such that,

$$M(ABx, BAx, t) \geq M\left(Ax, Bx, \frac{t}{R}\right), for \; all \; x \in X \; and \; t > 0.$$

Theorem 2.1.: Let (X, M, $*$) be a T-orbit ally complete fuzzy metric space, if A,B,S,T be self-mapping of X into itself such that

(i) $A(X) \subseteq T(X) \; and \; B(X) \subseteq S(X), T(X) \; or \; S(X)$ are closed subset of X.

(ii) The pair (A, S) and (B, T) are R-weakly commuting and generalized weakly contractive mappings.

(iii) For all $x, y \in \overline{O(x_o)}$ and $k \in [0,1)$.we define

$$M(Ax, By, kt) \leq k \max\left\{\frac{M(Ax, Sx, t). M(By, Ty, t)}{1 + M(Sx, Ty, t)}, \frac{M(Sx, By, t). M(Ax, Ty, t)}{1 + M(Sx, Ty, t)}, M(Sx, Ty, t)\right\}$$

For all $x, y \in X \; and \; t > 0$, then A, B, S and T have unique fixed point in $\overline{O(x_o)}$.

Proof: we suppose that $x_o \in X$ arbitrary and we choose a point $x \in X$ such that

$$y_o = Ax_o = Tx_1 \; and \; y_1 = Bx_1 = Sx_2$$

In general there exists a sequence,

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \; and \; y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$$

For n=1, 2, 3,…

First we claim that the sequence $\{y_n\}$ is a Cauchy sequence.

For this from (iii) we have

$$M(y_{2n}, y_{2n+1}, t) \leq k. M(Ax_{2n}, Bx_{2n+1}, t) - \psi(M(Ax_{2n}, Bx_{2n+1}, t)$$

$$M(y_{2n}, y_{2n+1}, t) \leq k \max\left\{\begin{array}{c}\frac{M(Ax_{2n}, Sx_{2n}, t). M(Bx_{2n+1}, Tx_{2n+1}, t)}{1 + M(Sx_{2n}, Tx_{2n+1}, t)}, \\ \frac{M(Sx_{2n}, Bx_{2n+1}, t). M(Ax_{2n}, Tx_{2n+1}, t)}{1 + M(Sx_{2n}, Tx_{2n+1}, t)}, M(Sx_{2n}, Tx_{2n+1}, t)\end{array}\right\}$$

$$M(y_{2n},y_{2n+1},t) \leq k \max\left\{\begin{array}{c}\frac{M(y_{2n},y_{2n-1},t).M(y_{2n+1},y_{2n},t)}{1+M(y_{2n-1},y_{2n},t)}, \\ \frac{M(y_{2n-1},y_{2n+1},t).M(y_{2n},y_{2n},t)}{1+M(y_{2n-1},y_{2n},t)}, M(y_{2n-1},y_{2n},t)\end{array}\right\}$$

$$\leq k \max\{M(y_{2n+1},y_{2n},t),0,M(y_{2n-1},y_{2n},t)\}$$

There arise three cases:

Case 1: If we take

$\max\{M(y_{2n+1},y_{2n},t),0,M(y_{2n-1},y_{2n},t)\} = M(y_{2n-1},y_{2n},t)$

then we have $M(y_{2n},y_{2n+1},t) \leq k.M(y_{2n-1},y_{2n},t)$

Case 2: If we take

$\max\{M(y_{2n+1},y_{2n},t),0,M(y_{2n-1},y_{2n},t)\} = M(y_{2n+1},y_{2n},t)$

Then we have $M(y_{2n},y_{2n+1},t) \leq k.M(y_{2n+1},y_{2n},t)$

Which contradiction.

Case 3: If we take

$\max\{M(y_{2n+1},y_{2n},t),0,M(y_{2n-1},y_{2n},t)\} = 0$

Then we have $M(y_{2n},y_{2n+1},t) \leq 0$

Which contradiction.

From the above all three cases we have

$$M(y_{2n},y_{2n+1},t) \leq k.M(y_{2n-1},y_{2n},t)$$

Processing the same way we have

$$M(y_{2n},y_{2n+1},t) \leq k^{2n}.M(y_0,y_1,t)$$

Or

$$M(y_n,y_{n+1},t) \leq k^n.M(y_0,y_1,t)$$

For any $m > n$, we have

$$M(y_n,y_m,t) \leq M(y_n,y_{n+1},t) + M(y_{n+1},y_{n+2},t) + \cdots + M(y_{m-1},y_m,t)$$

$$M(y_n,y_m,t) \leq (k^n + k^{n+1} + \cdots + k^{m-1}).M(y_0,y_1,t)$$

$$M(y_n,y_m,t) \leq \frac{k}{1-k}.M(y_0,y_1,t)$$

As $n \to \infty$, it follows that $\{y_n\}$ is a Cauchy sequence and by the completeness of X, $\{y_n\}$ converges to $y \in X$. That is we can write;

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = y.$$

Let $T(X)$ is closed subset of X such that $Tv = y$.

We prove that $Bv = y$ for this again from (iii)

$$M(Ax_{2n}, Bv, t) \leq k \max \left\{ \begin{matrix} \dfrac{M(Ax_{2n}, Sx_{2n}, t). M(Bv, Tv, t)}{1 + M(Sx_{2n}, Tv, t)}, \\ \dfrac{M(Sx_{2n}, Bv, t). M(Ax_{2n}, Tv, t)}{1 + M(Sx_{2n}, Tv, t)}, M(Sx_{2n}, Tv, t) \end{matrix} \right\}$$

$$M(y, Bv, t) \leq k \max\{M(Bv, y, t), M(y, Bv, t), 0\}$$

$$M(y, Bv, t) \leq kM(y, Bv, t)$$

Which contradiction,

Hence $Bv = y = Tv$ and that $BTv = TBv$ implies that $By = Ty$.

Now we proof that $By = y$ for this again from (iii)

$$M(Ax_{2n}, By, t) \leq k \max \left\{ \begin{matrix} \dfrac{M(Sx_{2n}, By, t). M(Ax_{2n}, Ty, t)}{1 + M(Sx_{2n}, Ty, t)}, \\ \dfrac{M(Ax_{2n}, Sx_{2n}, t). M(By, Ty, t)}{1 + M(Sx_{2n}, Ty, t)}, M(Sx_{2n}, Ty, t) \end{matrix} \right\}$$

$$\lim_{n\to\infty} M(Ax_{2n}, By, t) \leq kM(y, By, t)$$

$$By = y = Ty.$$

Since $B(X) \subseteq S(X)$

For, $w \in X$ such that $Sw = y$.

Now we show that $Aw = y$

$$M(Aw, By, t) \leq k \max \left\{ \begin{matrix} \dfrac{M(Aw, Sw, t). M(By, Ty, t)}{1 + M(Sw, Ty, t)}, \\ \dfrac{M(Sw, By, t). M(Aw, Ty, t)}{1 + M(Sw, Ty, t)}, M(Sw, Ty, t) \end{matrix} \right\}$$

It follows that,

$$M(Aw, y, t) \leq kM(Aw, y, t)$$

Which contradiction $M(Aw, y, t) > 0$ thus

$$Aw = y = Sw.$$

Since A and S are R-weakly compatible, so that

$$ASw = SAw \text{ this implies}, Ay = Sy.$$

Now we show that, $Ay = y$ for $Ay = y$ for this again from (iii),

$$M(Aw, By, t) \leq k \max \left\{ \begin{matrix} \dfrac{M(Ay, Sy, t). M(By, Ty, t)}{1 + M(Sy, Ty, t)}, \\ \dfrac{M(Sy, By, t). M(Ay, Ty, t)}{1 + M(Sy, Ty, t)}, M(Sy, Ty, t) \end{matrix} \right\}$$

It follows that,

$$M(Ay, y, t) \leq kM(Ay, y, t)$$

Which contradiction thus $Ay = y$ and then, we write

$$Ay = Sy = By = Ty = y$$

Uniqueness: we suppose that x is another fixed point for A, B, S, T then by using (iii) then we have

$$M(x, y, kt) \leq kM(x, y, t)$$

Which contradiction, so that $x = y$ and y is unique fixed point of A, B, S, T.

If we omit the completeness of the space then we get following corollary.

Corollary 2.2: Let (X, M, $*$) be a T-orbit ally complete fuzzy metric space, if A,B,S,T be self-mapping of X into itself such that

(i) $A(X) \subseteq T(X)$ *and* $B(X) \subseteq S(X), T(X)$ *or* $S(X)$ are closed subset of X.

(ii) The pair (A, S) and (B, T) are weakly compatible and generalized weakly contractive map.

(iii) For all $x, y \in \overline{O(x_o)}$ and $k \in [0,1)$.we define,
$M(Ax, By, kt) \leq k.M(x, y, t) - \psi(M(x, y, t)$

where, $M(Ax, By, kt) \leq$
$k \max\left\{\frac{M^2(Ax,Sx,t)+M^2(By,Ty,t)}{1+M(Sx,Ty,t)}, \frac{M^2(Sx,By,t)+M^2(Ax,Ty,t)}{1+M(Sx,Ty,t)}, \frac{M(Ax,Sx,t).M(By,Ty,t)}{1+M(Sx,Ty,t)}, \frac{M(Sx,By,t).M(Ax,Ty,t)}{1+M(Sx,Ty,t)} M(Sx, Ty, t)\right\}$

For all $x, y \in X$ *and* $t > 0$, then A, B, S and T have unique fixed point in $\overline{O(x_o)}$

Corollary 2.3.: Let (X, M, $*$) be a T-orbitally complete fuzzy metric space, if A,B be self-mapping of X into itself such that

(i) $A(X) \subseteq X$ *and* $B(X) \subseteq X,$

(ii) The pair (A, B) weakly compatible and generalized weakly contractive map.

(iii) For all $x, y \in \overline{O(x_o)}$ and $k \in [0,1)$.we define,
$M(Ax, By, kt) \leq k.M(x, y, t) - \psi(M(x, y, t)$

where, $M(Ax, By, kt) \leq$
$k \max\left\{\frac{M^2(Ax,x,t)+M^2(By,y,t)}{1+M(x,y,t)}, \frac{M^2(x,By,t)+M^2(Ax,y,t)}{1+M(x,y,t)}, \frac{M(Ax,x,t).M(By,y,t)}{1+M(x,y,t)}, \frac{M(x,By,t).M(Ax,y,t)}{1+M(x,y,t)} M(x, y, t)\right\}$

then A, B have unique fixed point in $\overline{O(x_o)}$.

Proof. It is enough if we take $S = T = I$(identity mapping) in Theorem 2.1 then we get the result.

Corollary 2.4: Let (X, M, $*$) be a T-orbitally complete fuzzy metric space, if A,B,S,T be self-mapping of X into itself such that

(i) $A(X) \subseteq T(X)$ *and* $B(X) \subseteq S(X), T(X)$ *or* $S(X)$ are closed subset of X.

(ii) The pair (A, S) and (B, T) are R-weakly commuting and generalized weakly contractive mappings.

(iii) For all $x, y \in \overline{O(x_o)}$ and $k \in [0,1)$.we define

$$M(Ax, By, kt) \leq k \max\left\{\begin{array}{c}\frac{M^2(Ax, Sx, t) + M^2(By, Ty, t)}{1 + M(Sx, Ty, t)}, \frac{M^2(Sx, By, t) + M^2(Ax, Ty, t)}{1 + M(Sx, Ty, t)}, \\ \frac{M(Ax, Sx, t).M(By, Ty, t)}{1 + M(Sx, Ty, t)}, \frac{M(Sx, By, t).M(Ax, Ty, t)}{1 + M(Sx, Ty, t)}, M(Sx, Ty, t)\end{array}\right\}$$

For all $x, y \in X$ *and* $t > 0$, then A, B, S and T have unique fixed point in $\overline{O(x_o)}$.

Proof: It is immediate to see that if we take $\psi(t) = 0$ in Theorem 2.1, then we get the result.

Conclusion: in this paper, we prove some common fixed point theorems for two pairs of R-weakly commuting maps in fuzzy metric space using the E.A. property. The existence and uniqueness of solution for certain system of functional equations arising in dynamic programming are also presented as an application.

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