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Some More Results On $^{\delta}$R-Norm Information Measure and Conditional $^{\delta}$R-Norm Information Measure in Quantum Logics

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Abstract: In the present communication, I deal with the mathematical modelling of $^{\delta}$R-Norm Information Measure ($^{\delta}$RIM) in quantum logics. Some results are extended concerning $^{\delta}$RIM, Joint $^{\delta}$RIM and Conditional $^{\delta}$RIM to the quantum logics. I studied the concepts of $^{\delta}$RIM, Joint $^{\delta}$RIM and Conditional $^{\delta}$RIM in quantum logics. I proved the Concavity property for $^{\delta}$RIM in quantum logics. I showed that $^{\delta}$RIM does not satisfy the property of additivity. I proved the monotonicity property for conditional $^{\delta}$RIM.

***Keywords* – R-Norm Information measure, Conditional R-Norm Information measure, Quantum Logic.**

I. INTRODUCTION

Birkhoff and Neumann introduce the approach of quantum logic. With the help of information measure, the uncertainty in random events can be measured. The concept of information measure has been applied in many areas such as in physics, information theory, computer science, general systems theory, sociology, statistics, biology, chemistry and many other fields. There are several information measures presented in history such as Shannon's information measure and the measures of order δ and of type β, introduced by Renyi [1], Havrda-Charvat [2] and Daroczy [3] respectively. There are many applications of them in Statistics, Pattern recognition and Coding theory.

We define $\triangle_n$ as the set of all n-ary probability distributions

$$\Delta_n = \left\{P = (\Upsilon_1, \Upsilon_2, \Upsilon_3, \dots, \Upsilon_n); \Upsilon_i \geq 0; i = 1,2, \dots n; \sum_{i=1}^{n} \Upsilon_i = 1\right\}$$

Information measure given by Shannon is

$$H(P) = -\sum_{i=1}^{n} \Upsilon_i \log \Upsilon_i \tag{1}$$

where D>0.

The RIM of the distribution P is given by Boekke and Lubbe [4] and is defined for $R \in \Re$ by

$$H_R(P) = \frac{R}{R-1}\left[1 - \left[\sum_{i=1}^{n} \Upsilon_i^R\right]^{\frac{1}{R}}\right] \tag{2}$$

Where $\Re = \{R: R > 0, R \neq 1\}$ and the R-Norm information measure (RIM) is a real function from Δ_n to $\Re$ defined on Δ_n, $(n \geq 2)$ R is real numbers set (see [4]). When $R \to 1$, RIM approaches to Shannon's information measure (see [5]) and when $R \to \infty, H_R(P) \to (1 - \max \Upsilon_i); i = 1,2, \dots, n$. Satish Kumar [6] studied $^{\delta}$RIM which is given by

$$H_R^{\delta} = \frac{R}{R-\delta}\left[1 - \left(\sum_{i=1}^{n} \Upsilon_i^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right]; \delta \neq R, R \in \Re. \tag{3}$$

In [7], [8] and [9] the concepts of information measure, joint information measure and conditional information measure of partitions and dynamical systems have been studied in quantum logics. We can study the quantum information theory in an easy way by using the concepts of information measures in quantum logics (see [10]). In this paper, we study the notion of $^{\delta}$RIM, joint $^{\delta}$RIM and conditional $^{\delta}$RIM in quantum logics. Some of the results of the information measures are generalized to the quantum logics.

Whole paper is divided into 4 sections. In section 2, some facts and some basic concepts are provided that will be used in other sections in this paper. In section 3, I define a partition and then corresponding $^{\delta}$RIM in quantum logics with respect to a state ζ. Concavity property of $^{\delta}$RIM in quantum logics is proved. Additivity property is also studied. In section 4, We define joint and conditional $^{\delta}$RIM of partitions in quantum logics. Monotonicity property for this conditional $^{\delta}$RIM is also studied. We study $^{\delta}$RIM, joint and conditional $^{\delta}$RIM of ζ-independent partitions.

II. Some definitions

Definition 1.[8] *Consider a quantum logic* QL. *This is a σ-orthomodular lattice, i.e., a lattice $\Omega(\Omega, R, \vee, \wedge, 0, 1)$ where 0 is the minimal element and 1 is the maximal element. We can define an operation ‘: $\Omega \to \Omega$ such that the following properties hold $\forall u, v \in \Omega$:*
(i) $(u')' = u$.
(ii) $uRv \Rightarrow v'Ru'$.
(iii) Given any finite sequence $< u_i >, u_i R u_j'$, then $V_{i \in \mathbf{N}} u_i$ lies in Ω.
(iv) Ω is orthomodular: $uRv \Rightarrow v = u \vee (v \wedge u')$.
In quantum logics we have $u \wedge u' = 0$ and $u \vee u' = 1$.

Definition 2. [8] *Consider a quantum logic Ω. A state ζ is a mapping $\Omega \to [0,1]$ such that:*
(i) $\zeta(1) = 1$.
(ii) $\zeta(u \vee v) = \zeta(u) + \zeta(v)$ *if* $u \perp v$.
It may be observed that $\zeta(0) = 0, \zeta$ is monotone and $\zeta(u') = 1 - \zeta(u), u \in \Omega$.

Definition 3. [8] *Consider $u_1, \dots, u_n$ are the finite elements of a quantum logic. $P = \{u_1, \dots, u_n\}$ is $\vee$-orthogonal iff $\vee_{i=1}^{k} u_i \perp u_{k+1}$, $\forall k = 1,2, \dots, n-1$.*

Definition 4. [8] *Consider Ω is a quantum logic. $P = \{u_1, \dots, u_n\} \subset \Omega$ is called a partition of Ω w.r.t. a state ζ on Ω then*
(i) P is $\vee$-orthogonal.
(ii) $\zeta(\vee_{i=1}^{n} u_i) = 1$.
Remarks 1. *From the above definitions, we can find that $\sum_{i=1}^{n} \zeta(u_i) = 1$.*

Definition 5. [11] *Let (Ω, ζ) be any couple in quantum logics and $P = \{u_1, u_2, \dots, u_n\}$ and $Q = \{v_1, v_2, \dots, v_m\}$ be any two partitions of it. Q is said to be a ζ-refinement of P if $\exists$ a partition $I(1), \dots, I(n)$ of the set $\{1, \dots, m\}$ such that $u_i = \vee_{j \in I(i)} v_j$, $\forall\, i = 1, \dots, n$.*

Definition 6. [8] *Consider a partition Q=$\{v_1, \dots, v_m\}$ of a couple (Ω, ζ) and $u \in \Omega$. The state ζ satisfy Bayes' property if*

$$\zeta\left(\bigvee_{j=1}^{m} (u \wedge v_j)\right) = \zeta(u).$$

In this case, we get

$$\sum_{j=1}^{m} \zeta(u \wedge v_j) = \zeta(u).$$

Let us consider $P = \{u_1, u_2, \dots, u_n\}$ and $Q = \{v_1, \dots, v_m\}$ are the two partitions of (Ω, ζ). Now, $P \vee Q$ is a partition of (Ω, ζ) if ζ has Bayes's property. Then the refinement of this partition is given by

$$P \vee Q = \{u_i \wedge v_j | u_i \in P, v_j \in Q\}.$$

Definition 7. *Two partitions P and Q are said to be ζ-independent if $\zeta(u \wedge v) = \zeta(u)\zeta(v)$ for $u \in P$ and $v \in Q$.*
Remarks 2. [12] *The well-known Minkowski inequality is given by*

Case1. *If* $m > 1$,

$$\left(\sum_{i=1}^{k} x_i^m\right)^{\frac{1}{m}} + \left(\sum_{i=1}^{k} y_i^m\right)^{\frac{1}{m}} \geq \left(\sum_{i=1}^{k} (x_i + y_i)^m\right)^{\frac{1}{m}}$$

Case2. *If* $0 < m < 1$,

$$\left(\sum_{i=1}^{k} x_i^m\right)^{\frac{1}{m}} + \left(\sum_{i=1}^{k} y_i^m\right)^{\frac{1}{m}} \leq \left(\sum_{i=1}^{k} (x_i + y_i)^m\right)^{\frac{1}{m}}$$

where x_i, y_j are non negative numbers.

Remarks 3. [12] *Also, the well-known Jensen inequality is given by*

Case1. *If ϕ is a real convex function,*

$$\phi\left(\sum_{i=1}^{n} u_i x_i\right) \leq \sum_{i=1}^{n} u_i \phi(x_i)$$

Case2. *If ϕ is a real concave function,*

$$\boldsymbol{\phi}\left(\sum_{i=1}^{n} \boldsymbol{u_i x_i}\right) \geq \sum_{i=1}^{n} \boldsymbol{u_i \phi(x_i)}$$

where x_i's are real numbers and u_i's are non negative real numbers with condition $\sum_{i=1}^{n} u_i = 1$.

III. $^{\delta}$RIM IN QUANTUM LOGICS

In this section, we define $^{\delta}$RIM for a finite partition on a quantum logic. Some of the properties of suggested measure are proved on Quantum logic. An example is also given to give prove of obtained results.

Definition 8. *Let us consider a partition $P = \{u_1, u_2, \dots, u_n\}$ of a couple (Ω, ζ). We can define the $^{\delta}$RIM w.r.t. state ζ as:*

$$H_{R,\delta}^{\zeta}(P) = \frac{R}{R-\delta}\left(1 - \left(\sum_{i=1}^{n}(\zeta(u_i))^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right) \tag{4}$$

for $R \in (0, \delta) \cup (\delta, +\infty)$.

Theorem 1. *For a partition $P = \{u_1, u_2, \dots, u_n\}$ of a couple (Ω, ζ), $H_{R,\delta}^{\zeta}(P) \geq 0$.*

Proof. Case1. If $0 < R < \delta$,
then we have

$$(\zeta(u_i))^{\frac{R}{\delta}} \geq \zeta(u_i), i = 1,2, \dots, n.$$

therefore,

$$\sum_{i=1}^{n}(\zeta(u_i))^{\frac{R}{\delta}} \geq \sum_{i=1}^{n}\zeta(u_i) = 1$$

it follows that

$$\left(\sum_{i=1}^{n}(\zeta(u_i))^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}} \geq 1$$

Also, for $0 < R < \delta$, we have $\frac{R}{R-\delta} < 0$. So, from eq. (4), we have $H_{R,\delta}^{\zeta}(P) \geq 0$.

Case2. if $R > \delta$,
then we have

$$(\zeta(u_i))^{\frac{R}{\delta}} \leq \zeta(u_i), i = 1,2, \dots, n.$$

therefore,

$$\sum_{i=1}^{n}(\zeta(u_i))^{\frac{R}{\delta}} \leq \sum_{i=1}^{n}\zeta(u_i) = 1$$

it follows that

$$\left(\sum_{i=1}^{n}(\zeta(u_i))^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}} \leq 1$$

Also, for $R > \delta$, we have $\frac{R}{R-\delta} > 0$. So, from eq. (4), we have $H_{R,\delta}^{\zeta}(P) \geq 0$.

Theorem 2. *Let us consider a family of all the states denoted by K, on quantum logics. Let $\zeta,\eta \in K$ and P is a partition of (Ω, ζ),(Ω, η). Then for $\lambda \in [0,1]$, we have*

$$\lambda H_{R,\delta}^{\zeta}(P) + (1-\lambda)H_{R,\delta}^{\eta}(P) \leq H_{R,\delta}^{\lambda\zeta+(1-\lambda)\eta}(P).$$

Proof. Let $P = \{u_1, u_2, \dots, u_n\}$ be a partition. In the Minkowski inequality put $x_i = \lambda\zeta(u_i)$ and $y_i = (1-\lambda)\eta(u_i)$, we have for $R > \delta$,

$$\lambda\left(\sum_{i=1}^{n}(\zeta(u_i))^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}} + (1-\lambda)\left(\sum_{i=1}^{n}(\eta(u_i))^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}} \geq \left(\sum_{i=1}^{n}(\lambda\zeta(u_i) + (1-\lambda)\eta(u_i))^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}},$$

and for $0 < R < \delta$,

$$\lambda\left(\sum_{i=1}^{n}(\zeta(u_i))^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}} + (1-\lambda)\left(\sum_{i=1}^{n}(\eta(u_i))^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}} \leq \left(\sum_{i=1}^{n}(\lambda\zeta(u_i) + (1-\lambda)\eta(u_i))^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}.$$

Since, $\frac{R}{R-\delta} > 0$ for $R > \delta$ and $\frac{R}{R-\delta} < 0$ for $0 < R < \delta$, therefore we get

$$\lambda H_{R,\delta}^{\zeta}(P) + (1-\lambda)H_{R,\delta}^{\eta}(P) \leq H_{R,\delta}^{\lambda\zeta+(1-\lambda)\eta}(P).$$

Theorem 3. *Let us consider a couple (Ω, ζ). Let $P = \{u_1, u_2, \dots, u_n\}$ and $Q = \{v_1, v_2, \dots, v_m\}$ are two partitions of it and Q is a ζ-refinment of P. Then, $H_{R,\delta}^{\zeta}(P) \leq H_{R,\delta}^{\zeta}(Q)$.*

Proof. Since, Q is a ζ-refinment of P, therefore ∃ a partition $I_1, I_2, \dots, I_n$ of the set $\{1,2, \dots, m\}$ s.t. $u_i = \vee_{j\in I_i} v_j$ for every i ∈ $\{1,2, \dots, m\}$. From Definition 2, we have $\zeta(u_i) = \sum_{j\in I_i}\zeta(v_j)$.

Case1. Now, for $R > \delta$,

$$(\zeta(u_i))^{\frac{R}{\delta}} = \left(\sum_{j\in I_i}\zeta(v_j)\right)^{\frac{R}{\delta}} \geq \sum_{j\in I_i}(\zeta(v_j))^{\frac{R}{\delta}} \tag{5}$$

Since, the partition of the set $\{1,2,\dots,m\}$ is $I_1, I_2, \dots, I_n$, therefore, we have

$$\bigcup_{i=1}^{n} I_i = \bigcup_{j=1}^{m} \{j\}$$

and

$$I_p \cap I_q = \phi; \forall p,q \in \{1,2,\dots,n\}.$$

So,

$$\sum_{i=1}^{n}\sum_{j\in I_i}\left(\zeta(v_j)\right)^{\frac{R}{\delta}} = \sum_{j=1}^{m}\left(\zeta(v_j)\right)^{\frac{R}{\delta}} \tag{6}$$

Taking summation on both sides of Eq (5) w.r.t. i varies from 1 to n and using Eq. (6), we get

$$\sum_{i=1}^{n}(\zeta(u_i))^{\frac{R}{\delta}} \geq \sum_{i=1}^{n}\sum_{j\in I_i}\left(\zeta(v_j)\right)^{\frac{R}{\delta}} = \sum_{j=1}^{m}\left(\zeta(v_j)\right)^{\frac{R}{\delta}} \tag{7}$$

Thus, we have

$$\sum_{i=1}^{n}(\zeta(u_i))^{\frac{R}{\delta}} \geq \sum_{j=1}^{m}\left(\zeta(v_j)\right)^{\frac{R}{\delta}} \tag{8}$$

Raising power to $\frac{\delta}{R}$ on both sides, we have

$$\left(\sum_{i=1}^{n}(\zeta(u_i))^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}} \geq \left(\sum_{j=1}^{m}\left(\zeta(v_j)\right)^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}} \tag{9}$$

Now, $\frac{R}{R-\delta} > 0$ (for $R > \delta$), so we have

$$\frac{R}{R-\delta}\left(1-\left(\sum_{i=1}^{n}(\zeta(u_i))^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right) \leq \frac{R}{R-\delta}\left(1-\left(\sum_{j=1}^{m}\left(\zeta(v_j)\right)^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right)$$

i.e.

$$H_{R,\delta}^{\zeta}(P) \leq H_{R,\delta}^{\zeta}(Q). \tag{10}$$

Case2. Now, for $0 < R < \delta$,

$$(\zeta(u_i))^{\frac{R}{\delta}} = \left(\sum_{j\in I_i}\zeta(v_j)\right)^{\frac{R}{\delta}} \leq \sum_{j\in I_i}\left(\zeta(v_j)\right)^{\frac{R}{\delta}} \tag{11}$$

Continuing as in above case, we get

$$\sum_{i=1}^{n}(\zeta(u_i))^{\frac{R}{\delta}} \leq \sum_{j=1}^{m}\left(\zeta(v_j)\right)^{\frac{R}{\delta}}$$

Using $\frac{R}{R-\delta} < 0$ (for $0 < R < \delta$) and Eq. (4), we get the same result as in Eq. (10).

Theorem 4. *Let $P = \{u_1, u_2, \dots, u_n\}$ and $Q = \{v_1, v_2, \dots, v_m\}$ are two partitions of a couple (Ω, ζ) satisfying Bayes' property, then*

$$\max\{H_{R,\delta}^{\zeta}(P), H_{R,\delta}^{\zeta}(Q)\} \leq H_{R,\delta}^{\zeta}(P \vee Q). \tag{12}$$

Proof. As we know that (P ∨ Q) is ζ-refinement of both P and Q Therefore, by using result of Theorem. 3, we have

$$H_{R,\delta}^{\zeta}(P) \leq H_{R,\delta}^{\zeta}(P \vee Q)$$

and

$$H_{R,\delta}^{\zeta}(Q) \leq H_{R,\delta}^{\zeta}(P \vee Q).$$

Hence, the result.

Theorem 5. *Let $P = \{u_1, u_2, \dots, u_n\}$ and $Q = \{v_1, v_2, \dots, v_m\}$ be partitions of a couple (Ω, ζ) having Bayes' property, then the $^{\delta}$RIM with respect to state ζ is sub-additive.*
i.e.

$$H_{R,\delta}^{\zeta}(P \vee Q) = H_{R,\delta}^{\zeta}(P) + H_{R,\delta}^{\zeta}(Q) - \frac{R-\delta}{R}H_{R,\delta}^{\zeta}(P)H_{R,\delta}^{\zeta}(Q). \tag{13}$$

where P and Q are ζ-independent partitions.

Proof. Consider

$$H_{R,\delta}^{\zeta}(P) + H_{R,\delta}^{\zeta}(Q) - \frac{R-\delta}{R}H_{R,\delta}^{\zeta}(P)H_{R,\delta}^{\zeta}(Q)$$
$$= \frac{R}{R-\delta}\left(1-\left(\sum_{i=1}^{n}(\zeta(u_i))^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right) + \frac{R}{R-\delta}\left(1-\left(\sum_{j=1}^{m}\left(\zeta(v_j)\right)^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right)$$

$$-\frac{R-\delta}{R}\left(\frac{R}{R-\delta}\right)^2\left(1-\left(\sum_{i=1}^{n}(\zeta(u_i))^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right)\left(1-\left(\sum_{j=1}^{m}(\zeta(v_j))^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right)$$

$$=\frac{R}{R-\delta}\left(1-\left(\sum_{i=1}^{n}(\zeta(u_i))^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\left(\sum_{j=1}^{m}(\zeta(v_j))^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right)$$

$$=\frac{R}{R-\delta}\left(1-\left(\sum_{i=1}^{n}\sum_{j=1}^{m}(\zeta(u_i)\zeta(v_j))^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right)$$

$$=\frac{R}{R-\delta}\left(1-\left(\sum_{i=1}^{n}\sum_{j=1}^{m}(\zeta(u_i\wedge v_j))^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right)$$

$$=H_{R,\delta}^{\zeta}(P\vee Q)$$

Hence, the result.

IV. JOINT AND CONDITIONAL $^{\delta}$RIM IN QUANTUM LOGICS

Let us assume that Ω is a quantum logic. Let $P=\{u_1,u_2,\ldots,u_n\}$ and $Q=\{v_1,v_2,\ldots,v_m\}$ are two partitions of Ω corresponding to a state ζ on Ω. Let us consider a set π given by

$$\pi=\{\pi_{11},\pi_{12},\ldots,\pi_{nm}\},$$

where $\pi=x_{ij}y_j=y_{ji}x_i;\ \forall\ i=1,2,\ldots,n; j=1,2,\ldots,m.$
Also,

$$x_{ij}=\frac{\zeta(u_i\wedge v_j)}{\zeta(v_j)},\quad y_{ji}=\frac{\zeta(v_j\wedge u_i)}{\zeta(u_i)},\qquad \forall\ i=1,2,\ldots,n; j=1,2,\ldots,m. \tag{14}$$

And

$$x_i=\sum_{j=1}^{m}\pi_{ij},\quad y_i=\sum_{i=1}^{n}\pi_{ij},\quad \forall\ i=1,2,\ldots,n; j=1,2,\ldots,m. \tag{15}$$

Definition 9. *The Joint $^{\delta}$RIM of partition P and Q w.r.t. state ζ is defined by:*

$$H_{R,\delta}^{\zeta}(P,Q)=\frac{R}{R-\delta}\left(1-\left(\sum_{i=1}^{n}\sum_{j=1}^{m}(\pi_{ij})^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right) \tag{16}$$

Definition 10. *The Conditional $^{\delta}$RIM of P given Q w.r.t. state ζ is defined by:*

$$'H_{R,\delta}^{\zeta}(Q|P)=\frac{R}{R-\delta}\left[1-\sum_{i=1}^{n}\zeta(u_i)\left(\sum_{j=1}^{m}\left(\frac{\zeta(v_j\wedge u_i)}{\zeta(u_i)}\right)^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right] \tag{17}$$

and

$$''H_{R,\delta}^{\zeta}(Q|P)=\frac{R}{R-\delta}\left[1-\left(\sum_{i=1}^{n}\zeta(u_i)\sum_{j=1}^{m}\left(\frac{\zeta(v_j\wedge u_i)}{\zeta(u_i)}\right)^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right] \tag{18}$$

for $R>0(R\neq\delta)$.

Theorem 6. *Let Ω is a quantum logic w.r.t. a state ζ. If P and Q are two partitions of it satisfying Bayes' property. Then the following results hold:*

(a) $'H_{R,\delta}^{\zeta}(Q|P)\leq H_{R,\delta}^{\zeta}(Q)$,

(b) $''H_{R,\delta}^{\zeta}(Q|P)\leq H_{R,\delta}^{\zeta}(Q)$,

(c) $''H_{R,\delta}^{\zeta}(Q|P)\leq {}'H_{R,\delta}^{\zeta}(Q|P)$.

Proof. (a) **Case 1**. For $R>\delta$,
Use Jensen's inequality [12] for $\frac{R}{\delta}>1$, we get

$$\left[\sum_{j=1}^{m}\left(\sum_{i=1}^{n}\pi_{ij}\right)^{\frac{R}{\delta}}\right]^{\frac{\delta}{R}} \le \left[\sum_{i=1}^{n}\left(\sum_{j=1}^{m}(\pi_{ij})^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right]$$

Using Eq. (15) on LHS and $\pi_{ij} = y_{ji}x_i$ on RHS, we have

$$\left[\sum_{j=1}^{m}(y_j)^{\frac{R}{\delta}}\right]^{\frac{\delta}{R}} \le \left[\sum_{i=1}^{n}\left(\sum_{j=1}^{m}(y_{ji}x_i)^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right]$$

i.e.

$$1-\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{m}(y_{ji}x_i)^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right] \le 1-\left[\sum_{j=1}^{m}(y_j)^{\frac{R}{\delta}}\right]^{\frac{\delta}{R}}$$

i.e.

$$1-\left[\sum_{i=1}^{n}x_i\left(\sum_{j=1}^{m}(y_{ji})^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right] \le 1-\left[\sum_{j=1}^{m}(y_j)^{\frac{R}{\delta}}\right]^{\frac{\delta}{R}}$$

Putting the values from Eq.(14), we get

$$1-\sum_{i=1}^{n}\zeta(u_i)\left(\sum_{j=1}^{m}\left(\frac{\zeta(v_j\wedge u_i)}{\zeta(u_i)}\right)^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}} \le 1-\left(\sum_{j=1}^{m}\left(\zeta(v_j)\right)^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}$$

Using $\frac{R}{R-\delta} > 0$ (for $R > \delta$), we have

$$'H_{R,\delta}^{\zeta}(Q|P) \le H_{R,\delta}^{\zeta}(Q) \tag{19}$$

Case2. On the same line we can prove the same result for $0 < R < \delta$.
Also, the equality sign holds iff $\pi_{ij} = x_i y_j$.

(b) **Case 1**. For $R > \delta$,
Using Jensen's inequality [12], we have

$$\sum_{i=1}^{n}x_i\, y_{ji}^{\frac{R}{\delta}} \ge \left[\sum_{i=1}^{n}x_i\, y_{ji}\right]^{\frac{R}{\delta}}$$

use Eq. (14) and (15), we get

$$\sum_{i=1}^{n}\zeta(u_i)\left(\frac{\zeta(v_j\wedge u_i)}{\zeta(u_i)}\right)^{\frac{R}{\delta}} \ge \left(\sum_{i=1}^{n}\zeta(u_i)\left(\frac{\zeta(v_j\wedge u_i)}{\zeta(u_i)}\right)\right)^{\frac{R}{\delta}} = \zeta(v_j)^{\frac{R}{\delta}}$$

i.e.

$$\sum_{i=1}^{n}\zeta(u_i)\left(\frac{\zeta(v_j\wedge u_i)}{\zeta(u_i)}\right)^{\frac{R}{\delta}} \ge \zeta(v_j)^{\frac{R}{\delta}}$$

Take summation w.r.t j varying from 1 to m and then raising power to $\frac{\delta}{R}$ on both sides, we have

$$\left(\sum_{i=1}^{n}\zeta(u_i)\sum_{j=1}^{m}\left(\frac{\zeta(v_j\wedge u_i)}{\zeta(u_i)}\right)^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}} \ge \left(\sum_{j=1}^{m}\zeta(v_j)^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}$$

i.e.

$$1-\left(\sum_{i=1}^{n}\zeta(u_i)\sum_{j=1}^{m}\left(\frac{\zeta(v_j\wedge u_i)}{\zeta(u_i)}\right)^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}} \le 1-\left(\sum_{j=1}^{m}\zeta(v_j)^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}$$

Using $\frac{R}{R-\delta} > 0$ (for $R > \delta$), we have

$$''H_{R,\delta}^{\zeta}(Q|P) \le H_{R,\delta}^{\zeta}(Q) \tag{20}$$

Case2. On the same line we can prove the same result for $0 < R < \delta$.

(c) **Case 1.** For $R > \delta$,
Using jensen's inequality [12], we have

$$\left[\sum_{i=1}^{n} x_i \left(\sum_{j=1}^{m} y_{ji}^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right] \le \left[\sum_{i=1}^{n} x_i \sum_{j=1}^{m} y_{ji}^{\frac{R}{\delta}}\right]^{\frac{\delta}{R}}$$

use Eq. (14) and (15) in above eq., we get

$$\left[\sum_{i=1}^{n} \zeta(u_i) \left(\sum_{j=1}^{m} \left\{\frac{\zeta(v_j \wedge u_i)}{\zeta(u_i)}\right\}^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right] \le \left[\sum_{i=1}^{n} \zeta(u_i) \sum_{j=1}^{m} \left\{\frac{\zeta(v_j \wedge u_i)}{\zeta(u_i)}\right\}^{\frac{R}{\delta}}\right]^{\frac{\delta}{R}}$$

i.e.

$$1 - \left[\sum_{i=1}^{n} \zeta(u_i) \left(\sum_{j=1}^{m} \left\{\frac{\zeta(v_j \wedge u_i)}{\zeta(u_i)}\right\}^{\frac{R}{\delta}}\right)^{\frac{\delta}{R}}\right] \ge 1 - \left[\sum_{i=1}^{n} \zeta(u_i) \sum_{j=1}^{m} \left\{\frac{\zeta(v_j \wedge u_i)}{\zeta(u_i)}\right\}^{\frac{R}{\delta}}\right]^{\frac{\delta}{R}}$$

Using $\frac{R}{R-\delta} > 0$ (for $R > \delta$), we have

$$''H_{R,\delta}^{\zeta}(Q|P) \le {}'H_{R,\delta}^{\zeta}(Q|P) \tag{21}$$

Case2. On the same line we can prove the same result for $0 < R < \delta$. Hence, Theorem is proved.

V. Conclusion

In this paper, I have defined the ${}^{\delta}$RIM, joint ${}^{\delta}$RIM and conditional ${}^{\delta}$RIM in quantum logics. I generalize R-norm information measure introduced by (Boekee and Van der Lubbe 1980) and studied a new function in quantum logics.

VI. References

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