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SET-DOMINATION IN LINE GRAPHS

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Abstract: In this paper, we initiate the study of set-domination in line graph L(G) of a graph G.The line graph L(G) of a graph G is the graph whose vertex set is in one-to-one correspondence with the elements of the set E such that two vertices of L(G) are adjacent if and only if they correspond to two adjacent edge of G. In this paper, many bounds of $\gamma_{sd}[L(G)]$ were obtained. Also its exact values for some standard graphs were found.

Keywords and Phrases: Line Graph, Set-domination, co-total domination, perfect domination, total domination.

Mathematics Subject Classification: AMS-05C69, 05C70, 05C76.

1. INTRODUCTION

We consider all non-trivial, simple, finite, connected and undirected graphs only as our research object. The general notation and terms are taken from [4]. Let $n = |V|$ be the order of graph $G = (V, E)$. The open neighbourhood of a vertex $v \in V$ is $N(v) = \{u|u, v \in E\}$ whereas its closed neighbourhood is $N[v] = N(u) \cup \{v\}$. The vertex degree can be denoted by $\deg(v) = |N(v)|$ and $\Delta(G)$ and $\delta(G)$ are the maximum and minimum degree of vertices of G.

A Line graph $L(G)$ is the graph whose vertices relate to the edges of $G$and vertices of $L(G)$ are contiguous if and provided that the comparing edges in G are neighbouring.

In a graph G, a set $S \subseteq V(G)$ is dominating set of G if each $v \in V(G)$ is either in S or neighbouring vertex in S.The minimum cardinality of this set is known as domination number $\gamma(G)$. See [6].

The set of vertices that covers all the edges of G is called the Vertex cover in G. The Vertex covering number $\alpha_0(G)$ is the minimum cardinality of a vertex cover in a graph G. The set of edges that covers all the vertices of G is called the Edge cover in G. The Edge covering number $\alpha_1(G)$ is the minimum cardinality of an edge cover in G.

The maximum cardinality of independent set of vertices in G is called the vertex independence number denoted by $\beta_0(G)$ whereas the maximum cardinality of independent set of edges in G is called the Edge independence number denoted by $\beta_1(G)$.The maximum eccentricity of any vertex in the connected graph G is called Diameter of G and its denoted by $diam\ (G)$.

We start by reviewing some standard fundamental definitions from the theory of domination.

A set D subset of vertex set of $L(G)$ is supposed to be dominating set of $L(G)$, if for each vertex which is not in D is an adjoining vertex in D. The domination number signified by $\gamma[L(G)]$ is the minimum cardinality of dominating set in G. Refer [6].

A set F of edges in a graph G is called an edge dominating set of G if each degree in $E - F$ where E is the arrangement of edges in G is adjoining atleast one edge in F. The edge domination number $\gamma'(G)$ of a graph G is the minimum cardinality of an edge dominating set of G. The idea of Edge domination number in graphs was concentrated by S.Mitchell and S.T. Hedetniemi [10].

Let D be a total dominating set if each vertex v of G is adjacent to some vertex $u \neq v$ of D.The size of the smallest total dominating set is known as Total Domination Number denoted by $\gamma_t(G)$ [2].

In [7], Hedetniemi and Laskar introduced a connected dominating set. If any induced subgraph $< D >$ induced by a dominating set D is connected in G is called Connected Dominating set. The minimum cardinality of a CD-set of G is called Connected Domination number and it is denoted by $\gamma_c(G)$.

If the induced subgraph $< V - D >$ is disconnected then the graph is said to be a Split Dominating set and the minimum cardinality of this split dominating set is called Split Domination number. It is signified by $\gamma_s(G)$. See for details [9].

If the induced subgraph $< V - S >$is totally disconnected with the minimum of two vertices, then this dominating set S is called Strong Split Dominating set. The minimum cardinality of this set is called Strong Split Domination number $\gamma_{ss}(G)$ of G. Refer [9].

Comparably, we elucidate Set-Domination number in Line Graph as follows:

A dominating set is called a Set-Dominating set of a Line Graph, if for every set which is subset of $< V - D >$, there exist a set S subset of dominating set which is non-empty such that subgraph of this set and$S$ is connected. Furthermore, the minimum cardinality of this dominating set is called SD-set of Line graph. It is denoted by $\gamma_{sd}[L(G)]$. Refer [6] for domination sets in graphs.

In [12], Sampath Kumar and PushpaLatha have presented the idea of set-domination in graphs. In this paper, we start the connection between $\gamma_{sd}[L(G)]$ with other known distinctive domination parameters, additionally its exact values of some standard graphs were found.

2. RESULTS

Presently, we give Set-Domination number of line graph for some standard graphs, which are straight forward in the accompanying proposition.

2.1 Theorem:

a) For any Path P_p with $p \geq 6$,

$$\gamma_{sd}[P_p] \leq \gamma_{ns}(G) - \Delta(G).$$

b) For any Path P_p, with $p > 2$ vertices,

$$\gamma_{sd}[P_p] = diam(G) - \gamma_{dd}(G) + 2.$$

c) For any star S_n with $p \geq 3$,

i) $\gamma_{sd}[S_n] = p - n$ where $n = 2,3, \ldots$

ii) $\gamma_{sd}[S_n] = \alpha_o(G) = \beta_1(G) = \gamma(G) = \gamma_c(G) = \gamma_s(G) = \gamma_p(G) = i(G) = \delta(G).$

iii) $\gamma_{sd}[S_n] = \gamma_{dd}(G) - \Delta(G).$

iv) $\gamma_{sd}[S_n] = \gamma_{cot}(G) - q(G).$

v) $\gamma_{sd}[S_n] = p(G) - \gamma_r(G).$

d) For any wheel W_n with $p \geq 4$,

i) $\gamma_{sd}[W_n] = \gamma'(G).$

ii) $\gamma_{sd}[W_n] \leq \gamma_{lw}(G) \leq \beta_1(G) \leq \alpha_o(G) \leq \Delta'(G) \leq q(G).$

e) For any Cycle C_p with $p \geq 3$,

i) $\gamma_{sd}[C_p] = \gamma(G) = \gamma_{lw}(G).$

ii) $\gamma_{sd}[C_p] \leq \gamma_{cot}(G) \leq \alpha_1(G).$

iii) $\gamma_{sd}[C_p] \geq \frac{n}{\Delta+1}$ where n is order of graph.

f) For any Fan graph $F_n, n \geq 3$,

i) $\gamma_{sd}[F_n] = \left\lfloor \frac{n}{2} \right\rfloor$

ii) $\gamma_{sd}[F_n] = \frac{\beta_o(G)}{2}.$

In the accompanying theorem, we build up the upper bounds for $\gamma_{sd}[L(G)]$ in terms of edges and independent domination of G.

2.2 Theorem: For any connected (p,q)graph G,

$$\gamma_{sd}[L(G)] \geq \left\lfloor \frac{q-i(G)}{4} \right\rfloor \text{where} G \neq K_{1,p} \text{for } p \geq 10.$$

Proof: Let $E = \{e_1, e_2, e_3, \ldots \ldots e_n\} \subset E(G)$ such that $E(G) = V[L(G)]$.Suppose$D = \{v_1, v_2, v_3, \ldots \ldots v_m\} \subset V(G)$ and assume $N[D] = V(G)$. Then D is a minimal dominating set of G. If $\forall v_i \in D$ and does not incident with any $e_i \in E(G)$ then D is an independent dominating set of G.

Now $A = \{v_1, v_2, v_3, \ldots . v_n\}$ be the vertex set of $L(G)$ corresponding to the edges of E in $L(G)$ such that $A = V[L(G)]$. Now we consider a subset $A_1 \subset A$ such that $N[A_1] = V[L(G)]$. Clearly $A_1$is a dominating set of $L(G)$. Suppose there exists a subset $B \subseteq V[L(G)] - A_1$and $A_1' \subset A_1$ where $A_1', B \neq \emptyset$ and $< A_1' \cup B >$ is connected. Hence $A_1$is a set-dominating set of $L(G)$.Since $|E| = q$ and we have $|A_1| \geq \frac{|E|-|D|}{4}$ gives $\gamma_{sd}[L(G)] \geq \left\lfloor \frac{q-i(G)}{4} \right\rfloor$

Suppose $G = K_{1,p}$for $p \geq 10$. Then $L(K_{1,p})$=K_p, Since for any star $K_{1,p}, q = p - 1$ and $i(G) = 1$ which gives $\left\lfloor \frac{q-i(G)}{4} \right\rfloor \geq 2$ and also $|A_1| = 1$ which gives $\gamma_{sd}[K_{1,p}] \leq \left\lfloor \frac{q-i(G)}{4} \right\rfloor$

Hence $G \neq K_{1,p}$ with $p \geq 10$.

The accompanying proposition relates total domination and weak domination with set domination in line graph.

2.3 Theorem: For any connected (p,q) graph G,

$$\gamma_{sd}[L(G)] \leq \gamma_t(G) + \gamma_w(G) - 1.$$

Proof: Suppose $D = \{v_1, v_2, v_3, \ldots \ldots v_n\} \subseteq V(G), 1 \leq i \leq n$ and $N[D] = V(G)$.Then D is a minimal dominating set of G. Suppose$< D >$ has adjacent vertex. Then D itself is a total dominating set of G. Or else there exists an isolated vertex v of zero degree $v \in \{V(G) - D\}$ and if $< \{D\} \cup \{v\} >$ has adjacent vertex then $\{D\} \cup \{v\}$ forms a total dominating set of G.

Further, Let $F = \{v_1, v_2, v_3, \ldots \ldots v_k\} \subseteq V(G)$ be the set of vertices with $\deg(v_j) \geq 1, 1 \leq j \leq k$.Suppose there exist a vertex set $D_1 \subseteq F$ with $N[D_1] = V(G)$ and if $|\deg(x) - \deg(y)| \leq 1, \forall\, x \in [V(G)] - D_1 \forall\, y \in D_1$. Then D_1forms a weak dominating set in G.

Suppose $H = \{v_1, v_2, v_3, \ldots \ldots v_n\} = V[L(G)]$. Let D_1' be the minimal dominating set of $L(G)$ such that $N[D_1'] = V[L(G)]$. Then D_1'forms a $\gamma_{sd} - set$ of $L(G)$.Then $|D_1'| = \gamma_{sd}[L(G)]$.

Hence $|D_1'| \leq |\{D\} \cup \{v\}| + |D_1| - 1$ gives $\gamma_{sd}[L(G)] \leq \gamma_t(G) + \gamma_w(G) - 1$.

The accompanying proposition shows the connection between set-domination of line graph and domination number of graph.

2.4 Theorem: For any connected (p,q) graph G,

$$\gamma_{sd}[L(G)] \leq \gamma(G) + 3.$$

Proof: Let $V = \{v_1, v_2, v_3, \ldots .. v_m\} \subseteq V(G)$ be the set of all vertices in G.Assume there exists a minimal set of vertices $D = \{v_1, v_2, v_3, \ldots . v_k\} \subseteq V(G)$such that $N[D] = V(G)$.Then D forms a minimal dominating set of G. Now we consider $D_1 = \{u_1, u_2, u_3, \ldots .. u_i\}\, where\, m \geq k \geq i$ be the minimal set of $L(G)$such that $N[D_1] = V[L(G)]$. Let $P \subseteq V[L(G)] - D_1$and $Q \subseteq D_1$ where $P, Q \neq \emptyset$ and $< Q \cup P >$ is connected. Then D_1 is a set dominating set of $L(G)$. Hence $|D_1| \leq |D| + 3$.

It follows $\gamma_{sd}[L(G)] \leq \gamma(G) + 3$.

The accompanying proposition correlate perfect domination number of graph and set domination of line graph.

2.5 Theorem : For any connected (p,q) graph G,

$$\gamma_{sd}[L(G)] \leq \gamma_p(G) + 1.$$

Proof: Suppose $H = \{v_1, v_2, v_3, \ldots . v_n\}$ be the vertex set of G. Now assume there exist $A_1 \subset A$ such that each vertex of $< A - A_1 >$ is adjacent to minimum one vertex of A_1.Then A_1 is a minimal dominating set of G. If every vertex of $A_1$is adjacent to exactly one vertex of $< A - A_1 >$, then A_1 is a $\gamma_p - set$ of G. Now we consider $D_1 = \{v_1, v_2, v_3, \ldots . v_m\}$ be the vertex set of $L(G)$ corresponding to the edges of E in $L(G)$ such that $D_1 = V[L(G)]$. Let a subset $D_1' \subset D_1$ such that $N[D_1] = V[L(G)]$. Clearly D_1' is a dominating set of $L(G)$. Suppose there exists a subset $S \subseteq V[L(G)] - D_1'$ and $T \subset D_1'$ where $S, T \neq \emptyset$ and $< S \cup T >$is connected. Then D_1' is a set-dominating set of $L(G)$such that $|D_1'| = \gamma_{sd}[L(G)]$ which gives $|D_1'| \leq |A_1| + 1$.

Hence $\gamma_{sd}[L(G)] \leq \gamma_p(G) + 1$.

The next proposition gives an upper bound of $\gamma_{sd}[L(T)]$ of a tree of fixed order and diameter with domination number of a tree.

2.6 Theorem : For any tree T, with $p > 2$ vertices and $diam \geq 4$ then

$$\gamma_{sd}[L(T)] + 1 \leq \gamma(T).$$

Proof: Let $S = \{v_1, v_2, v_3, \ldots . v_n\} \subseteq V(T)$ be the set of vertices with $\deg(v_i) \geq 2, \forall v_i \in S, 1 \leq i \leq n$. Further, let there exists a set $S_1 \subseteq S$ of vertices with $diam(u,v) \geq 4\ \forall\ u, v \in S$, which covers all the vertices in T, clearly S_1forms a dominating set of T. Otherwise, if $diam\ (u,v) < 3$, then there exists atleast one vertex $x \notin S_1$ such that $S' = S_1 \cup \{x\}$ forms a minimal $\gamma - set$ of T. Now suppose D_1 be a $\gamma_{sd} - set$ of tree T and assume $V = \{v_1, v_2, v_3, \ldots .. v_n\} = V(T)$ such that $P \subseteq D_1$and $Q \subseteq V - D_1$ where $P, Q \neq \emptyset$ and $< P \cup Q >$ is connected. Then $|D_1| = \gamma_{sd}(T)$. Hence $|D_1| \leq |S'|$ which gives $\gamma_{sd}(T) + 1 \leq \gamma(T)$.

Presently we build up relationship among edges and maximum degree of G with set-domination in line graph.

2.7 Theorem : For any connected (p, q) graph G,

$$\gamma_{sd}[L(G)] \leq q(G) - \Delta'(G) \ \forall \ p \geq 3.$$

Proof: Let $\{e_1, e_2, e_3, \ldots .. e_n\} = E(G)$. For any graph G, there exists atleast one edge $e \in E(G)$ with $\deg(e) = \Delta'(G)$. Suppose $|E(G)| = q$. Further in $L(G)$, suppose $D' = \{u_1, u_2, u_3, \ldots . u_n\} \subseteq V[L(G)]$ be the set of vertices such that $N[D'] = V[L(G)]$.Let$F \subseteq V[L(G) - D'$ and $H \subseteq D'$ where $H, F \neq \emptyset$ and $< F \cup H >$ is connected. Then $|D'| = \gamma_{sd}[L(G)]$. It follows $|D'| \leq |E(G)| - |\deg(e)|$.

Hence $\gamma_{sd}[L(G)] \leq q - \Delta'(G)$.

2.8 Theorem: For any connected (p, q) graph G,

$$\gamma_{sd}[L(G)] \leq \gamma_c(G) + 2.$$

Proof: Since $\gamma(G) \leq \gamma_c(G)$,henceby theorem 2.4, the result follows.

Hence $\gamma_{sd}[L(G)] \leq \gamma_c(G) + 2$.

The accompanying propostion correlates between set-domination in line graph and connected domination in graph.

2.9 Theorem: For any connected (p, q) graph G,

$$\gamma_{sd}[L(G)] \leq \gamma_{ss}(G), G \neq K_p, p \geq 2.$$

Proof: Let $D = \{v_1, v_2, v_3, \ldots \ldots v_n\} \subseteq V(G)$ be the set of non-end vertices such that $N[D] = V(G)$.Then$D$ is minimal dominating set of G.

Suppose $G = K_p, p \geq 2$. Then by the definition of γ_{ss} of G, the strong split domination does not exist. Hence there exist a minimal dominating set of D such that for every $v_i \in V - D$, with $\deg(v_i) = 0$ and $< V - D >$ has atleast two vertices. Then D is a $\gamma_{ss} - set$ of G. Otherwise if there exists a vertex set $H = \{v_1, v_2, v_3, \ldots \ldots v_k\}$ and every vertex of H is incident to atleast one edge, where $H \in V - D$.

Now consider $H_1 \subseteq H \ \forall \ v_i \in < H - H_1 >, \deg(v_i) = 0$.Then$< V - \{D \cup H_1\} >$ has a set with atleast two isolated vertices. Clearly $\{D \cup H_1\}$ is a $\gamma_{ss} - set$ of G. Further, let $D_1 = \{u_1, u_2, u_3, \ldots . u_i\} \subseteq V[L(G)]$ be the minimal set of $L(G)$ and every vertex of $V[L(G)] - D_1$is adjacent to atleast one vertex of D_1.Suppose$P \subseteq V[L(G)] - D_1$and $Q \subseteq D_1$where $P, Q \neq \emptyset$ and $< Q \cup P >$ is connected. Then $D_1$is a sd-set of $L(G)$ which implies that $|D_1| \leq |\{D \cup H_1\}|$.

Thus $\gamma_{sd}[L(G)] \leq \gamma_{ss}(G)$.

In the next theorem, we establish relationship between co-total domination, minimum vertex degree of a graph and set-domination of line graph.

2.10 Theorem: For any connected (p,q)graph G,

$$\gamma_{sd}[L(G)] \le \gamma_{cot}(G) + \delta(G) + 1.$$

Proof: Let $A = \{v_1, v_2, v_3, \ldots . v_n\}$ be the set of vertices in G and $B = V(G) - A$. Suppose there exists a subset $B_1 \subset B$ and if $A \cap B_1 = \emptyset$ or $u, u \in V(G) - A$ such that $N[A \cup B_1] = V(G)$. Then $A \cup B_1$ is a dominating set of G. Further if $< V(G) - \{A \cup B_1\} >$ has no isolates, then $\{A \cup B_1\}$ is a $\gamma_{cot} - set$ of G. Otherwise select $B_2 \subset V(G) - \{A \cup B_1\}$ so that $< V(G) - \{A \cup B_1 \cup B_2\} >$ has no isolates. Hence $\{A \cup B_1 \cup B_2\}$ is a $\gamma_{cot} - set$ of G. Let there exists a vertex v of minimum degree $\delta(G)$.

Now assume $E = \{e_1, e_2, e_3, \ldots .. e_m\} = E(G)$. Then $V[L(G)] = \{v_1, v_2, v_3, \ldots . v_m\}$ be the set of vertices corresponding to the set E. Suppose $D \subset V[L(G)]$ and if $\forall v_i \in D$ is adjacent to at least one vertex of $V[L(G)] - D$. Then D itself is a minimal dominating set of $L(G)$. If there exists a subset $D_1 \subset D$ and $D_2 \subset V[L(G)] - D_1$ so that $D_1, D_2 \neq \emptyset$ such that $< D_1 \cup D_2 >$ is connected. Then D is a $\gamma_{sd} - set$ of $L(G)$. Now with respect to $\{A \cup B_1 \cup B_2\}$,we have $|D| \le |\{A \cup B_1 \cup B_2\}| + \delta(G) + 1$, which gives $\gamma_{sd}[L(G)] \le \gamma_{cot}(G) + \delta(G) + 1$.

The following theorem shows relationship between $\gamma_s(G)$, $\beta_0(G)$ and $\gamma_{sd}[L(G)]$.

2.11 Theorem: For any connected (p,q) graph G,

$$\gamma_{sd}[L(G)] \le \gamma_s(G) + \beta_0(G).$$

Proof: Let $A = \{v_1, v_2, v_3, \ldots .. v_n\} \subseteq V(G)$ if $N(v_i) \cap N(v_j) = \{u\}, 1 \le i \le n$ and $1 \le j \le n$. Then A is a maximal independent vertex set of G.Hence $|A| = \beta_0(G)$. Further consider a minimum set of vertices D_1such that $N[D_1] = V(G)$ and $< V - D_1 >$ is disconnected. It follows that $D_1$is a split dominating set of G. Otherwise there exists atleast one vertex $\{u\} \in V(G) - D_1$ such that $< V(G) - D_1 \cup \{u\} >$ has more than one component. Hence $D_1 \cup \{u\}$ forms a minimal $\gamma_s - set$ of G.

Suppose $B \subset V[L(G)]$where $\forall v_i \in B$ are the edges of G which are incident to $\forall v_j \in A$. Now there exists a subset $B_1 \subset B$ and if $N[B_1] = V[L(G)]$. Hence B_1 is a dominating set of $L(G)$. Suppose there exists $B_2 \subseteq B_1$and $B_3 \subset V[L(G)] - \{B_1\}$.If $< B_2 \cup B_3 >$ is connected, then $B_1$is a $\gamma_{sd} - set$ of $L(G)$. Hence $|B_1| \le |D_1 \cup \{u\}| + |A|$ which gives $\gamma_{sd}[L(G)] \le \gamma_s(G) + \beta_0(G)$

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