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GRADATION OF MULTIPLICATION MODULES AND IDEALLY GRADED θg(M)

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Abstract : Assume that G is a group and that R is a commutative ring that belongs to G. The concept of the related graded ideal g (M) of R is defined for a graded R-module. It has been established that studying graded multiplication modules requires knowledge of the graded ideal g (M). The conclusions are as follows, which are supported by the numerous applications given: if M is a graded faithful multiplication module, then g (M) is an idempotent graded multiplication ideal of R such that g (g (M)) = g (M), and any graded representable multiplication R -module is finitely produced.

***IndexTerms* - Commutative Ring, Graded Ideal, Graded R Module, Multiplication R-module**

I. INTRODUCTION

Calculations are typically made easier by focusing on the homogeneous elements, which are probably easier to understand or more manageable than random elements. To make this work, one must understand that the constructs under study are ranked. A possible solution to this problem is to completely rewrite the constructs in terms of the category of graded modules, omitting all mention of non-graded modules or non-homogeneous parts in the process. Sharp provides such a treatment of attached primes in [12]. Unfortunately, while this method aids in understanding the graded modules themselves, understanding the original construction will only be aided if the graded version of the notion coincides with the original one. Thus, it should be noted that studying graded modules is crucial.

In this paper we study the concepts of graded multiplication modules and graded representable modules over a G -graded commutative ring. We study these concepts in analogous way to that done for graded modules in [4, 5, 12]. However, if G is a finitely generated abelian group then G is isomorphic to the direct sums of some copies of Z_m and Z^n and, for this case, the results are well-known [4, 5, 12]. Throughout this paper G is a non-finitely generated abelian group. So, our work is a new direction in the study of graded multiplication modules and related results.

A module M over a commutative ring R is called a multiplication module if for any submodule N of M there exists an ideal I of R such that $N = IM$. Let M be a multiplication module. Anderson [1], defines $\theta(M) = \Sigma_{m \in M} (Rm : M)$. In case M is faithful, it is proved in [2] that $\theta(M)$ is an idempotent multiplication ideal such that $\theta(\theta(M)) = \theta(M)$. Let G be a group. Graded modules over a commutative G -graded ring have been studied by many authors [4-12] for example. Here we study graded multiplication$R$ –modules. In the present paper we show that the graded module structures of M and $\theta_g(M)$ are closely related. The main aim of this paper is that of extending some results obtained by [2, 10] to the theory of graded modules.

For the sake of completeness, we recall some definitions and notations used throughout. Let G be an arbitrary group. A commutative ring R with non-zero identity is G -graded if it has a direct sum decomposition (as an additive group) $R = \bigoplus_{g\in G} R_g$ such that for all $g, h \in G$, $R_g R_h \subseteq R_{gh}$. If R is G -graded, then an R -module M is said to be G -graded if it has a direct sum decomposition $M = \bigoplus_{g\in G} M_g$ such that for all $g, h \in G$, $R_g M_h \subseteq M_{gh}$. An element of some R_g or M_g is said to be a homogeneous element. A submodule $N \subseteq M$, where M is G -graded, is called G -graded if $N = \bigoplus_{g\in G}(N \cap M_g)$ or if, equivalently, N is generated by homogeneous elements. Moreover, M/N becomes a G -graded module with g -component $(M/N)_g = (M_g + N)/N$ for $g \in G$. Clearly, 0 is a graded submodule of M. We write $h(R) = \cup_{g\in G} R_g$ and $h(M) = \cup_{g\in G} M_g$.

Let R be a G -graded ring R. A graded ideal I of R is said to be a graded prime ideal if $I /= R$; and whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a, b \in h(R)$. The graded radical of I, denoted by Gr(I), is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x \in I$. A proper graded submodule N of a graded R-module M is called graded prime if $rm \in N$, then $m \in N$ or $r \in (N : M) = \{r \in R : rM \subseteq N\}$, where $r \in h(R)$, $m \in h(M)$. The set of all graded prime submodules of M is called the graded spectrum of M and denoted by Spec (M). A graded R-module M is called graded finitely generated if $M = \sum^{n} Rx_{g}$, where $x_{gi} \in h(M)$ ($1 \le i \le n$). It is clear that a graded module is finitely generated if and only if it is graded finitely generated.

Definition 1.1 Let R be a G-graded ring. A graded R-module M is defined to be a graded multiplication module if for each graded submodule N of M, N = IM for some graded ideal I of R [9]. Graded multiplication ring is defined in a similar way.

One can easily show that if N is a graded submodule of a graded multiplication module M, then $N = (N : M)M$. It is clear that every graded module which is multiplication is a graded multiplication module. Moreover, the class of graded multiplication domains has been characterized in [5] as the class of graded Dedekind domains which is the class of graded domains in which every graded ideal is graded invertible (a graded ideal I of a graded ring R is called graded invertible ideal if there exists a graded ideal J of R such that $IJ = R$). In [14], we can see an example of a graded multiplication ring which is not multiplication. Indeed, the group ring $R[Z]$, where R is a Dedekind domain is a graded Dedekind domain and so it is a graded multiplication domain. On the other hand, if R is not a field, then $R[Z]$ is not a Dedekind domain and so it is not a multiplication domain. So a graded multiplication module need not be multiplication. We need the following lemma proved in [9].

Lemma 1.2 Let M be a graded module over a G-graded ring R. Then the following hold:

(i) If N is a graded submodule of M, a ∈ h(R) and m ∈ h(M), then Rm, IN and aN are graded submodules of M and Ra is a graded ideal of R.

(ii) If $\{N_i\}_{i\in\Lambda}$ is a collection of graded submodules of M, then $\sum_{i\in\Lambda} N_i$ and $\bigcap_{i\in\Lambda} N_i$ are graded submodules of M.

(iii) M is graded multiplication if and only if for each m in h(M) there exists a graded ideal I of R such that Rm = IM.

2.0 The graded ideal $\theta_g(M)$

In this section we study the graded ideal $\theta_g(M)$ where R is a commutative G-graded ring with identity and M is a graded multiplication R-module.

Remark 2.1 Let M be a graded module over a G-graded ring R.

(i) Assume that M is a finitely generated R-module and that I is be a graded ideal of R such that IM = M. Then by standard determinant arguments, we have that $(1 - t)M = 0$ for some $t \in I$ (note that every graded finitely generated R-module is finitely generated), so $R = I + (0 : M)$. Moreover, if I is finitely generated ideal of R, then IM is a finitely generated submodule of M.

Proof. By Remark 2.1, $M = \sum_{m\in h(M)} Rm = \sum_{m\in h(M)}(Rm : M)M = (\sum_{m\in h(M)}(Rm : M))M = \theta_g(M)M$. Moreover, $N = (N : M)M = (N : M)(\theta_g(M)M) = \theta_g(M)((N : M)M) = \theta_g(M)N$.

Proposition 2.3 Let M be a graded multiplication module over a G-graded ring R. If I is a finitely generated ideal of R with $I \subseteq \theta_g(M)$, then IM is finitely generated. Conversely, if I is a graded ideal of R with IM finitely generated, then $I \subseteq \theta_g(M)$.

Proof. Let $I = \langle a_1, \dots, a_n \rangle$, where $a_i \in I \cap h(R)$. Then there exist $x_i \in h(M)$ $(1 \le i \le n)$ such that $a_i \in (Rx_i : M)$ (note that a_i is a homogeneous element); hence $I \subseteq \sum_{i=1}^{n} (Rx_i : M)$. Therefore, $IM \subseteq \sum_{i=1}^{n} Rx_i = N$. It follows from Remark 2.1 that $\theta_g(M)N = N$, so $R = \theta_g(M) + (0 : N)$. There are elements $a \in \theta_g(M)$ and $b \in (0 : N)$ such that $1 = a + b$. Hence there exist $y_1, \dots, y_s \in h(M)$ such that $a \in \sum_{j=1}^{s} (Ry_j : M)$; thus $R = (0 : N) + \sum_{j=1}^{s} (Ry_i : M)$. It follows that $IM = IRy_1 + \cdots + IRy_s$ (since $IM(0 : N) = 0$); hence IM is finitely generated by Remark 2.1. Conversely, let I be a graded ideal of R and suppose that IM is finitely generated. First we show that $I(0 : IM) \subseteq (0 : M)$. It suf- fices to show that for each $a \in I \cap h(R)$, $b \in (0 : IM) \cap h(R)$, $abM = 0$. As $bIM = 0$, we must have $abM = 0$. Since IM is finitely generated and $IM = \theta_g(M)IM$, so $R = \theta_g(M) + (0 : IM)$. Hence $I = I\theta_g(M) + I(0 : IM) \subseteq \theta_g(M) + (0 : M) \subseteq \theta_g(M)$ because $(0 : M) \subseteq \theta_g(M)$.

Theorem 2.4 Let R be a G-graded ring and M a graded multiplication R-module. Then the followingconditions are equivalent:

(i) M is finitely generated.

(ii) $\theta_g(M) = R$.

(iii) $\theta_g(M)$ is finitely generated.

Proof. (i) → (ii). Apply the second part of Proposition 2.3. (ii) → (iii). Clear. (iii) → (i). Set $I = \theta_g(M)$. Then by 2.3, $M = \theta_g(M)M$ is graded finitely generated.

3.Graded representable modules

Dual to the more familiar theory of primary decomposition and associated primes, the theory of secondary representations and attached primes is a useful tool for studying Artinian modules, particularly the local cohomology H. (M) of finitely generated modules relative to the maximal ideal of a local ring [11, 12]. In reality, a module's collection of associated prime ideals provides a wealth of information about the module itself. One solution to the graded situation is to simply limit all language to homogeneous parts and graded submodules. Assume R is a G-graded ring. A non-zero graded module M is said to be secondary if the endomorphism a,M (i.e., multiplication by an in M) is either surjective or nilpotent for any a h(R). Gr(annM) = P is an obvious graded prime ideal of R, and M is said to be graded P -secondary (see [12, Proposition 2.2]). If a graded module M can be expressed as a sum M = M1 +•••+ Mk with each Mi graded secondary, and if such a representation exists (and is irredundant), then the graded attached primes of M are Attg(M) = Gr(annM1),..., Gr(annMk). It should be noted that a graded secondary module is not, in general, secondary.

Assume R is a G-graded ring. If M = 0 and the sum of any two suitable graded submodules of M is always a proper submodule, a graded R -module M is sum-irreducible. M is graded Noetherian if any non-empty set of graded submodules of M has a maximum member with regard to set inclusion. The ascending chain condition on graded submodules of M is identical to this formulation. The definitions of graded Noetherian rings and graded Artinian rings are similar.

Proposition 3.1 If R is a G-graded Noetherian (resp. Artinian) ring, then any graded multiplication R-module is graded Noetherian (resp. Artinian).

Proof. Consider a chain of graded submodules of M:

$$N_1 \subseteq N_2 \subseteq \cdots \subseteq N_k \subseteq \ldots$$

Then, there exist graded ideals $(N_i : M)$ such that $N_i = (N_i : M)M$ for each i. So we can have a chainof graded ideals in R:

$$(N_1 : M) \subseteq \cdots \subseteq (N_k : M) \subseteq \ldots.$$

Since R is graded Noetherian, there exists n such that $(N_n : M) = (N_{n+1} : M) = \ldots$. Therefore, $N_n = N_i$ for each $\geq n$, as required.

Lemma 3.2 Let R be a G-graded ring. Then a finite sum of graded P -secondary modules is graded P -secondary.

Proof. Let $M = M_1 + \cdots + M_k$, where for each i ($1 \leq i \leq k$), M_i is graded P -secondary. Let $a \in h(R)$. If $a \in P$, then there is a positive integer n such that $a^n M_i = 0$ for every i; hence $a^n M = 0$. Similarly, if $a \notin P$, then $aM = M$. Thus M is graded P -secondary.

Theorem 3.3 Let R be a G-graded ring. Then every graded Artinian R-module M has a graded secondary representation.

Proof. First, we show that if M is sum-irreducible, then M is graded secondary. Suppose M is not graded secondary. Then there is an element $r \in h(R)$ such that rMM and $r^n M$ /= 0 for every positive integer n. By assumption, there exists a positive integer k such that $r^k M = r^{k+1} M = \ldots$. Set $M_1 = \mathrm{Ker}\phi_r k,_M$ and $M_2 = r^k M$. Then M_1 and M_2 are proper graded submodules of M. Let $x \in M$. Then $r^k x = r^{2k} y$ for some $y \in M$; hence $x - r^k y \in M_1$ and therefore $x \in M_1 + M_2$. Hence $M = M_1 + M_2$, and therefore M is not sum-irreducible. Next, suppose that M is not graded representable. Then the set of non-zero graded submodules of M which are not graded representable has a minimal element N. Certainly N is not graded secondary and N /= 0 ; hence N is the sum of two strictly smaller graded submodules N_1 and N_2. By the minimality of N, each N_1, N_2 is graded representable, and therefore so also is N, which is a contradiction.

Let R be a G-graded ring and M, N graded R-modules. Let $f : M \to N$ be an R-module homomor- phism. Then f is said to be graded homomorphism if $f(M_g) \subseteq N_g$ for all $g \in G$. It is easy to see that Ker(f) is a graded submodule of M and Im(f) is a graded submodule of N. A graded R-module M is said to be graded Hopfian if each graded R-epimorphism $f : M \to M$ is graded isomorphism.

Proposition 3.4 If M is a graded multiplication module over a G-graded ring R, then M is a graded Hopfian.

Proof. Let $f : M \to M$ be a graded epimorphism. By assumption, there exist a graded ideal I of R such that $N = \mathrm{Ker}(f) = IM$. Hence $0 = f(N) = If(M) = IM = N$, as needed.

Proposition 3.5 Let R be a G-graded ring, M a graded multiplication R-module and N a graded P -secondary R-submodule of M. Then there exists $a \in h(R)$ such that $a \in \theta_g(M)$ and $a \notin P$. In particular, aM is a finitely generated R-submodule of M.

Proof. Suppose not. Then $\theta_g(M) \subseteq P$. Let $x \in h(N)$. Then by Lemma 2.2, $Rx = \theta_g(M)Rx \subseteq Px \subseteq Rx$, so $x = px$ for some $p \in P \cap h(R)$. There is a positive integer m such that $p^m x = x = 0$, which is a contradiction. Finally, aM is graded finitely generated

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