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SOLUTION OF NON-LINEAR EXPONENTIAL DIOPHANTINE EQUATION $x^{\alpha} + (1 + my)^{\beta} = z^2$

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ABSTRACT: In this article, authors discussed the solution of non-linear Exponential Diophantine equation $x^{\alpha} + (1 + my)^{\beta} = z^2$, where x is a number of the form $mr + 1$, m is an odd prime such that either $m \equiv 3 \; or \; 5 (mod \; 8)$ and $\alpha, \beta, r, y, z \in W$, here W is the set of whole numbers. Results illustrate that this Diophantine equation has no solution in whole numbers.

KEY WORDS: Exponential Diophantine equations; Quadratic residue; Legendre symbol.

MATHEMATICS SUBJECT CLASSIFICATION: 11D61, 11D72.

1. INTRODUCTION

Theory of numbers is avaluable part of Mathematics in which we generally talk about integers. The Diophantine equation having a variable occurring as exponent is known as exponential Diophantine equation. Exponential Diophantine equation is a type of Diophantine equation.

In present years, Diophantine equation of the form $p^x + q^y = z^2$ is of central object and many researchers works on the solution of this type Diophantine equation, where p, q are distinct primes and $x, y, z \in W$, here W is the set of whole numbers. In [1], Kumar et al. discussed the Exponential Diophantine equation $601^p + 619^q = r^2$, where $p, q, r \in W$.In [2], Pakapongpun and Chattae discussed the solution of Diophantine equation $a^x + (a + 2)^y = z^2$, where $a \equiv 3 (mod \; 20)$. In [3], Sroysang proved that the Diophantine equation $2^x + 19^y = z^2$ has unique solution in W, which is $\{x = 3, y = 0, z = 3\}$. In [4], Sroysang proved that the Diophantine equation $8^x + 13^y = z^2$ has unique solution in W, which is $\{x = 1, y = 0, z = 3\}$.

In [5], Terai showed that the equation $(12m^2 + 1)^x + (13m^2 - 1)^y = (5m)^z$ has unique solution $\{x = 1, y = 1, z = 2\}$ under some conditions. In [6], Gupta and Kumar discussed the Exponential Diophantine equation $n^x + (n + 3m)^y = z^{2k}$, where n is a number of the form $6r + 1$ and $x, y, z, m, k, r \in W$. In [7], Gupta and Kumar discussed the Exponential Diophantine equation$a^u + (a + 5b)^v = c^{2w}$, where a is a number of the form$5r + 1$and$u, v, w, b, c, r \in W$. In [8], Burshtein proved that the equation $p^x + (p + 4)^y = z^2$, where $x, y, z \in W$ and $p, p + 4$ are primes with $p > 3$ has no solution.

In [9], Borah and Dutta discussed on the solution of exponential Diophantine equation $n^x + 24^y = z^2$, for $n \equiv 5$ or $7(mod\ 8)$. In [10], Mishra et al. studied the Diophantine equation $211^\alpha + 229^\beta = \gamma^2$, where $\alpha, \beta, \gamma \in W$. In [11], Kumar et al. discussed on the equation$p^x + (p+12)^y = z^2$and show that under some conditions this equation has no non-negative integer solution. In [12], Kumar et al. discussed the equations $31^x + 41^y = z^2$and $61^x + 71^y = z^2$, where $x, y, z$are natural numbers. In [13], Orosram et al. discussed the solution of exponential Diophantine equation $(p+2)^x + (2p+1)^y = z^2$, where $p, p+2, 2p+1$ are prime and $p \equiv 1(mod\ 4)$.

In present article, authors proved that the Exponential type non-linear Diophantine equation

$$x^\alpha + (1+my)^\beta = z^2, \qquad (1)$$

where x is a number of the form $mr+1$, m is an odd prime such that either $m \equiv 3\ or\ 5(mod\ 8)$ and $\alpha, \beta, r, y, z \in W$, has no solution in W.We will use congruence method and Quadratic residue to prove that this Diophantine equation has no solution in W.

2. PRELIMINARIES

Quadratic Residue: If a is an integer coprime to p, where p is a prime then a is said to be quadratic residue $(mod\ p)$ iff $x^2 \equiv a(mod\ p)$ has a solution otherwise a is said to be quadratic non-residue $(mod\ p)$.

Legendre Symbol: If p is an odd prime and a is any integer such that g.c.d. $(a,p) = 1$, then Legendre symbol $\left(\frac{a}{p}\right)$ is defined as

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & if\ a\ is\ a\ quadratic\ residue\ (mod\ p) \\ -1, & if\ a\ is\ a\ quadratic\ non-residue\ (mod\ p) \end{cases}$$

Proposition 2.1 If p is an odd prime, then

$$\left(\frac{2}{p}\right) = \begin{cases} 1, & if\ p \equiv 1(mod\ 8)\ or\ p \equiv 7(mod\ 8) \\ -1, & if\ p \equiv 3(mod\ 8)\ or\ p \equiv 5(mod\ 8) \end{cases}$$

3. MAIN THEOREM

Theorem3.1. If x is a number of the form $mr+1$ and$m$ is an odd prime such that either $m \equiv 3\ or\ 5(mod\ 8)$, then the Exponential type equation $x^\alpha + (1+my)^\beta = z^2$, has no solution in W, where $\alpha, \beta, r, y, z \in W$.

Proof: We have, $x = mr+1$, where $m$an is odd prime such that either $m \equiv 3\ or\ 5(mod\ 8)$

$\Rightarrow x \equiv 1(mod\ m) \Rightarrow x^\alpha \equiv 1(mod\ m)$.

Also, $1+my \equiv 1(mod\ m) \Rightarrow (1+my)^\beta \equiv 1(mod\ m)$

Thus, we obtain $x^\alpha + (1+my)^\beta \equiv 2(mod\ m)$.

Now, If given equation has a solution, then we must have $z^2 \equiv 2(mod\ m)$.

Since m is an odd prime such that either $m \equiv 3\ or\ 5(mod\ 8)$, therefore we have $\left(\frac{2}{m}\right) = -1$

$\Rightarrow$ 2 is a quadratic non-residue $(mod\ m)$

$\Rightarrow z^2 \equiv 2(mod\ m)$ does not have a solution.

Hence, the equation $x^\alpha + (1+my)^\beta = z^2$, has no solution in W.

Corollary3.1.1. If x is a number of the form $mr+1$ and$m$ is an odd prime such that either $m \equiv 3\ or\ 5(mod\ 8)$, then the Exponential type equation $x^\alpha + (1+my)^\beta = k^{2\gamma}$, has no solution in W, where $\alpha, \beta, \gamma, r, y, k \in W$.

Proof: Let $k^\gamma = z$, then this equation become$x^\alpha + (1+my)^\beta = z^2$, which has no solution by Theorem3.1.

Corollary3.1.2. If x is a number of the form $mr+1$ and$m$ is an odd prime such that either $m \equiv 3\ or\ 5(mod\ 8)$, then the Exponential type equation $x^\alpha + (1+my)^\beta = k^{2\gamma\delta}$, has no solution in W, where $\alpha, \beta, \gamma, \delta, r, y, k \in W$.

Proof. Let $k^{\gamma\delta} = z$, then this equation become $x^{\alpha} + (1 + my)^{\beta} = z^2$, which has no solution by Theorem 3.1.

Note: In this article, authors determined that the Diophantine equation $x^{\alpha} + (1 + my)^{\beta} = z^2$, where x is a number of the form $mr + 1$, m is an odd prime such that either $m \equiv 3 \; or \; 5(mod\ 8)$ and $\alpha, \beta, r, y, z \in W$, has no solution in W. If we give some particular value to r, m and y, then we find infinite many Exponential Diophantine equations which has no solution in W. Some Exponential Diophantine equation for particular values of r, m and y, which has no solution in W are given in the following tables.

r	m	y	$x^{\alpha} + (1 + my)^{\beta} = z^2$
0	3	1	$1^{\alpha} + 4^{\beta} = z^2$
1	3	1	$4^{\alpha} + 4^{\beta} = z^2$
2	3	1	$7^{\alpha} + 4^{\beta} = z^2$
3	3	1	$10^{\alpha} + 4^{\beta} = z^2$
4	3	1	$13^{\alpha} + 4^{\beta} = z^2$

r	m	y	$x^{\alpha} + (1 + my)^{\beta} = z^2$
0	3	2	$1^{\alpha} + 7^{\beta} = z^2$
1	3	2	$4^{\alpha} + 7^{\beta} = z^2$
2	3	2	$7^{\alpha} + 7^{\beta} = z^2$
3	3	2	$10^{\alpha} + 7^{\beta} = z^2$
4	3	2	$13^{\alpha} + 7^{\beta} = z^2$

r	m	y	$x^{\alpha} + (1 + my)^{\beta} = z^2$
0	5	1	$1^{\alpha} + 6^{\beta} = z^2$
1	5	1	$6^{\alpha} + 6^{\beta} = z^2$
2	5	1	$11^{\alpha} + 6^{\beta} = z^2$
3	5	1	$16^{\alpha} + 6^{\beta} = z^2$
4	5	1	$21^{\alpha} + 6^{\beta} = z^2$

r	m	y	$x^{\alpha} + (1 + my)^{\beta} = z^2$
0	5	2	$1^{\alpha} + 11^{\beta} = z^2$
1	5	2	$6^{\alpha} + 11^{\beta} = z^2$
2	5	2	$11^{\alpha} + 11^{\beta} = z^2$
3	5	2	$16^{\alpha} + 11^{\beta} = z^2$
4	5	2	$21^{\alpha} + 11^{\beta} = z^2$

r	m	y	$x^{\alpha} + (1 + my)^{\beta} = z^2$
0	11	1	$1^{\alpha} + 12^{\beta} = z^2$
1	11	1	$12^{\alpha} + 12^{\beta} = z^2$
2	11	1	$23^{\alpha} + 12^{\beta} = z^2$

3	11	1	$34^{\alpha} + 12^{\beta} = z^2$
4	11	1	$45^{\alpha} + 12^{\beta} = z^2$

r	m	y	$x^{\alpha} + (1 + my)^{\beta} = z^2$
0	11	2	$1^{\alpha} + 23^{\beta} = z^2$
1	11	2	$12^{\alpha} + 23^{\beta} = z^2$
2	11	2	$23^{\alpha} + 23^{\beta} = z^2$
3	11	2	$34^{\alpha} + 23^{\beta} = z^2$
4	11	2	$45^{\alpha} + 23^{\beta} = z^2$

4. CONCLUSION: In this article, authors proved that the Exponential Diophantine equation $x^{\alpha} + (1 + my)^{\beta} = z^2$, where x is a number of the form $mr + 1$, m is an odd prime such that either $m \equiv 3$ *or* $5 (mod\ 8)$ and $\alpha, \beta, r, y, z \in W$, has no solution in W.

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