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SOLUTION OF NON-LINEAR EXPONENTIAL DIOPHANTINE EQUATION $(x^a+1)^m+\left(y^b+1\right)^n=z^2$

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ABSTRACT: In this paper, authors discussed the solution of non-linear exponential Diophantine equation $(x^a+1)^m+\left(y^b+1\right)^n=z^2$, where x is a prime divisor of y such that either $x\equiv 3\ or\ 5(mod\ 8)$ and $a,b,m,n,y,z\in W$, here W is the set of whole numbers. Authors illustrate that this Diophantine equation has no solution in whole numbers.

KEY WORDS: Exponential Diophantine Equations; Quadratic Residue; Legendre Symbol.

MATHEMATICS SUBJECT CLASSIFICATION: 11D61, 11D72.

1.INTRODUCTION: Diophantine equation is an equation in which we discussed about only integer's solutions. The study of Diophantine equations is a major part of theory of numbers. In current years, many researchers work on the solution of exponential Diophantine equation of the form $p^x+q^y=z^2$, where p,q are distinct primes and x,y,z are non-negative integers. In [1], Acu proved that the Diophantine equation $2^x+5^y=z^2$ has only two solutions $\{x=3, y=0, z=3\}$ and $\{x=2, y=1, z=3\}$. In [2], Asthana and Singh proved that the Diophantine equation $3^x+13^y=z^2$, where x,y,z are non-negative integers, has only four solutions which are $(x,y,z)=(1,0,2),(1,1,4),(3,2,14),(5,1,16)$.

In [3], Burshtein proved that the Diophantine equation $p^x+(p+4)^y=z^2$, where x,y,z are positive integers such that $x+y=2,3,4$ and $p,(p+4)$ are primes has unique solution $(p,x,y,z)=(3,2,1,4)$. In [4], Burshtein find all the solutions of the Diophantine equation $p^x+(p+6)^y=z^2$, where x,y,z are positive integers such that $x+y=2,3,4$ and $p,(p+6)$ are primes. In [5], Gupta and Kumar discussed the exponential Diophantine equation $n^x+(n+3m)^y=z^{2k}$, where n is a number of the form $(6r+1)$ and $x,y,z,m,k,r\in W$.

In [6], Gupta and Kumar discussed the solution of exponential Diophantine equation $a^u+(a+5b)^v=c^{2w}$, where a is a number of the form $(5r+1)$ and $u,v,w,b,c,r\in W$. In [7], Gupta et al. discussed the non-linear Diophantine equation $p^x + (p + 6)^y = z^2$, where p and $p + 6$ both are primes and x, y, z are positive integers. In [8], Kumar et al. discussed the exponential Diophantine equation $601^p + 619^q = r^2$, where $p, q, r \in W$. In [9], Kumar et al. discussed on the equation $p^x + (p + 12)^y = z^2$ and show that under some conditions this equation has no solution in W. In [10], Kumar et al. discussed the non-linear Diophantine equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$, where x, y, z are positive integers.

In [11], Mishra et al. studied the Diophantine equation $211^\alpha + 229^\beta = \gamma^2$, where $\alpha, \beta, \gamma \in W$. In [12], Sroysang studied the Diophantine equation $31^x + 32^y = z^2$ and show that this equation has no solution in W. In [13], Sroysang proved that the Diophantine equation $2^x + 19^y = z^2$ has unique solution in W, which is $x = 3, y = 0, z = 3$. In [14], Oliveria proved that the Diophantine equation $p^x + (p + 8)^y = z^2$, where x, y, z are positive integers and $p, p + 8$ are primes with $p > 3$ has no solution.

In the present paper, authors proved that exponential non-linear Diophantine equation $(x^a + 1)^m + \left(y^b + 1\right)^n = z^2$, where x is a prime divisor of y such that either $x \equiv 3 \; or \; 5 (mod\; 8)$ and $a, b, m, n, y, z \in W$, has no solution in W.

2. PRELIMINARIES:

Quadratic Residue

If a is an integer coprime to p, where p is a prime then a is said to be quadratic residue $(mod\; p)$ iff $x^2 \equiv a (mod\; p)$ has a solution otherwise a is said to be quadratic non-residue $(mod\; p)$.

Legendre Symbol

If p is an odd prime and a is any integer such that g.c.d.$(a, p) = 1$, then Legendre symbol $\left(\frac{a}{p}\right)$ is defined as

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & if\ a\ is\ a\ quadratic\ residue\ (mod\ p) \\ -1, & if\ a\ is\ a\ quadratic\ non-residue\ (mod\ p) \end{cases}$$

Proposition: 2.1 If p is an odd prime, then

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & if\ p \equiv 1 (mod\ 8)\ or\ p \equiv 7 (mod\ 8) \\ -1 & if\ p \equiv 3 (mod\ 8)\ or\ p \equiv 5 (mod\ 8) \end{cases}$$

3. MAIN THEOREM:

Theorem: 3.1 If x is a prime divisor of y such that either $x \equiv 3 \; or \; 5 (mod\; 8)$ then the non-linear exponential Diophantine equation $(x^a + 1)^m + \left(y^b + 1\right)^n = z^2$, has no solution in W, where $a, b, m, n, y, z \in W$.

Proof: We have $x \equiv 0 (mod\; x)$

$\Rightarrow x^a \equiv 0 (mod\; x)$

$\Rightarrow x^a + 1 \equiv 1(mod\ x)$

$\Rightarrow (x^a + 1)^m \equiv 1(mod\ x)$.

Since x is a factor of y, therefore, $y \equiv 0(mod\ x)$

$\Rightarrow y^b \equiv 0(mod\ x)$

$\Rightarrow y^b + 1 \equiv 1(mod\ x)$

$\Rightarrow (y^b + 1)^n \equiv 1(mod\ x)$.

Thus we obtain, $(x^a + 1)^m + (y^b + 1)^n \equiv 2(mod\ x)$

$\Rightarrow z^2 \equiv 2(mod\ x)$.

Now, since x is an odd prime such that either $x \equiv 3\ or\ 5(mod\ 8)$, therefore we have $\left(\frac{2}{x}\right) = -1$

$\Rightarrow$ 2 is a quadratic non-residue $(mod\ x)$

$\Rightarrow z^2 \equiv 2(mod\ x)$ does not have a solution.

Hence, the equation $(x^a + 1)^m + (y^b + 1)^n = z^2$ has no solution in W.

Corollary: 3.1.1 The exponential Diophantine equation $(x^a + 1)^m + (y^b + 1)^n = w^{2k}$, where x is a prime divisor of y such that either $x \equiv 3\ or\ 5(mod\ 8)$, has no solution in W, where $a, b, m, n, y, w, k \in W$.

Proof: Let $w^k = z$, then the exponential Diophantine equation $(x^a + 1)^m + (y^b + 1)^n = w^{2k}$ becomes $(x^a + 1)^m + (y^b + 1)^n = z^2$, which has no solution by Theorem 3.1.

Corollary: 3.1.2 The exponential Diophantine equation $(x^a + 1)^m + (y^b + 1)^n = (w + 1)^{2k}$, where x is a prime factor of y such that either $x \equiv 3\ or\ 5(mod\ 8)$, has no solution in W, where $a, b, m, n, y, w, k \in W$.

Proof: Let $(w + 1)^k = z$, then the non-linear exponential Diophantine equation $(x^a + 1)^m + (y^b + 1)^n = (w + 1)^{2k}$ becomes $(x^a + 1)^m + (y^b + 1)^n = z^2$, which has no solution by Theorem 3.1.

Note: In this paper, we have shown that the Diophantine equation$(x^a + 1)^m + (y^b + 1)^n = z^2$, where x is a prime factor of y such that either $x \equiv 3\ or\ 5(mod\ 8)$ and $a, b, m, n, y, z \in W$, has no solution in W.

4. CONCLUSION: In this article, authors proved that the non-linear exponential Diophantine equation $(x^a+1)^m + \left(y^b+1\right)^n = z^2$, where x is a prime divisor of y such that either $x \equiv 3\ or\ 5(mod\ 8)$ and $a, b, m, n, y, z \in W$, has no solution in W.

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