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Left Invariant Topology on a Subgroup of a Group G Defined by Idempotents of G^*

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Abstract: The Stone - Čech compactification βS of a discrete semigroup S is the set of ultrafilters on S on which the binary operation can be extended uniquely making it a compact right topological semigroup. The idempotent elements in βS have some extra ordinary algebraic structures which are applied in combinatorial problems, especially in Ramsey theory. The idempotent elements of βS, which are not in S induced a topology on the group S which are left invariant. In this paper we study subspace topology on a subgroup T of this left invariant topological semigroup and topology induced by idempotent in the Stone - Čech compactification βT.

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1 Introduction:

Given a nonempty set X, a family $\mathfrak{F}$ of subsets of X will be a filter on X if $\emptyset \notin \mathfrak{F}$, $A \cap B \in \mathfrak{F}$ when $A, B \in \mathfrak{F}$ and if $A \in \mathfrak{F}$ and $A \subseteq B$ then $B \in \mathfrak{F}$. A filter $\mathfrak{U}$ is called an ultrafilter on X if $\mathfrak{U}$ is not properly contained in any filter. An ultrafilter $\mathfrak{U}$ which contains a singletone set $\{x_0\}$ as a member is called a principal ultrafilter. We denote this ultrafilter by x_0. For a principal ultrafilter $\mathfrak{U}$, $\bigcap \mathfrak{U} =$ singletone set. For further studies on ultrafilters we refer [3]. A compactification of a space X is a compact Hausdorff space Y such that there is a topological embedding $e: X \to Y$ with $e(X)$ dense in Y. Whereas the Stone - Čech compactification of a Tychonoff space X is a compactification βX having the property that: if $f: X \to Y$ is a continuous function on any compact Hausdorff space Y then there is a unique continuous function $\tilde{f} : \beta X \to Y$ with $\tilde{f}|_X = f$.The Stone - Čech compactification of a Tychonoff space X taking maximal Z-ideals of X as points has been studied in [6] which is equivalent for a discrete space X by taking all its ultrafilters.For a discrete semigroup $(S, \cdot)$ its Stone - Čech compactification βS consisting of all ultrafilters of S is topologized by taking the collection $\{\hat{A} : \ A \subseteq S\}$ as a base for the topology on βS, where $\hat{A} = \{p \in \beta S: \ A \in p\}$. It is a routine matter to check that the above mentioned base is a base for the closed sets also. Thus the Stone - Čech compactification βS of a discrete semigroup S is a zero- dimensional space. We can identify each element $s \in S$ with a principal ultrafilter, so that $S \subseteq \beta S$. Then the semigroup operation $\cdot$ on S can be extended to a binary operation on βS as follows: for $s \in S$, $q \in \beta S$, $s \cdot q = \widetilde{\lambda_s}(q)$ where $\widetilde{\lambda_s} : \beta S \to \beta S$ is the continuous extension of $\lambda_s : S \to S \subseteq \beta S$ defined by $\lambda_s(x) = s \cdot x$. Now for $p, q \in \beta S$, $p \cdot q = \widetilde{\rho_q}(q)$, where $\widetilde{\rho_q} : \beta S \to \beta S$ is the continuous extension of $\rho_q : S \to \beta S$ defined by $\rho_q(x) = x \cdot q$ and $(\beta S, \cdot)$ is also a semigroup. Also it is a compact right topological (follows form [2]) semigroup. From Elli's theorem [2] it is clear that $(\beta S, \cdot)$ contains an idempotent element.

We can use the following definitions and results from semigroup theory. In a semigroup $(S,\cdot)$ an element $x \in S$ is said to be invertible if there is(unique) $x' \in S$ such that $x = xx'x$ and $x' = x'xx'$ and $xx' = x'x$. The element $x\,x'$ is an idempotent element denoted by x^0. A subset $I \subseteq S$ is a left(right) ideal of S if $SI \subseteq S$ $(IS \subseteq S)$. It is called an ideal if it is both left and right ideal of S. Using Zorn's lemma we can say that every left (right) ideal contains a minimal left(right) ideal. The smallest ideal (if exists) of a semigroup $(S,\cdot)$ is denoted by $K(S)$ and it is the intersection of a minimal left ideal and a mainimal right ideal. The set of idempotents of S is denoted by $E(S)$.

If $(S,\ \cdot)$ is a semigroup with a topology τ then its topological center is $\Lambda\,(S) = \{a \in S :\ \lambda_a$ is a continuos mapping$\}$. For the Stone - Čech compactification of a discrete semigroup $(S,\ \cdot)$, $S \subseteq \Lambda\,(S)$.

A topological semigroup $(S,\ \tau)$ is a pair where S is a semigroup, τ is a topology on S and the binary operation is continuous. It is a topological inverse semigroup if S is an inverse semigroup and the map $x \to x'$ is continuous. In a topological semigroup $(S,\ \tau)$ the closure of any (left, right) ideal is also an (left, right) ideal.

A homomorphism between two semigroups $S\ and\ T$ is a mapping $\psi: S \to T$ satisfying $\psi(ab) = \psi(a)\psi(b)$. If it is bijective then it is an isomorphism. An isomorphism $: S \to T$, where $S\ \ and\ T$ are semigroups endowed with topologies is called a topological isomorphism if it is a homeomorphism also. Henceforth we will consider the discrete topology on the semigroup $(S,\ \cdot)$.

Definition 1.1 Suppose $(S,\ \ \cdot)$ is a discrete inverse semigroup, $A \subseteq S$ and $p \in \beta S$. Define $A^{-1} = \{x' :\ \ x \in A\}$, $p' = \{A^{-1} \subseteq S : A \in p\}$.

Obviously $(A^{-1})^{-1} = A$ and $(p')' = p$.

Proposition 1.2 $p \in \beta S$ if and only if $p' \in \beta S$.

Proof: Suppose $p \in \beta S$. Clearly then $\emptyset \notin p'$. Now if $A, B \in p'$ then $A^{-1}, B^{-1} \in p$. p being an ultrafilter, $A^{-1} \cap B^{-1} \in p$. Then $(A \cap B) = (A^{-1} \cap B^{-1})^{-1} \in p'$. Again if $A \in p'$ and $A \subseteq B$ then $A^{-1} \subseteq B^{-1}$ implies $B^{-1} \in p$ implies $B \in p'$. So p' is a filter on S. Suppose $\mathfrak{F}$ be a filter such that $p' \subseteq \mathfrak{F}$. Then $p = (p')' \subseteq \mathfrak{F}'$. Since p is an ultrafilter, $\mathfrak{F}' = p$ and hence $p' = \mathfrak{F}$.

Definition 1.3 Suppose $(S,\ \ \cdot)$ is a discrete semigroup, $Y(\subseteq S)$ is a subsemigroup of S. For $p \in \beta S \cap \hat{Y}$, we define, $p_Y = \{A \cap Y : A \in p\}$.

Proposition 1.4 p_Y is an ultrafilter on Y and hence $p_Y \in \beta Y$.

Proof: Since $Y \in p$, $A \cap Y \neq \emptyset$ for all $A \in p$. Suppose $B \in p_Y$ and $B \subseteq C \subseteq Y$. Then $B = B_1 \cap Y$ for some $B_1 \in p$. Then there is some $C_1 \supseteq B_1$ such that $C = C_1 \cap Y$. Clearly then $C_1 \in p$ and so $C \in p_Y$. Now if $A, B \in p_Y$ then $A = A_1 \cap Y, B = B_1 \cap Y$, for some $A_1, B_1 \in p$. p being an ultrafilter, $A_1 \cap B_1 \in p$. Then $(A \cap B) = (A_1 \cap B_1) \cap Y \in p_Y$ implies p_Y is a filter on Y. Suppose $\mathfrak{F}$ be a filter on Y such that $p_Y \subseteq \mathfrak{F}$. Suppose $\mathfrak{F}_1 = \{A \subseteq S: A \cap Y \in \mathfrak{F}\}$is a filter on S containing p. Since p is an ultrafilter, $\mathfrak{F}_1 = p$. Then $p_Y = \mathfrak{F}$, which implies p_Y is an ultrafilter on Y.

Definition 1.5 For a subsemigroup T of the semigroup $(S,\ \cdot)$ the set $\{p_T:\ p \in \beta S\}$ is a set of ultrafilters on T which will be denoted by $\beta_S T$.

Proposition 1.6 For a subsemigroup T of the semigroup $(S,\ \cdot)$, $card(\beta_S T) = card(\beta S \cap \hat{T})$.

Proof: Suppose $f: \beta S \cap \hat{T} \to \beta_S T$ be defined by $f(p) = p_T, \forall p \in \beta S \cap \hat{T}$ … … … (1).

We shall show that f is a bijection. Suppose $p, q \in \beta S \cap \hat{T}\ and\ p \neq q$. Then there is some $A \in p$ such that $A \notin q$. Then $A \cap T \in p_T$ but $A \cap T \notin q_T$ showing that f is injective. Again if $r \in \beta_S T$ then $r_S = \{A \subseteq S : A \cap T \in r\}$ is an ultrafilter in S and $(r_S)_T = r$. So f is an onto mapping.

Theorem 1.7 If T is a subsemigroup of a discrete semigroup S then the mapping $f: \beta S \cap \hat{T} \to \beta_S T\,(\subseteq \beta T)$ as defined in $(1)\ of\ proposition$ 1.6 is a homeomorphism.

Proof: Using proposition 1.6, to prove the theorem it is only to show that $f\ is\ a$ continuous open map.

If $p \in \beta S \cap \hat{T}$ and $p_T \in \hat{U} \subseteq \beta_S T$ then $U \in p_T$ and so $U = A \cap T$, for some $A \in p$. Clearly then $\hat{A} \cap \hat{T}$ is an open neighbourhood of p and $f(\hat{A} \cap \hat{T}) \subseteq \hat{U}$ implies that f is continuous at p. Now if $\hat{A} \cap \hat{T}$ is an open set in $\beta S \cap \hat{T}$ then $f(\hat{A} \cap \hat{T}) = (\widehat{A \cap T})_{\beta T}$ is an open set in βT implies f is an open map.

Henceforth we use the symbol $\hat{T}$ for $\beta S \cap \hat{T}$.

From the above theorem (1.7) it is clear that $\hat{T}$ is topologically embedded in $\beta_S T$.

Notation: For a subset A of a semigroup $(S, \ \cdot)$ if $s \in S$ then

(a) $s^{-1}A = \{t \in S: \quad s \cdot t \in A\}$

(b) $A\, s^{-1} = \{t \in S: \quad t \cdot s \in A\}$.

The following theorem follows from the continuity of $\widetilde{\lambda_s}$ and $\widetilde{\rho_q}$.

Theorem 1.8[1] For a discrete semigroup $(S, \ \cdot)$ if $x \in S, p, q \in \beta S$ then

(a) $x \cdot p = \{A \subseteq S : x^{-1}A \in p\}$

(b) $p \cdot q = \{A \subseteq S : \{x \in S: \quad x^{-1}A \in q\} \in p\}$.

Theorem 1.9 Suppose T be a sub semigroup of the discrete semigroup $(S, \ \cdot)$, $x \in T$, $p, q \in \hat{T}$. Let $\star$ be the binary operation on βT extended from $\cdot$ on T. Then

(a) $(x \cdot p)_T = x \star p_T$

(b) $(p \cdot q)_T = p_T \star q_T$

Proof : (a) Since $T \in p, x \cdot T \in x \cdot p$. Also $T \supseteq x \cdot T$. This implies $T \in x \cdot p$.

(b) Suppose $A \in \ (p \cdot q)_T$. Then $A = B \cap T$ for some $B \in p.q$. Let $C = \{x \in S: x^{-1}B \in q\}$. Then $C \in p$. Now for any $x \in T \cap C$, $x^{-1}A \in q_T$ implies $\{x \in T: x^{-1}A \in q_T\} \in p_T$. Hence $A \in p_T \star q_T$. Consequently $(p \cdot q)_T = p_T \star q_T$

Definition 1.10 A topology τ on a semigroup $(S,\cdot)$ is called a right(left) invariant topology if for every $U \in \tau$ and every $a \in S$, $Ua = \{u \cdot a : u \in U\} \in \tau$ $(aU \in \tau)$. A topology is called invariant if it is both left and right invariant.

From the definition, it is clear that, if τ on a group $(S,\cdot)$ is a right(left) invariant topology then $(S,\cdot)$ is a right(left) topological group. Indeed we get stronger relation as follows from the theorem:

Theorem 1.11 A topology τ on a group $(G,\cdot)$ is left invariant if and only if for every $a \in G$, λ_a is a homeomorphism, where $\lambda_a: G \to G$ is defined by $\lambda_a(g) = \ a \cdot g \ \ \forall \ g \in G$.

Proof: If τ is left invariant then for each $a \in G$ and each open set U, $\lambda_a(U) = aU$ is open implies λ_a is an open map.Again for any open set U of G, $\lambda_a^{-1}(U) = \ a^{-1}U$ is also open showing that λ_a is continuous. λ_a being a bijective mapping, it is a homeomorphism. Conversely, suppose λ_a is a homeomorphism. Then τ is left invariant follows from the fact that $\lambda_a$is an open mapping.

For constructions of left invariant topologies on an infinite group we use the following definitions from **[1].**

Definition 1.12 If G is a discrete group with identity e and $C \subseteq G^*$ is a finite subsemigroup then define

(a) $\tilde{C} = \{x \in \beta G: xC \subseteq C\}$.

(b) ϕ is the filter of subsets U of G for which $C \subset cl(U)$.

(c) $\tilde{\phi}$ is the filter of subsets U of G for which $\tilde{C} \subset cl(U)$.

It is clear that $\tilde{C}$ is a semigroup and $e \in \tilde{C}$. Also $\phi = \cap\, C$ and $\tilde{\phi} = \cap\, \tilde{C}$.

Definition 1.13 Suppose ϕ and $\tilde{\phi}$ are defined as in Definition 1.12. For any $U \in \phi$, let $\widetilde{U} = \{a \in G: aC \subseteq \ cl(U)\}$.

Then $e \in \widetilde{U} \in \tilde{\phi}$ and the family $\{\widetilde{U}: U \in \phi\}$ is a base for the filter $\tilde{\phi}$.

It can be easily verified that the family $\mathfrak{B} = \{a\,\widetilde{U}: a \in G, U \in \phi\}$ is a base for some topology on G.

Theorem 1.14 [1] For any discrete group $(G,\cdot)$ there is a left invariant topology generated by $\mathfrak{B}$, as defined in Def 1.13 on G for which $\tilde{\phi}$ is the filter of neighbourhoods of e. Furthermore, G will be zero dimensional if $xC = C$ for every $x \in \tilde{C}$,

Theorem 1.15 [1] If $xC = C$ for every $x \in \tilde{C}$, then the following are equivalent:
(a) The topology defined in Theorem 1.14 is Hausdorff.
(b) $\{x \in G: xC \subseteq C\} = \{e\}$.
(c) $\{e\} = \cap\,\tilde{\phi}$.

Theorem 1.16 [1] If G has no nontrivial finite subgroup, then $\cap\,\tilde{\phi} = \{e\}$.

Example 1.17 Suppose p be an idempotent in $\mathbb{Q}^*$. Define $\tilde{C}_p = \{x \in \beta\mathbb{Q}: x + p = p\}$.Then $\phi = \{U \subseteq \mathbb{Q}: A \cap U \neq \emptyset \; for\ all\ A \in p\} = p$ from the properties of an ultrafilter. Now, for any $U \in \phi$, $\widetilde{U} = \{r \in \mathbb{Q}: \ r + p \in cl(U)\} = \{r \in \mathbb{Q}: (-r + A) \cap U \neq \emptyset \; for\ all\ A \in p\}$.The family $\{a + \widetilde{U}: a \in \mathbb{Q},\ U \in p\}$ is a base for the left invariant topology on $\mathbb{Q}$ generated by the idempotent p. As $\mathbb{Q}$ is commutative under addition, it follows immediately that the topology is an invariant topology on $\mathbb{Q}$.

From Theorem 1.14 and 1.15 the following theorem follows immediately:

Theorem 1.18 If G is a discrete group with identity e, there is a left invariant topology on G with a basis of clopen sets such that $\tilde{\phi}$ is the filter of neighbourhoods of e. Further more this topology is Hausdorff if G has no nontrivial finite subgroups.

Definition 1.19 [6] A topological space is said to be extremally disconnected if closure of every open set is open.

Theorem 1.20[6] If S is a discrete space then βS is extremally disconnected.

Lemma 1.21[1] For a discrete semigroup $(S,\ \cdot)$ if p is an idempotent in βS then $S \cdot p$ is extremally disconnected subspace of βS.

Theorem 1.22 [1] Suppose p be an idempotent in S^* where $(S, +)$ is an infinite discrete group with the identity 0. Then there is a left invariant zero dimensional Hausdorff topology on S such that the filter $\tilde{\phi}$ of neighbourhoods of 0 consists of $U(\subseteq S)$ for which $\{x \in \beta S: x + p = p\} \subseteq cl_{\beta S}U$. This topology is extremally disconnected and is the same as the weak topology on S induced by the mapping $(\rho_p)_{|S}: S \to S^*$. Furthermore, if $(\rho_p)_{|S}$ and $(\rho_q)_{|S}$ induce the same topology on S then $p + \beta S = q + \beta S$.

The topology as in Theorem 1.22 is called the left invariant topology on S induced by the idempotent p and will be denoted by τ_p.

2. Topology on a subgroup of a discrete group G induced by the idempotents

In this section we study the left invariant topologies induced on the subgroup $(H, +)$ by idempotents in H^* by two ways. One is by the subspace topology of G which is induced by an idempotent element p of G^* and the other is the right invariant topology on H induced by the idempotent p_H of H^*.

Theorem 2.1 Let $(G,\ \cdot)$ be a group endowed with a left invariant topology τ. Then any subgroup H of the group $(G,\ \cdot)$ is also left invariant with respect to the subspace topology of τ.

Proof: Suppose $U \in \tau_H$, the subspace topology on H and $a \in H$. Then $U = V \cap H$ for some $V \in \tau$. Since τ is left invariant $a \cdot V \in \tau$. We first show that $a \cdot U = (a \cdot V) \cap H$. Clearly $a \cdot U \subseteq (a \cdot V) \cap H$. Suppose $x \in (a \cdot V) \cap H$. Then $x = a \cdot v$ for some $v \in V$. This implies $v = a^{-1} \cdot x \in H$. Thus $v \in V \cap H$ showing that $x \in a \cdot (V \cap H)$ which implies $(a \cdot V) \cap H \subseteq a \cdot U$. Consequently $a \cdot U = (a \cdot V) \cap H$ and so $a \cdot U \in \tau_H$.

Theorem 2.2 Let $(Y,\ \sigma)$ be a topological space and τ be the topology on X induced by the mapping $f: X \to Y$. Let S be a nonempty subset of X. Then the subspace topology of τ on S is same as the topology induced by $f|_S: S \to Y$ on S.

Proof: Suppose U be a basic open set of $(S,\ \tau_S)$, the subspace topology on S. Then $U = V \cap S$ for some basic open set $V \in \tau$. Since τ is induced by the mapping $f: X \to Y$, $V = \bigcap_{i=1}^{n} f^{-1}(W_i)$, for some $W_i \in \sigma$ and for some $n \in \mathbb{N}$. Then $U = (\bigcap_{i=1}^{n} f^{-1}(W_i)) \cap S = \bigcap_{i=1}^{n}(f^{-1}(W_i) \cap S) = \bigcap_{i=1}^{n} f|_S{}^{-1}(W_i)$, which implies that U is a basic open set with respect to the topology induced by $f|_S: S \to Y$ on S. Again if U is a basic open set with respect to the topology induced by $f|_S: S \to Y$ on S, then $U = \bigcap_{i=1}^{n} f|_S{}^{-1}(W_i) = \bigcap_{i=1}^{n}(f^{-1}(W_i) \cap S) = (\bigcap_{i=1}^{n} f^{-1}(W_i)) \cap S = V \cap S$ where $V = \bigcap_{i=1}^{n} f^{-1}(W_i)$ is a basic open set of τ. Consequently the subspace topology of τ on S is same as the topology induced by $f|_S: S \to Y$ on S.

Theorem 2.3 Let $(G,+)$ be an infinite group with the identity 0 and H be an infinite subgroup of G. Then $p \in G^* \cap \widehat{H}$ is an idempotent element of βG if and only if p_H(as defined by definition 1.3) is an idempotent element of $\beta H \setminus H$.

Proof : Suppose p be an idempotent element of βG **.** Since $p \in G^*$, p is not a principal ultrafilter and so p_H is a non principal ultrafilter of βH. Then from Theorem 1.9 $p_H = (p+p)_H = p_H + p_H$, it follows that p_H is an idempotent element of $\beta H \setminus H$.

Conversely, let p_H be an idempotent element of $\beta H \setminus H$. Then $p \in G^*$ and $(p+p)_H = p_H + p_H = p_H$. Then from Proposition 1.6 it follows that $p+p=p$. Consequently $p \in G^* \cap \widehat{H}$ is an idempotent element of βG.

Theorem 2.4 Let $(G,+)$ be an infinite group with the identity 0 and H be an infinite subgroup of G. Suppose τ_p and τ_{p_H}be the left invariant topologies on G and H induced by an idempotent p of $G^* \cap \widehat{H}$and p_H of H^* respectively. Then τ_{p_H} is the subspace topology of G on H, ie $(\tau_p)_H = \tau_{p_H}$.

Proof : Suppose $\phi = \{U \subseteq G\colon A \cap U \neq \emptyset \,\forall A \in p\}$. Then from theorem 1.22, $\tilde{\phi} = \{U \subseteq G\colon \{x \in \beta G\colon x+p=p\} \subseteq cl_{\beta G}U\}$ is a filter base for neighbourhoods of 0 in G with respect to the topology τ_p . Therefore $\tilde{\phi}|_H = \{U \cap H\colon U \subseteq G \text{ and } \{x \in \beta G\colon x+p = p\} \subseteq cl_{\beta G}U\}$ is a filter base for neighbourhoods of 0 in H with respect to the subspace topology. Now, if $K \in \tilde{\phi}|_H$, then $K = U \cap H$ for some $U \subseteq G$ *such that* $\{x \in \beta G\colon x+p=p\} \subseteq cl_{\beta G}U$. Suppose $x \in \beta H$ be such that $x + p_H = p_H$. Then $x+p=p$, where $x \in \widehat{H}$. So $\{x \in \beta H\colon x+p_H = p_H\} \subseteq \{x \in \beta G\colon x+p=p\}$. Again $cl_{\beta H}K = cl_{\beta H}(U \cap H) = cl_{\beta G}U \cap \beta H$. So $\{x \in \beta H\colon x+p_H = p_H\} \subseteq cl_{\beta H}K$. Therefore $K \in \tilde{\psi}$, where $\tilde{\psi}$ is the filter base for neighbourhoods of 0 in H with respect to the topology τ_{p_H} induced by the idempotent p_H on H. So $\tilde{\phi}|_H \subseteq \tilde{\psi}$. Therefore $(\tau_p)_H \subseteq \tau_{p_H}$.

Again, if $\tilde{\psi}$ is the filter base for neighbourhoods of 0 in H with respect to the topology τ_{p_H} induced by the idempotent p_H on H then from theorem 1.22, for any $K = U \cap G \in \tilde{\psi}$, $\{x \in \beta H\colon x+p_H = p_H\} \subseteq cl_{\beta H}K$. This implies $\{x \in \beta G\colon x+p=p\} = \beta G \cap \{x \in \beta G\colon x+p_H = p_H\} \subseteq cl_{\beta G}K \cap \beta H = cl_{\beta G}U \cap \beta H$. This implies $U \in \tilde{\phi}$. So $\tilde{\psi} \subseteq \tilde{\phi}|_H$. Consequently $\tau_{p_H} \subseteq (\tau_p)_H$. Hence $(\tau_p)_H = \tau_{p_H}$.

From theorem 1.22 and theorem 2.4 we can conclude the main result:

Corollary 2.5 Let H be an infinite subgroup of $(G,+)$ and p be an idempotent in $G^* \cap \widehat{H}$. If $(\rho_p)_{|G}\colon G \to G^*$ and $(\rho_{p_H})_{|H}\colon H \to G^*$ are the right translations respectively on G and H with respect to p and p_H then the subspace of the weak topology on G induced by $(\rho_p)_{|G}\colon G \to G^*$ on H is same as the weak topology on H induced by $(\rho_{p_H})_{|H}\colon H \to G^*$.

Definition 2.6 For a discrete semigroup $(S,\cdot)$ if $p,q \in E(\beta S)$, the set of idempotents of βS then define

(a) $p \leq_L q$ if $p = p \cdot q$

(b) $p \leq_R q$ if $p = q \cdot p$

(c) $p \leq q$ if $p = p \cdot q = q \cdot p$

Theorem 2.7 For any group $(G,+)$ and for any two idempotents p and q in βG the following statements are equivalent:

(a) $p \leq_L q$

(b) The function $\psi = (\rho_p)_{|G}\left((\rho_q)_{|G}\right)^{-1} : q+G \to p+G$ is continuous.

(c) The topology induced on G by $(\rho_q)_{|G}$ is finer than that one induced by $(\rho_p)_{|G}$.

Proof: (a) ⇒ (b):For any $g \in G$, suppose $\{q+g_\alpha\}_{\alpha \in \Lambda}$ be any net converging to $q+g$. Then $\{p+g_\alpha\}_{\alpha \in \Lambda} = \{p+q+g_\alpha\}_{\alpha \in \Lambda}$ converges to $p+q+g = +g$. So ψ is continuous at $q+g$.

(b) ⇒ (a): Suppose ψ is continuous. Then $p = \psi(q) = \psi(q+q) = \lim_{g \to q} \psi(q+g) = \lim_{g \to q}(p+g) = p+q$ implies $p \leq_L q$.

(b) ⇒ (c): Given that $\psi = \left(\rho_p\right)_{|G} \left(\left(\rho_q\right)_{|G}\right)^{-1}$ is continuous. Suppose τ_q be the topology on G induced by $\left(\rho_q\right)_{|G}$. Then $\left(\rho_p\right)_{|G} = \psi \circ \left(\rho_q\right)_{|G}$ is also a continuous mapping from G to $p + \beta G$. Since the topology τ_p induced by $\left(\rho_p\right)_{|G}$ is the weakest topology for which $\left(\rho_p\right)_{|G}$ is continuous, τ_q is finer than τ_p .

(c) ⇒ (b): This is obvious.

Corollary 2.8 Suppose H be an infinite subgroup of a group $(G, +)$ and p and q are two idempotents in $G^* \cap \widehat{H}$. Then the topology induced on G by $\left(\rho_q\right)_{|G}$ is finer than that one induced by $\left(\rho_p\right)_{|G}$ if and only if the topology induced on H by $\left(\rho_{q_H}\right)_{|H}$ is finer than that one induced by $\left(\rho_{p_H}\right)_{|H}$.

Proof: Since p and q are two idempotents in $G^* \cap \widehat{H}$, p_H and q_H are two idempotents in H^*.

Suppose the topology induced on G by $\left(\rho_q\right)_{|G}$ is finer than that one induced by $\left(\rho_p\right)_{|G}$. Then from theorem 2.7 $p \leq_L q$ in $E(\beta G)$. Therefore $p = p + q$. From theorem 1.9 $p_H = p_H + q_H$ in βH . This implies $p_H \leq_L q_H$ in $E(\beta H)$. Again from theorem 2.7 the topology induced on H by $\left(\rho_{q_H}\right)_{|H}$ is finer than that one induced by $\left(\rho_{p_H}\right)_{|H}$. The converse part can be proved in a similar way.

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