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Positivity and Locally Attractivity Solutions of Quadratic Integral Equation of Fractional order in Partially Ordered Normed Linear Space

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Abstract: In this paper we proved that Positive and existence the solution for a fractional order nonlinear quadratic functional integral equation in partially ordered in Banach algebra. Also we proved that the solution of the equation in locally attractivity and Positivity solutions.

Keyword: Quadratic Integral Equation, Norm Linear Space, Hybrid Fixed Point Theorem, Banach Algebra.

1. Introduction

Nonlinear quadratic functional integral equations is developing with the help of several tools of functional analysis , fixed point theory and topology, with bounded intervals have been studied of extensively literature and properties with include existence uniqueness stability boundedness, monotonically and extremely solutions.[4] The hybrid fixed point theorem in Banach algebra find numerous applications in the theory of nonlinear functional differential and integral equations see Dhage [3] and there references but they formulation of functional analytic methods in particular fixed point theory partially ordered normed linear space developed in integral equations[15]. The theory of the integral equation of fractional order plays a very important role in describing the some real world problem.[9,19,24,25] It has recently received a lot of attention and now constitute a significant branch of nonlinear analysis. In recent year differential and integral equations of fractional order have found applications in physics, chemistry, mechanics, economics, and other fields. [19-24]. Meanwhile numerous research papers and monographs have appeared devoted to differential and integral equation of fractional orders.[15]..

In this paper we investigate the monotonic solution for a quadratic integral equation of fractional order in QIEF (1.1) partially ordered normed linear space by using hybrid fixed point theorem due to B.C. Dhage.

Given Let R be the line

$$x(t) = K\left(t, x(\mu(t))\right) + \left[f\left(t, x(\theta(t))\right)\right]\left[q(t) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{v(t,s)}{(t-s)^{\alpha-1}} g\left(s, x(\eta(s))\right) ds\right] \quad (1.1)$$

For all $t \in R_+$ and $\propto \in [0,1]$ where $\mu, \theta, \eta: R_+ \to R$, $q: R_+ \to R$ and $f, g, K: R_+ \times R \to R_+$. and $v: J \times J \to R$ continuous function, sand Γ is the Euler gamma function.

By the solution of the FIE (1.1) we mean the function $x \in C(R_+, R)$, that satisfies the equation (1.1), where $C \in (R_+, R)$ is the continuous of real valued function on R_+.

Observe that the function above integral equations has rather general from an include several classes of the functional integral equations considered in the literature. And their references [7].In this paper we prove the existence of extermal and locally attractivity of solution for the above nonlinear functional integral equation. The main tool are used hybrid fixed point theorem established by Dhage iteration method under weak partial Lipschtiz and compactness type conditions [5-12]. In fact, our result in this paper is motivated by the extension of the work of S. Dhage and Dhage [11-13] also, we proceed and generalize the result obtained in the papers [13].

In the next section we collect the preliminary, definition and auxiliary results that will be used follows.

2.0 Auxiliary results

In this section we give definition notations, hypothesis and preliminary tools which be used in the sequel

Definition 2.1[2]: Let E be denote a partially ordered real normed linear space with ordered relation $\preccurlyeq$ and the norm $\|\bullet\|$ which is addition and scalar multiplication by positive real numbers are presented by $\preccurlyeq$. A relation in E is said to be the partial ordered norm space if its satisfy the following properties.

Let $a, b, c, d \in E$ and $\lambda \in R$, then

a) Reflexivity $a \preccurlyeq a \ \forall a \in E,$

b) Antisymmetry $a \preccurlyeq b \ and \ b \preccurlyeq a \ then \ implies \ a \Rightarrow b \quad \forall a, b \in E$

c) Transitivity $a \preccurlyeq b \ and \ b \preccurlyeq c \quad then \ implies \quad a \preccurlyeq c \quad \forall a, b, c \in E.$

d) Order linearity $a \preccurlyeq b \ and \ b \preccurlyeq d \quad then \ implies \quad a + c \preccurlyeq b + d.$

Two element x and y in E are said to be comparable if either relation $x \preccurlyeq y$ or $y \preccurlyeq x$ holds. A non-empty subset C of E is called chain or totally ordered if all the element of C are comparable is known that E is regular if $\{x_n\}$ is non-decreasing sequence in E such that $x_n \to x^*$ as $n \to \infty$ then $x_n \leq x^*$ for all $n \in N$. Order Banach space and operator theoretic technique are given in papers of Dhage [14] Lakshmikanatham and Heikkila [21] and there references.

We need the following definitions (see Dhage [5-8]) and there references there in)

Definition2.2 [2]: A mapping $H: E \to E$ is called isotone or monotone non-decreasing if it preserves the order relation $\preccurlyeq$ that is if $x \preccurlyeq y \Longrightarrow Hx \preccurlyeq Hy \ \forall x, y \in E$. Similarly H is called monotone non increasing if $x \preccurlyeq y \Longrightarrow Hx \succcurlyeq$

$Hy \ \forall x, y \in E$.Finally H is called simply monotone if it is either monotone non-decreasing or monotonic non-increasing on $\preccurlyeq$, E.

Defination2.3 (Dhage[18]) : A mapping $H: E \to E$ is called partially continuous at a point $a \in E$ if for $\in > 0$ there exists $a\delta > 0$ such that $\|Hx - Ha\| < \in$ whenever x is comparable to a and $\|x - a\| < \delta$. H is called partially continuous on E if it is partially continuous at every point of it. It is clear that if H is partially continuous on E. then it's continuous on every chain C contained in E.

Defination2.4[7]: A nonempty subset S of the partially ordered Banach space and E is called Partially compact if every chain C in S is bounded and relatively compact subset of E. then H is called the uniformly partially bounded and compact if H is partially completely continuous function then it is called as partially completely continuous function on E.

Remark 2.5: H is the non-decreasing operator on E into itself then H is partially bounded or partially compact if H(C) is bounded or relatively compact subset of E for each chain C in E.

Definition2.6 [11]: the order relation $\preccurlyeq$ and metric d is non-empty set E are said to be D-Compatible if $\{x_n\}_{n\in N}$ is a monotone sequence in E and if subsequence $\{x_{n_k}\}_{n\in N}$ of $\{x_n\}_{n\in N}$ converges to x^* similarly, given a partially ordered normed linear space$(E, \preccurlyeq, \|\bullet\|)$, the order relation $\preccurlyeq$ *and* $\|\bullet\|$ are said to be D-compatible if $\preccurlyeq$ and the metric d defined through the norm $\|\bullet\|$ are D-compatible.

Theorem2.1 (Dhage [6]): Let S be a non-empty ,closed and partially bounded subset of regular partially ordered complete normed linear space$(E, \preccurlyeq, \|\bullet\|)$ such that order relation $\preccurlyeq$ and the norm $\|\bullet\|$ are compatible . Let $H: S \to S$ be the partially continuous non-decreasing and partially K-set contraction with K < 1.If there exist an element $x_0 \in S$ such that $x_0 \preccurlyeq Hx_0 \ or \ x_0 \succcurlyeq Hx_0$ then H has fixed point x^* and sequence $\{H^n x_0\}$ of successive iteration converges to x^*.

Remark2.7: The regularity of E and the partial continuity of H may be replaced strongly continuity condition of the operator H on E.

The set R of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\bullet|$ has the property. Similarly the finite dimensional Euclidean space R^n with usual component wise order relation and the standard norm possesses the D-compatibility property. Similarly every partially compact subset of the space $C(R_+R)$ with usual order relation defined by $x \leq y$ if and only if $x(t) \leq y(t) \quad \forall \quad t \in R_+$ with usual standard norm $\|\bullet\|$ defined by $\|x\| = sup_{t\in R_+}|x(t)|$ are compatible.

Definition 2.8[17]: A mapping $\varphi: R_+ \to R_+$ is called dominating function or D-short D-function if it is an upper semi continuous and monotonic non-decreasing function satisfies $\varphi(0) = 0$.

Definiation2.9:let$(E, \preccurlyeq, \|\bullet\|)$ be a partially ordered normed linear space A mapping $H: E \to E$ is called partially nonlinear D-Lipschitz or partially nonlinear D-lipschtiz if there exist an upper semi-continuous non-decreasing function $\varphi: R_+ \to R_+$ such that

$$\|Hx - Hy\| \leq \varphi\|x - y\| \qquad (2.1)$$

For all comparable elements $x, y \in E$ where $\varphi(0) = 0$ if $\varphi(r) = kr,\ k > 0$ then H is called partially lipschitz with lipschitz constant K.

If K < 1, H is called partially contraction with contraction constant K.Finally H is called nonlinear D-contraction if it is nonlinear D-Lipschtiz with $\varphi(r) < r$ for $r > 0$.

Definition2.10: Let $(E, \preccurlyeq, \|\bullet\|)$ be partially normed linear algebra Denote

$E^+ = \{x \in E | x, x \succcurlyeq \theta, where\ \theta\ is\ the\ zero\ element\ of\ E\}$ and

$$K = \{E^+ \subset E | uv \in E^+\ \ for\ all\ u, v \in E^+\} \qquad (2.2)$$

The element of K is called the positive vectors of the normal linear algebra E.

Lemma2.1 (Dhage[8]): If $u_1, u_2, v_1, v_2 \in K$ are such that,

$u_1 \preccurlyeq v_1$ and $u_2 \preccurlyeq v_2$ then $u_1, u_2 \preccurlyeq v_1, v_2$.

Defination2.11: An operator $H: E \to E$ is said to be positive if the range $R(H)$ of H is such that $R(H) \subset K$ for any two chain $\wp_1\ and\ \wp_2$ in E denote,

$$\wp_1 \wp i_2 = \{x = c_1 c_2, \wp_1\ and\ \wp_2\}$$

Theorem2.2 [7]: Let S be non-empty partially and closed subset of a regular partially ordered complete algebra $(E, \preccurlyeq, \|\bullet\|)$ such the order relation $\preccurlyeq$ and norm $\|\bullet\|$ are compatible in every chain $\wp$ of S. Let $A, B: S \to K$ and $C: E \to E$ being two non-decreasing operators, such that

a) A and C are partially nonlinear D-function φ_A andφ_C respectively,

b) B is partially continuous and compact

c) $A(x).B(x) + C(x)\ \ \forall x \in S$

d) $M\varphi_A(r) + \varphi_C(r) < r, r > 0\ where\ M = \|B(S)\|\ and$

f) There exist an element $x_0 \in S$ such that $x_0 \preccurlyeq A(x_0)B(x_0) + C(x_0)$ or

$$x_0 \succcurlyeq A(x_0)B(x_0) + C(x_0)$$

Then the operator equation is $A(x)B(x) + C(x) = x$ has solution x^* in S and the sequence$\{x_n\}$ of successive iteration defined by $x_{n+1} = A(x_n)B(x_n) + C(x_n)$,

$n = 0, 1, 2\ldots$ Converges monotonically to x^*.

Remark 2.11: The compatibility of the order relation $\preccurlyeq$ and the norm $\|\bullet\|$ in every compact chain of E holds if every partially compact subset of E possesses the compatibility property with respect to $\preccurlyeq\ and\ \|\bullet\|$.

Remark 2.12: The hypothesis (a) of theorem (2.2) implies that the operator A and C are partially continuous and consequently all three operators A, B and C in the partially continuous on E it replaced strongly continuity conditions of the operators A, B and $C\ on\ E$.

3.0 Existence theory

Defination3.1: We place the problem QFIE (1.1) in the function space $BC(R_+, R)$ of continuous real-valued function $x = x(t)$, defined continuous and bounded on R_+

$$\|x\| = \sup_{t \in J} |x(t)| \quad For\ \ all\ \ t \in R_+ \qquad (3.1)$$

Clearly $BC(R_+, R)$. In Banach algebra with the maximum norm $\|\bullet\|$ We define the order relation $\leq$ in $BC(R_+, R)$ with help of cone K in$BC(R_+, R)$ ie

$K = \{x \in BC(R_+,R) \mid x(t) \geq 0 \ \forall \, t \in R_+ \ \}$ (3.2)

Clearly K is positive and norm cone in$BC(R_+,R)$. Let $L^1(R_+,R)$ we denote the lebseguge measurable real-valued function defined on R_+ by $\|\bullet\|_{L^1}(R_+,R)$ defined by

$$\|x\|_{L^1} = \int_0^T |x(S)| \, ds \qquad (3.3)$$

Define the order relation $\leq$ in $BC(R_+,R)$ as follows. Let $x, y \in BC(R_+,R)$ then $x \leq y$ we mean $x(t) < y(t) \text{ for all } t \in R_+$. It is clear that $(BC(R_+,R), \preccurlyeq, \|\bullet\|)$ is regular and the order relation $\leq$ and the norm $\|\bullet\|$ are compatible in$BC(R_+,R)$.

In order to introduce further the concept used in this paper let's assume that $\boldsymbol{\Omega}$ is nonempty chain of space$BC(R_+,R)$. Moreover let Q be an operator defined by $\boldsymbol{\Omega}$ with the value in $BC(R_+,R)$.

Consider the operator equation of the form

$$x(t) = Q(t), \quad t \in R_+ \qquad (3.4)$$

Definition 3.2[2]: We say that the approximate solution of the equation (3.4) are called locally attractive if there exists an open ball $B(x_0,r)$ in the space $BC(R_+,R)$ such that arbitrary approximate solution $x = x(t) \, and \, y = y(t)$ of the equation (3.4) belonging $\bar{B}(x_0,r) \cap \Omega$ we have,

$$\lim_{t\to\infty}[x(t) - y(t)] = 0 \qquad (3.5)$$

In case when limit (3.4) is uniform with respective to the set $\bar{B}(x_0,r) \cap \Omega$

i.e. when for each $\in > 0$ there exists $T > 0$ such that

$$|x(t) - y(t)| \leq \in \qquad (3.6)$$

For all$x, y \in \bar{B}(x_0,r) \cap \Omega$ being the approximate solution of (3.4) and for $t \geq T$ we say that the approximate solutions of the operator equation (3.4) are uniformly locally ultimately attractive defined on R_+. .

Defination3.3[8]: We say that the approximate solution of the equation (3.4) are locally asymptotically stable to the line $x(t) = c$ for all $t \in R_+$ if there exist an open ball $B(x_0,r)$ in the space$BC(R_+,R)$ such that for arbitrary approximate solution $x = x(t)$ of the equation (3.4) belonging to $\bar{B}(x_0,r) \cap \Omega$ we have that,

$$\lim_{t\to\infty}[x(t) - c] = 0 \qquad (3.7)$$

In case when limit (3.4) is uniform with respect to the set $\bar{B}(x_0,r) \cap \Omega$,

i.e. when each $\in > 0$ there exist $T > 0$ such that,

$$|x(t) - c| \leq \in \qquad (3.8)$$

for all $x \in \bar{B}(x_0,r) \cap \Omega$ the approximate solution (3.4) and $t \geq T$, we say that the approximate solution of the operator equation (3.4) are uniformly locally asymptotically stable to the line $x(t) = c$ defined on R_+.

lemma3.1 (Dhage[7]): Let $(BC(R,R_+), \preccurlyeq, \|\bullet\|)$ be partially ordered Banach Space with the norm $\|\bullet\|$ the order relation $\leq$ is defined by (3.1) and (3.4) respectively. Then every partially compact subset S of $BC(R,R_+)$ is D-compatible in every compact chain C in S.

Theorem3.1 (Arezela- Asclio Theorem)[7]: If every uniformly bounded and equicontinuous sequence $\{f_n\}$ of function $BC(R,R_+)$then it has a convergent subsequence.

Definition3.4: A mapping $\theta: J \times R_+ \to R_+$ is said to satisfy condition of

$L^1 - Caratheodory$ if ,

1) $t \longmapsto g(t,x,y)$ is measurable for each $x \in R$

2) $t \longmapsto g(t,x,y)$ is continuous almost everywhere for $t \in J$

3) for each real number $r > 0$ there exists a function $h \in L^1(R_+,R)$ such that $|g(t,x,y)| \leq h_r(t)$ a.e. , $t \in R_+$ for all $x \in R$ with $|x| < r$ Finally, a caratheodory function $h(t,x)$ is called $L_R^1 -$ caratheodory if

4) There exists function $h_1 \in L^1(R_+,R)$ such that $|h_1(t,x,y)| \leq h_1(t)$

a.e. $t \in R_+$ for all $x, y \in R_+$ for convenience the function h_1 is referred to as bound function h.

The equation (1.1) will be considered under the following assumptions.

(H_0) The functions f is nonnegative on $J \times R_+ \times R$.

(H_1) The function D-function ψ_f such that,

$0 \le f(t,x_1,x_2) - f(t,y_1,y_2) \le \psi_f(\max\{x_1 - y_1, x_2 - y_2\})$For all $t \in R_+$ and $x_1, x_2, y_1, y_2 \in R_+$, $x_1 \ge y_1, x_2 \ge y_2$.

(H_2) There exist a constant $M_f > 0$ such that, $0 \le f(t,x,y) \le M_f, for\ all\ t \in R_+\ \ and\ x,y \in R_+$

(H_3) $g(t,x,y)$ is non-decreasing in $x\ and\ y\ for\ all\ t \in R_+$.

(H_4) $The\ function\ g$ define as L –caratheodoery function$g: J \times R \to R_+$.

(H_5) The function D-function ψ_f such that,

$$0 \le K(t,x_1,x_2) - K(t,y_1,y_2) \le \psi_K(\max\{x_1 - y_1, x_2 - y_2\})$$

For all $t \in R_+$ and$x_1, x_2, y_1, y_2 \in R_+$ $x_1 \ge y_1$, $x_2 \ge y_2$.

(H_6) There exists a constant $M_K > 0$ such that $|K(t,x,y)| \le M_k$ for all $t \in R_+$ and $x \in R_+$.

(H_7) The function v is continuous on $R_+ \times R$. Moreover $v = \sup_{t,s \in J} |v(t,s)|$

(H_8)The function $g: R_+ \times R \to R$ is continuous .moreover, there exists a function

$m: R_+ \to R_+$being continuous on R_+ and function $h: R_+ \to R_+$ being continuous on $R_+\ with\ h(0) = 0$ and such that, $|g(t,s,x) - g(t,s,y)| \le m(t)h(|x-y|)$

For all $t, s \in R_+$, such that $s \le t$ and for all $x, y \in R$ for further,

$|h(t,x,y)| \le h(t), (t,x,y) \in R_+$ Where $\lim_{t\to\infty} \int_0^t \frac{v(t,S)}{(t-s)^{1-\alpha}} h(s) ds = 0$

It follows we will assume additionally that the following conditions satisfied

(H_{10}) The uniform continuous function $v: R_+ \to R_+$ defined by the formula

$$v(t) = \lim_{n\to\infty} \int_0^t \frac{h_r(S)}{(t-s)^{1-\alpha}}\, ds$$

Is bounded on R_+ and vanish at infinity that is $\lim_{t\to\infty} v(t) = 0$.

Remark 3.1: note that (H_1) and (H_9) holds then there exist constant $K_1 > 0$ and $K_2 > 0$ such that $K_1 = sup\{q(t): t \in R_+\}$,

$$K_2 = sup \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h_r(S)}{(t-s)^{1-\alpha}}\, ds$$

Theorem 3.2: Assume that hypothesis (H1)-(H9) holds. If,

$$\left(\frac{VT^{\alpha-1}\|h\|_L}{\Gamma(\alpha)}\right)\psi_f(r) + \psi_k(r) \le r\ , r > 0,$$

Then the functional FIE (1.1) has at list one solution $x = x(t)$ in the space $BC(R_+, R)$ and sequence $\{x_n\}$of successive approximations defined by

$$x_{n+1}(t) = K\left(t, x_n(\mu(t))\right) + \left[f\left(t, x_n(\theta(t))\right)\right]$$
$$+\left[q(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(t,s)}{(t-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds\right] \qquad (3.9)$$

For each $n \in N$ with $x_0 = x$ and $x_n(t) = max_{0\le\xi\le t} x_n(\xi)$ converges monotonically to x^*, Moreover, the comparable solutions of the QFIE (1.1) are uniformly locally ultimately attractive defined on R_+.

Proof: we seek that the solution of the QFIE (1.1) in the space $BC(R_+, R)$.consider from the lemma (3.1) follows that every compact chain in E possesses the compatibility property with respect to the norm $\|\bullet\|$ and order relation $\le$ in E.

Define three operators A, B and C on E by

$$Ax(t) = f\left(t, x(\theta(t))\right) \quad forall\ t \in R_+ \qquad (3.10)$$

$$Bx(t) = q(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(t,s)}{(t-s)^{\alpha-1}} g\left(s, x(\eta(s))\right)\ , t \in R_+ \qquad (3.11)$$

$$Cx(t) = K\left(t, x(\mu(t))\right) \quad forall \quad t \in R_+ \tag{3.12}$$

Then the equation (1.1) is transformed in to an operator equation as ,

$$Cx(t) + Ax(t)Bx(t) = x(t),\ t \in J \tag{3.13}$$

We shall the operator A B and C satisfy all condition of theorem (2.2).This will be achieved in the series.

From the continuity of the integral and hypothesis (H_0)-(H_{10}) it follows that A and B define map $A, B: E \to K$ with equivalent to the operator equation to the fixed point $Ax(t)Bx(t) + Cx(t) = x(t)$ $forall\ t \in R_+$ we shall show the operators A, B and C satisfy all the condition of theorem (2.1) is achieved in the series of following steps;

Step I: A, B and C are non-decreasing on E.

Let $x, y \in E$ be such that $x \geq y$ then $t \in R_+$ then hypothesis (H_1)-(H_7), we obtain

$$Ax(t) = f\left(t, x(\theta(t))\right) \leq f\left(t, y(\theta(t))\right) \leq Ay(t)$$

$$Bx(t) = q(t) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{v(t,s)}{(t-s)^{\alpha-1}} g\left(s, x(\eta(s))\right) ds$$

$$\leq q(t) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{v(t,s)}{(t-s)^{\alpha-1}} g\left(s, y(\eta(s))\right) ds$$

$$\leq By(t)$$

And $Cx(t) = K\left(t, x(\mu(t))\right) \leq K\left(t, y(\mu(t))\right) \leq Cy(t)$ for all $t \in R_+$.

Thus A, B and C are non-decreasing positive operators on E into itself.

Step II: A and C are partially bounded and Partially D-Lipschtiz on E.

Let$x \in E$ be arbitrary, then by the hypothesis (H_1) and (H_2)

$$|Ax(t)| \leq |f(t, x(\emptyset(t))| \leq M_f$$

For all $t \in R_+$taking the superemum over, we obtain $\|Ax\| \leq M_f$ and so A is bounded. This further implies that is partially bounded on E. Similarly using hypothesis (H_6) it shown that $\|Cx\| \leq M_k$ and consequently C is the partially bounded on E.

Let $x, y \in E$ be such that $x \geq y$. Then we have

$$|Ax(t) - Ay(t)| = \left|f\left(t, x(\theta(t))\right) - f\left(t, y(\theta(t))\right)\right|$$

$$\leq \psi_f |x(\emptyset(t)) - y(\emptyset(t))|$$

$$\leq \psi_f \|x - y\| \text{ for all } t \in R_+$$

taking the supremum over t we obtain

$$\|Ax(t) - Ay(t)\| \leq \psi_f \|x - y\|$$

Hence A is partially nonlinear D-Lipchitz on S with D-Lipchitz constant$\|\psi_f\|$.

Similarly $$|Cx(t) - Cy(t)| = \left|K\left(t, x(\mu(t))\right) - K\left(t, y(\mu(t))\right)\right|$$

$$\leq \psi_K |x(\emptyset(t)) - y(\emptyset(t))|$$

$$\leq \psi_K \|x - y\| \text{ for all } t \in R_+$$

taking the supremum over t we obtain

$$\|Cx(t) - Cy(t)\| \leq \psi_K \|x - y\|$$

Hence A is partially nonlinear D-Lipschtiz on S with D-Lipschtiz constant $\|\psi_k\|$

Hence A and C are partially nonlinear D-function on S with $\|\psi_f\|$ and$\|\psi_k\|$ respectively.

Step III: To show that B is partially continuous and compact operator on S.

Firstly we show that B is the partially continuous and compact operator on S.

Let$\{x_n\}$ be the sequence chain $\wp$ in $S \subset E$ converging to point x then by the dominated converges theorem for all $t \in R_+$, we obtain,

Then,

$$\lim_{n\to\infty} Bx_n(t) = \lim_{n\to\infty}\left\{q(t) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{v(t,s)}{(t-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds\right\}$$

$$\le q(t) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{v(t,s)}{(t-s)^{\alpha-1}} g\left(s, x(\eta(s))\right) ds$$

$$\le Bx(t) \quad \text{for all } t \in R_+$$

This shows that Bx_n is convergence to Bx point-wise on S.

Next to show that the sequence $\{Bx_n\}$ is equicontinuous sequence of the function in E, Let $t_1, t_2 \in J$ be the arbitrary with $t_1 < t_2$ then

$$|Bx_n(t_2) - Bx_n(t_1)| = \left| \begin{array}{l} q(t_2) + \frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{v(t_2,s)}{(t_2-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds \\ -q(t_1) + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{v(t_1,s)}{(t_1-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds \end{array}\right|$$

$$\le |q(t_2) - q(t_1)| \left| \begin{array}{l} \frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{v(t_2,s)}{(t_2-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds \\ -\frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{v(t_1,s)}{(t_1-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds \end{array}\right|$$

$$\le |q(t_2) - q(t_1)| \left| \begin{array}{l} \frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{v(t_2,s)}{(t_2-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds \\ -\frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{v(t_1,s)}{(t_1-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds \end{array}\right|$$

$$+\left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{v(t_1,s)}{(t_2-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{v(t_1,s)}{(t_2-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds\right|$$

$$+\left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{v(t_1,s)}{(t_2-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_1} \frac{v(t_1,s)}{(t_1-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds\right|$$

$$\le |q(t_2) - q(t_1)| \frac{1}{\Gamma(\alpha)}\int_0^{t_2} \frac{|v(t_2,s) - v(t_1,s)|}{(t_2-s)^{\alpha-1}} \left|g\left(s, x_n(\eta(s))\right)\right| ds$$

$$+\frac{1}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2} \frac{|v(t_1,s)|}{(t_2-s)^{\alpha-1}} \left|g\left(s, x_n(\eta(s))\right)\right| ds\right|$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^{t_1} |v(t_1,s)||(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}||(t_2-s)^{\alpha-1}| ds$$

$$\le |q(t_2) - q(t_1)| \frac{1}{\Gamma(\alpha)}\int_0^{T} \frac{|v(t_2,s) - v(t_1,s)|}{(t_2-s)^{\alpha-1}} h(S) ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} \frac{v(t_1,s)}{(t_1-s)^{\alpha-1}} h(s) ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^{T} |v(t_1,s)||(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}||(t_2-s)^{\alpha-1}|\, h(s) ds$$

$$\le |q(t_2) - q(t_1)| \frac{T^{\alpha-1}}{\Gamma(\alpha)}\left(\int_0^T |v(t_2,s) - v(t_1,s)|^2 ds\right)^{\frac{1}{2}} \left(\int_0^T h^2(s) ds\right)^{\frac{1}{2}}$$

$$+\frac{V}{\Gamma(\alpha)}\left(\int_0^T |(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}|^2 ds\right)^{\frac{1}{2}} + \frac{VT^{\alpha-1}}{\Gamma(\alpha)}|p(t_1) - p(t_2)|$$

Where $p(t) = V\int_0^T h(S)\,ds$ since q, p and $k_s(t) = k(t)$ are continuous on J they are uniformly continuous and compact set$J \times J$, they uniformly continuous there. By using of uniform convergence that is uniform convergence imply continuity $|Bx_n(t_2) - Bx_n(t_1)| \to 0$ as $n \to \infty$ uniformly for all $n \in N$. This shows that the convergence $Bx_n \to Bx$ is uniform and hence B is partially continuous on E.

Step IV: To show that B is compact operator on S for this shows that B is uniformly bounded and equicontinuous in S.

Let $\wp$ be an arbitrary chain in E, then show that $B(\wp)$ is uniformly bounded and equicontinuous set in S.

First we show that $B(\wp)$ is uniformly bounded set in S. Let $y \in B(\wp)$ be any element .then there is an element $x \in \wp$ such that $y = B\wp$. Now hypothesis(H_5),

$$|Bx(t)| = \left| q(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(t,s)}{(t-s)^{\alpha-1}} g\left(s, y(\eta(s))\right) ds \right|$$

$$\leq q(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(t,s)}{(t-s)^{\alpha-1}} \left| g\left(s, y(\eta(s))\right) \right| ds$$

$$\leq q(t) + \frac{VT^{\alpha-1}\|h\|_L}{\Gamma(\alpha)}$$

$$\leq r = k_1$$

Taking supremum over t, we obtain $\|Bx\| \leq k_1 \quad \forall x \in \wp$

This shows that. $B(\wp)$ is uniformly bounded set in S.

Now we show that $B(\wp)$ is equicontinuous set in S.

Let in E, Let $t_1, t_2 \in J$ be the arbitrary with $t_1 < t_2$ then for any $y \in B(\wp)$

$$|Bx(t_2) - Bx(t_1)| = \left| \begin{array}{l} q(t_2) + \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{v(t_2,s)}{(t_2-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds \\ -q(t_1) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{v(t_1,s)}{(t_1-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds \end{array} \right|$$

$$\leq |q(t_2) - q(t_1)| \left| \begin{array}{l} \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{v(t_2,s)}{(t_2-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds \\ -\frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{v(t_1,s)}{(t_1-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds \end{array} \right|$$

$$\leq |q(t_2) - q(t_1)| \left| \begin{array}{l} \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{v(t_2,s)}{(t_2-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds \\ -\frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{v(t_1,s)}{(t_1-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds \end{array} \right|$$

$$+ \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{v(t_1,s)}{(t_2-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{v(t_1,s)}{(t_2-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds \right|$$

$$+ \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{v(t_1,s)}{(t_2-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{v(t_1,s)}{(t_1-s)^{\alpha-1}} g\left(s, x_n(\eta(s))\right) ds \right|$$

$$\leq |q(t_2) - q(t_1)| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \frac{|v(t_2,s) - v(t_1,s)|}{(t_2-s)^{\alpha-1}} \left| g\left(s, x_n(\eta(s))\right) \right| ds$$

$$+ \frac{1}{\Gamma(\alpha)} \left| \int_{t_1}^{t_2} \frac{|v(t_1,s)|}{(t_2-s)^{\alpha-1}} \left| g\left(s, x_n(\eta(s))\right) \right| ds \right|$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |v(t_1,s)| |(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}| |(t_2-s)^{\alpha-1}| ds$$

$$\leq |q(t_2)-q(t_1)|\frac{1}{\Gamma(\alpha)}\int_0^T \frac{|v(t_2,s)-v(t_1,s)|}{(t_2-s)^{\alpha-1}} h(S)ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{v(t_1,s)}{(t_1-s)^{\alpha-1}}h(s)ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^T |v(t_1,s)||(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}||(t_2-s)^{\alpha-1}|\, h(s)ds$$

$$\leq |q(t_2)-q(t_1)|\frac{T^{\alpha-1}}{\Gamma(\alpha)}\left(\int_0^T |v(t_2,s)-v(t_1,s)|^2\, ds\right)^{\frac{1}{2}}\left(\int_0^T h^2(s)ds\right)^{\frac{1}{2}}$$

$$+\frac{V}{\Gamma(\alpha)}\left(\int_0^T |(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}|^2 ds\right)^{\frac{1}{2}} + \frac{VT^{\alpha-1}}{\Gamma(\alpha)}|p(t_1)-p(t_2)|$$

$$\to 0 \; as\; t_1 \to t_2, \forall\, n \in N.$$

This shows that $B(\wp)$ is equicontinuous set in S and so $B(\wp)$ is relatively compact by Arzela-Ascoli theorem.

Hence $B(\wp)$ is compact subset of S and consequently B is partially compact operator on S.

Step V: To show that $x = AxBx + Cx\,,\ \forall\, x \in S$

Let $x \in S$ be an arbitrary element such that $x = Ax.Bx + Cx$.

By using the equation (3.13) we know

$$|x(t)| = |Cx(t) + Ax(t)Bx(t)|$$

$$\leq |Cx(t)| + |Ax(t)||Bx(t)|$$

$$\leq \left|K\left(t, x(\mu(t))\right)\right| + \left|f\left(t, x(\theta(t))\right)\right|\left|q(t) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{v(t,s)}{(t-s)^{\alpha-1}} g\left(\, s, x(\eta(s))\right) ds\right|$$

$$\leq G + F\left|q(t) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{v(t,s)}{(t-s)^{\alpha-1}} g\left(\, s, x(\eta(s))\right) ds\right|$$

$$\leq F + G\left||q(t)| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{v(t,s)}{(t-s)^{\alpha-1}} g\left(\, s, x(\eta(s))\right) ds\right|$$

$$\leq F + G\left[|q(t)| + \frac{VT^{\alpha-1}\|h\|_L}{\Gamma(\alpha)}\right]$$

$$\leq F + G[K_1 + K_2] = r\ \forall\ t\ in\ R_+$$

Taking superemum over t we obtain, $\|Ax(t)Bx(t) + Cx(t)\| \leq r$ for all $x \in B_r[0]$ hence Hypothesis(c) of theorem holds.

Also we have $M = \|B(B_r[0])\| = sup\{\|Bx\| : x \in B_r[0]\}$

$$= sup\left\{\sup_{t\geq 0}|q(t)| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{v(t,s)}{(t-s)^{\alpha-1}} g\left(\, s, x(\eta(s))\right) ds\right\} : x \in B_r[0]$$

$$= \ sup\left\{\sup_{t\geq 0}|q(t)| + \sup_{t\geq 0}\frac{VT^{\alpha-1}\|h\|_L}{\Gamma(\alpha)}\right\} \leq K_1 + K_2$$

There fore $M \propto +\beta = L(K_1 + K_2) + N < 1$

Now applying theorem (2.2) to show that QFIE (1.1) has a solution on R_+

Step VI: The D-function ψ_A and ψ_C the growth condition$M_{\psi_A}(r) + \psi_C(r) < r$, for all $r > 0$.

Finally, the D-function ψ_A and ψ_C of the operator A and C satisfy the inequality given hypothesis (d) of theorem (2.2),

$$M_{\psi_A}(r) + \psi_C(r) \leq \left(\frac{VT^{\alpha-1}\|h\|_L}{\Gamma(\alpha)}\right)\psi_f(r) + \psi_K(r) < r \ \text{ for all } r > 0.$$

Thus A,B and C satisfy all conditions of theorem (2.2) and we conclude that the operator equation $AxBx + Cx = x$ has a solution. Consequently the QFIE (1.1) with maxima has a solution $x^*$is defined on J. Furthermore, the sequence$\{x_n\}_{n\in N}$ of the successive approximations defined by (3.9) converges monotonically to x^*.

4.0 Locally attractivity of the solution

Finally we show that the locally attractivity of the solution for QFIE (1.1).Let $x \, and \, y$ be any solution of the QFIE (1.1) in $B_r[0]$ defined onR_+. Then we have,

$$|x(t) - y(t)| \leq \left| \begin{matrix} K\left(t, x(\mu(t))\right) + f\left(t, x(\theta(t))\right) \times \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(t,s)}{(t-s)^{\alpha-1}} g\left(s, x(\eta(s))\right) ds \\ -K\left(t, y(\mu(t))\right) + f\left(t, y(\theta(t))\right) \times \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(t,s)}{(t-s)^{\alpha-1}} g\left(s, y(\eta(s))\right) ds \end{matrix} \right|$$

$$\leq \left|K\left(t, x(\mu(t))\right)\right| + \left|f\left(t, x(\theta(t))\right)\right| \left|q(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(t,s)}{(t-s)^{\alpha-1}} g\left(s, x(\eta(s))\right) ds\right|$$

$$+ \left|K\left(t, y(\mu(t))\right)\right| + \left|f\left(t, y(\theta(t))\right)\right| \left|q(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(t,s)}{(t-s)^{\alpha-1}} g\left(s, y(\eta(s))\right) ds\right|$$

$$\leq F + G \left| |q(t)| + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(t,s)}{(t-s)^{\alpha-1}} g\left(s, x(\eta(s))\right) ds \right|$$

$$+F + G \left| |q(t)| + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(t,s)}{(t-s)^{\alpha-1}} g\left(s, x(\eta(s))\right) ds \right|$$

$$\leq 2F + 2G \left\| \left[|q(t)| + \frac{VT^{\alpha-1} \|h\|_L}{\Gamma(\alpha)} \right] \right\|$$

For all$t \in R_+$. Since $\lim_{n\to\infty} q(t) = 0 \, and \, \lim_{n\to\infty} v(t) = 0$ and by the hypothesis (b) the above inequality it follows that$\lim_{t\to\infty} |x(t) - y(t)| = 0$. Hence this is the complete the proof.

5.0 Positivity solutions

Next we prove that the positivity of the solution of the function QFIE (1.1) defined on J we need the following hypothesis

A₁) f, g, k define the function $f, g, k: J \times R \to R_+$

A₂) v define a function$v: J \times J \to R_+$.

Theorem 5.1: Assume that the all condition of theorem (3.1) hold .Furthermore if hypothesis (A_1) and (A_2) are hold, then the equation (1.1) has positive solution defined on J.

Proof: By theorem of (3.1), and equation (1.1) has solution x defined on J then we have to achieve this enough to prove that,

$$|x(t)| = x(t) \qquad \text{for all } t \in J \qquad (5.1)$$

$$||x(t)| - x(t)| = \left| \left| K\left(t, x(\mu(t))\right) + f\left(t, x(\theta(t))\right) \times \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(t,s)}{(t-s)^{\alpha-1}} g\left(s, x(\eta(s))\right) dx \right| - K\left(t, x(\mu(t))\right) - f\left(t, x(\theta(t))\right) \times \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(t,s)}{(t-s)^{\alpha-1}} g\left(s, x(\eta(s))\right) dx \right|$$

$$= 0$$

For all $t \in J$. Hence $|x(t)| - x(t)$ for all$t \in J$. Consequently x is positive solution of the functional QFIE (1.1) defined on J. This is the complete Proof

6.0 Conclusion:

In this paper we have studied the existence solutions for the nonlinear functional integral equation of fractional order .we have able to Lipscthiz condition to partially lipschtiz condition which otherwise is considered to be very strong condition in the given equations. We need the additionally assumptions on positivity on the nonlinearities involved integral equations of fractional order to require characterization of attractivity of the given solutions. Finally we conclude that this paper we mention to represent of attractivity results using some arguments with appropriate modification and some results in the directions of Dhages theorem.

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