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Some Common Fixed-Point Theorems In Hilbert Space

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Abstract: The objective of this paper is to obtain some common fixed-point theorems for expansive, contractive type mappings and for five mappings under asymptotic regularity at a point defined on a non-empty closed subset of a Hilbert Space. So, the purpose of this paper is to establish the existence and uniqueness of a fixed-point theorem in a Hilbert space. The presented theorems extend, generalize, and improve many existing results in the literature.

Keywords: Hilbert space, Expansive Mapping, Contractive Mapping, Five Mappings under asymptotic regularity, Fixed point.

1.INTRODUCTION

In recent years some fixed points of various type of expansive mapping in Hilbert space and Banach fixed point theorem and its application are well known. Fixed points of expansive mapping was initiated by Machuca [11]. Later Jungck discussed fixed points for other forms of expansive mapping [7]. In 1984, Wang et al. [14] presented some interesting work on expansive mapping in metric space which correspond to some contractive mapping. Also, Zhang [15] has done considerable work in this field. In order to generalize the results on fixed points of nonlinear operators, Zhang studied fixed point problems for expansive mapping. As applications, he also investigated the existence of solutions of equations for locally condensing mapping and locally accretive mappings. Recently the study about fixed points for expansive mapping is deeply explored and has extended too many other directions. Motivated and inspired the above work, we investigated fixed points for expansive mapping in Hilbert space. The existence and uniqueness of a fixed point was given by Banach [1] in 1922, which was acclaimed as Banach contraction principle and plays an important role in the development of various results connected with Fixed point Theory and Approximation Theory. The Banach fixed point theorem or the contraction principle concerns certain mappings of a complete metric space into itself. It lays down conditions; sufficient for the existence and uniqueness of a fixed point. Besides, this famous classical theorem gives an iteration process through which we can obtain better approximation to the fixed point. Banach's fixed point theorem has rendered a key role in solving systems of linear algebraic equations involving iteration process. Banach fixed point theorem and its applications are well known. Many authors have extended this theorem, introducing more general contractive conditions which imply the existence of a fixed point. Almost all of conditions imply the asymptotic regularity of the mappings under consideration. So, the investigation of the asymptotically regular maps plays an important role in fixed point theory. Sharma and Yuel [13] and Guay and Singh [9] were among the first who used the concept of asymptotic regularity to prove fixed point theorems for wider class of mappings than a class of mappings introduced and studied by Ciric [4]. The purpose of this short paper is to study a wide class of asymptotically regular mappings which possess fixed points in Hilbert spaces.

2.PRELIMINARIES

2.1 Norm: A norm on X is a real-valued function $\|.\|: X \to R$ defined on X such that for any $x, y \in X$ and for all $\lambda \in K$

a) $\|x\| = 0$ if and only if $x = 0$

b) $\|x + y\| \leq \|x\| + \|y\|$

c) $\|\lambda x\| = |\lambda|\|x\|$

2.2 Normed Linear Space: It is a pair $(X, \|.\|)$ consisting of a linear space X and a norm $\|.\|$. We shall abbreviate normed linear space as normed linear space.

2.3 Cauchy Sequence: A Sequence $\{x_n\}$ in a normed linear space X is a Cauchy sequence if for any given $\varepsilon > 0$, there exist $n_0 \in N$ such that $\|x_m - x_n\| < \varepsilon$ for $m, n \geq n_0$.

2.4 Convergence condition in Normed Linear Space: A sequence $\{x_n\}$ in a normed linear space X is said to be convergent to $x \in X$ if for any given

$\varepsilon > 0, \exists\, n_0 \in N$ such that $\|x_n - x\| < \varepsilon$ for $n \geq n_0$.

2.5 Completeness: A normed linear space X is said to be complete if for every Cauchy sequence in X converges to an element of X.

2.6 Inner Product Space: Let X be a linear space over the scalar field C of complex numbers. An inner product on X is a function $(.,.): X \times X \rightarrow C$ which satisfies the following conditions

a) $(x,x) \geq 0$; iff $x = 0$

b) $(x,y) = \overline{(y,x)}$ for $x, y \in X$

c) $(\lambda x + \mu y, z) = \lambda(x,z) + \mu(y,z)$ for $\lambda, \mu \in C, x, y, z \in X$.

2.7 Hilbert Space: An infinite dimensional inner product space which is complete for the norm induced by the inner product is called Hilbert Space.

2.8 Parallelogram Law: $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

Definition 2.1. A self-mapping T on a metric space (X,d) is said to be asymptotically regular at a point x in X, if $d(T^n x, T^n Tx) \rightarrow 0$ as $n \rightarrow \infty$

Where $T^n x$ denotes the nth iterate of T at x.

Definition 2.2. Let C be a closed subset of a Hilbert space H. A sequence $\{x_n\}$ in C is said to be asymptotically T-regular if $\|x_n - Tx_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Definition 2.3. A self-mapping T on a closed subset C of a Hilbert space H is said to be asymptotically regular at a point x in C if $\|T^n x - T^{n+1} x\| \rightarrow 0$ as $n \rightarrow \infty$.

Where $T^n x$ denotes the nth iterate of T at x.

Definition 2.4. Let T_1 and T_2 be two self-maps of a metric space (X,d) and $\{x_n\}$ is said to be asymptotically T_2-regular w.r.t T_1 if $\lim_{n\rightarrow\infty} \|T_1 x_n - T_2 x_n\|^2 = 0$

Definition 2.5. The pair $T_1 T_2$ is said to be compatible if $\lim_{n\rightarrow\infty} \|T_1 T_2 x_n - T_2 T_1 x_n\|^2 = 0$

Where $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} T_1x_n = \lim_{n\to\infty} T_2x_n = t$ for some $t \in X$

i.e., If $\|T_1T_2x - T_2T_1x\| \to 0$ as $\|T_2x - T_1x\| \to 0, T_1, T_2$ are compatible.

Lemma 2.6. (Jungck 1986). If T_1 and T_2 are compatible self-maps of a metric space (X, d) and $\lim_{n\to\infty} T_1x_n = \lim_{n\to\infty} T_2x_n = t$ for some $t \in X$ then $\lim_{n\to\infty} T_2T_1x_n = T_1t$ If T_1 is continuous.

3. MAIN RESULTS

Theorem 3.1. Let $(X, \|.,.\|)$ be a Hilbert space and T be a mapping of X into itself such that for every $x, y \in X$,

$$\|Tx - Ty\|^2 \geq a\|x - Tx\|^2\|x - y\|^2 + b\|y - Ty\|^2\|x - y\|^2 + c\|x - Tx\|^2\|y - Ty\|^2 \text{------------(3.1.1)}$$

For all distinct $x, y \in X$ where $a, c \geq 0, b > 0$ and $a + b + c > 1$.

Then T has a fixed point.

Proof: Construct a sequence $\{x_n\}$,

We claim that the inequality (3.3.3.1) for $x = x_{n+1}$ and $y = x_{n+2}$ implies that

$$\|Tx_{n+1} - Tx_{n+2}\|^2 \geq a\|x_{n+1} - Tx_{n+1}\|^2\|x_{n+1} - x_{n+2}\|^2 + b\|x_{n+2} - Tx_{n+2}\|^2\|x_{n+1} - x_{n+2}\|^2$$

$$+c\|x_{n+1} - Tx_{n+1}\|^2\|x_{n+2} - Tx_{n+2}\|^2$$

$$= a\|x_{n+1} - x_n\|^2\|x_{n+1} - x_{n+2}\|^2 + b\|x_{n+2} - x_{n+1}\|^2\|x_{n+1} - x_{n+2}\|^2$$

$$+c\|x_{n+1} - x_n\|^2\|x_{n+2} - x_{n+1}\|^2$$

$$\|x_n - x_{n+1}\|^2 \geq \|x_{n+1} - x_{n+2}\|^2(a + b + c)min\{\|x_n - x_{n+1}\|^2, \|x_{n+1} - x_{n+2}\|^2\}$$

Case 1: $\|x_n - x_{n+1}\|^2 \geq (a + b + c)\|x_{n+1} - x_{n+2}\|^2\|x_n - x_{n+1}\|^2$

$$\Rightarrow \|x_{n+1} - x_{n+2}\|^2 \leq \frac{1}{a + b + c}$$

$$\Rightarrow \|x_{n+1} - x_{n+2}\|^2 \leq k_1 \qquad \left[\text{where } k_1 = \frac{1}{a + b + c} < 1 \ \ (\text{As } a + b + c > 1)\right]$$

Case 2: $\|x_n - x_{n+1}\|^2 \geq (a + b + c)\|x_{n+1} - x_{n+2}\|^2\|x_{n+1} - x_{n+2}\|^2$

$$\Rightarrow \|x_{n+1} - x_{n+2}\|^4 \leq \frac{1}{a + b + c}\|x_n - x_{n+1}\|^2$$

$$\Rightarrow \|x_{n+1} - x_{n+2}\|^2 \leq k_2\|x_n - x_{n+1}\| \qquad \left[\text{where } k_2 = \sqrt{\frac{1}{a + b + c}} < 1 \ (\text{As } a + b + c > 1)\right]$$

Let $k = max\{k_1, k_2\}$ then $k < 1$

So, in general

From Case 1: $\|x_n - x_{n+1}\|^2 \leq k$ for $n = 1,2,3, \ldots$

From Case 2: $\|x_n - x_{n+1}\|^2 \leq k^n \|\mathrm{x_{n-1}} - \mathrm{x_n}\|$ ------------(3.1.2)

We can prove that $\{x_n\}$ is a Cauchy sequence using (3.1.2).

As X is a Hilbert space, so there exists a point $x \in X$ such that $\{x_n\} \to x$.

Existence of fixed point:

Since T is a surjective self map and hence there exist point y in X such that

$$x = Ty \qquad \text{------------(3.1.3)}$$

Consider, $\|x_n - x\| = \|Tx_{n+1} - Ty\|$

$$\|Tx_{n+1} - Ty\|^2 \geq a\|x_{n+1} - Tx_{n+1}\|^2\|x_{n+1} - y\|^2 + b\|y - Ty\|^2\|x_{n+1} - y\|^2$$

$$+c\|x_{n+1} - Tx_{n+1}\|^2\|y - Ty\|^2$$

$$\|x_n - x\|^2 \geq a\|x_{n+1} - x_n\|^2\|x_{n+1} - y\|^2 + b\|y - x\|^2\|x_{n+1} - y\|^2$$

$$+c\|x_{n+1} - x_n\|^2\|y - x\|^2$$

Since $\{x_{n+1}\}$ is a subsequence of $\{x_n\}$, so $\{x_n\} \to x, \{x_{n+1}\} \to x$ when $n \to \infty$

$$\|x - x\|^2 \geq a\|x - x\|^2\|x - y\|^2 + b\|y - x\|^2\|x - y\|^2 + c\|x - x\|^2\|y - x\|^2$$

$$0 \geq b\|x - y\|^2$$

$$\Rightarrow \|x - y\| = 0 \qquad [\text{As } b > 0]$$

$$\Rightarrow x = y \qquad \text{-------------(3.1.4)}$$

The fact (3.1.4) along with (3.1.3) shows that x is a common fixed point of T.

This completes the proof of the theorem 3.1.

Theorem 3.2. Let $(X, \|.,.\|)$ be a Hilbert space. Let T be a mapping $T: X \to X$ such that

$$\|Tx - Ty\|^2 \leq amax\{\|x - Tx\|^2, \|y - Ty\|^2, \|x - y\|^2\} + b\{\|x - Ty\|^2 +$$

$$\|y - Tx\|^2\} \qquad \text{--------------(3.2.1)}$$

Where $a, b > 0$ such that $a + 2b \leq 1 \ \forall x, y \in X$ then T has a unique fixed point.

Proof – Let $x_0 \in X$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X defined by recursion

$$x_n = Tx_{n-1} = T^n x_0 \quad n = 1,2,3,4.. \qquad \text{----------(3.2.2)}$$

By (3.2.1) and (3.2.2) we obtain that

$$\|x_n - x_{n+1}\|^2 = \|Tx_{n-1} - Tx_n\|^2$$

$$\leq amax\{\|x_{n-1} - Tx_{n-1}\|^2, \|x_n - Tx_n\|^2, \|x_{n-1} - x_n\|^2\} + b\{\|x_{n-1} - Tx_n\|^2 + \|x_n - Tx_{n-1}\|^2\}$$

$$\leq amax\{\|x_{n-1} - x_n\|^2, \|x_n - x_{n+1}\|^2, \|x_{n-1} - x_n\|^2\} + b\{\|x_{n-1} - x_{n+1}\|^2 + \|x_n - x_n\|^2\}$$

$$\leq amax\{\|x_{n-1} - x_n\|^2, \|x_n - x_{n+1}\|^2\} + b\{\|x_{n-1} - x_{n+1}\|^2\}$$

$$\leq amax\{\|x_{n-1} - x_n\|^2, \|x_n - x_{n+1}\|^2\} + b\{\|x_{n-1} - x_n\|^2 + \|x_n - x_{n+1}\|^2\}$$

$$\leq aM_1 + b\{\|x_{n-1} - x_n\|^2 + \|x_n - x_{n+1}\|^2\} \ \text{(By Parallelogram Law)}$$

where $M_1 = max\{\|x_{n-1} - x_n\|^2, \|x_n - x_{n+1}\|^2\}$

Case-1: If suppose that $M_1 = \|x_n - x_{n+1}\|^2$ then we have

$$\|x_n - x_{n+1}\|^2 \leq a\|x_n - x_{n+1}\|^2 + b\{\|x_{n-1} - x_n\|^2 + \|x_n - x_{n+1}\|^2\}$$

$$\leq a\|x_n - x_{n+1}\|^2 + b\|x_{n-1} - x_n\|^2 + b\|x_n - x_{n+1}\|^2$$

$$(1 - a - b)\|x_n - x_{n+1}\|^2 \leq b\|x_{n-1} - x_n\|^2$$

$$\|x_n - x_{n+1}\|^2 \leq \frac{b}{(1 - a - b)}\|x_{n-1} - x_n\|^2 \qquad \text{where } k = \left(\frac{b}{(1 - a - b)}\right)^{\frac{1}{2}}$$

$$\Rightarrow \|x_n - x_{n+1}\| \leq k^2\|x_{n-2} - x_{n-1}\|$$

$$\Rightarrow \|x_n - x_{n+1}\| \leq k^n\|x_0 - x_1\|$$

Case-2: If suppose that $M_1 = \|x_{n-1} - x_n\|^2$ then we have,

$$\|x_n - x_{n+1}\|^2 \leq a\|x_{n-1} - x_n\|^2 + b\{\|x_{n-1} - x_n\|^2 + \|x_n - x_{n+1}\|^2\}$$

$$\leq a\|x_{n-1} - x_n\|^2 + b\|x_{n-1} - x_n\|^2 + b\|x_n - x_{n+1}\|^2$$

$$(1 - b)\|x_n - x_{n+1}\|^2 \leq (a + b)\|x_{n-1} - x_n\|^2$$

$$\|x_n - x_{n+1}\|^2 \leq \frac{(a+b)}{(1-b)}\|x_{n-1} - x_n\|^2 \qquad \text{where } k = \left(\frac{(a+b)}{(1-b)}\right)^{\frac{1}{2}} < 1$$

$$\Rightarrow \|x_n - x_{n+1}\| \leq k^2\|x_{n-2} - x_{n-1}\|$$

$$\Rightarrow \|x_n - x_{n+1}\| \leq k^n\|x_0 - x_1\|$$

Thus T is a contractive mapping.

Now, we show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X.

Let $m, n \in N, m > n$

$$\|x_n - x_m\|^2 \leq \{\|x_n - x_{n+1}\|^2 + \|x_{n+1} - x_m\|^2\}$$

$$\leq \|x_n - x_{n+1}\|^2 + \{\|x_{n+1} - x_{n+2}\|^2 + \|x_{n+2} - x_m\|^2\}$$

$$\leq \|x_n - x_{n+1}\|^2 + \|x_{n+1} - x_{n+2}\|^2 + \|x_{n+2} - x_{n+3}\|^2 + \cdots$$

$$\leq k^{2n}\|x_0 - x_1\|^2 + k^{2n+2}\|x_0 - x_1\|^2 + k^{2n+4}\|x_0 - x_1\|^2 + \cdots$$

$$\leq k^{2n}\|x_0 - x_1\|^2[1 + k^2 + (k^2)^2 + (k^2)^3 + \cdots]$$

$$\|x_n - x_m\|^2 \leq \frac{k^{2n}}{1-k^2}\|x_0 - x_1\|^2$$

Then $\lim_{n\to\infty}\|x_n - x_m\|^2 = 0$ as $n, m \to \infty$, since $k < 1$.

$$\lim_{n\to\infty}\left(\frac{k^{2n}}{1-k^2}\right)\|x_0 - x_1\|^2 = 0 \text{ as } n, m \to \infty.$$

Hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy Sequence in X.

Since $\{x_n\}_{n=1}^{\infty}$ is a Cauchy Sequence, $\{x_n\}$ converges to $w \in X$.

Now, we show that w is the fixed point of T.

$$\|w - Tw\|^2 \leq \{\|w - x_{n+1}\|^2 + \|x_{n+1} - Tw\|^2\}$$

$$\leq \|w - x_{n+1}\|^2 + \|Tx_n - Tw\|^2$$

$$\leq \|w - x_{n+1}\|^2 + a\max\{\|x_n - Tx_n\|^2, \|w - Tw\|^2, \|x_n - w\|^2\}$$

$$+b\{\|x_n - Tw\|^2 + \|w - Tx_n\|^2\}$$

$$\leq \|w - x_{n+1}\|^2 + a\max\{\|x_n - x_{n+1}\|^2, \|w - Tw\|^2, \|x_n - w\|^2\}$$

$$+b\{\|x_n - w\|^2 + \|w - Tw\|^2\} + b\|w - x_{n+1}\|^2$$

$$(1-b)\ \|w - Tw\|^2 \leq (1+b)\|w - x_{n+1}\|^2 + aM_2 + b\|x_n - w\|^2$$

where $M_2 = max\{\|x_n - x_{n+1}\|^2, \|w - Tw\|^2, \|x_n - w\|^2\}$

Case-1: If suppose that $M_2 = \|x_n - x_{n+1}\|^2$ then we have,

$$(1-b)\|w - Tw\|^2 \leq (1+b)\|w - x_{n+1}\|^2 + a\|x_n - x_{n+1}\|^2 + b\|x_n - w\|^2$$

$$\leq (1+a+b)\|w - x_{n+1}\|^2 + (a+b)\|x_n - w\|^2$$

$$\|w - Tw\|^2 \leq \frac{(1+a+b)}{(1-b)}\|w - x_{n+1}\|^2 + \frac{(a+b)}{(1-b)}\|x_n - w\|^2$$

Taking $\lim_{n\to\infty}\|w - Tw\|^2 = 0 \ \Rightarrow w = Tw$

Case-2: If suppose that $M_2 = \|x_n - w\|^2$ then,

$$(1-b)\|w - Tw\|^2 \leq (1+b)\|w - x_{n+1}\|^2 + a\|x_n - w\|^2 + b\|x_n - w\|^2$$

$$\|w - Tw\|^2 \leq \frac{(1+b)}{(1-b)}\|w - x_{n+1}\|^2 + \frac{(a+b)}{(1-b)}\|x_n - w\|^2$$

Taking $\lim_{n\to\infty}\|w - Tw\|^2 = 0 \ \Rightarrow w = Tw$

Therefore w is the fixed point of T.

Case-3: If suppose that $M_2 = \|w - Tw\|^2$ then

$$(1-b)\|w - Tw\|^2 \leq (1+b)\|w - x_{n+1}\|^2 + a\|w - Tw\|^2 + b\|x_n - w\|^2$$

$$(1-a-b)\|w - Tw\|^2 \leq (1+b)\|w - x_{n+1}\|^2 + b\|x_n - w\|^2$$

$$\|w - Tw\|^2 \leq \frac{(1+b)}{(1-a-b)}\|w - x_{n+1}\|^2 + \frac{b}{(1-a-b)}\|x_n - w\|^2$$

Therefore w is the fixed point of T.

Uniqueness of Fixed point:

We must show that w is unique fixed point of T.

Assume that v is another fixed point of T then we have

$Tv = v$

and

$$\|w - v\|^2 = \|Tw - Tv\|^2$$

$$\leq amax\{\|w - Tw\|^2, \|v - Tv\|^2, \|w - v\|^2\} + b\{\|w - Tv\|^2 + \|v - Tw\|^2\}$$

$$\leq a\max\{\|w-w\|^2, \|v-v\|^2, \|w-v\|^2\} + b\{\|w-v\|^2 + \|v-w\|^2\}$$

$$\leq a\|w-v\|^2 + 2b\|w-v\|^2$$

$$\|w-v\|^2 \leq (a+2b)\|w-v\|^2$$

This is contradiction.

Therefore $w = v$. This completes the proof.

Hence w is the unique fixed point.

Theorem 3.3. Let $(X, \|.,.\|)$ be a Hilbert space and $T_i, i = 1,2,3,4,5$ be five self maps of X into itself such that for every $x, y \in X$.

(A) $\|T_5x - T_5y\|^2 \leq \alpha_1 \frac{\|T_3T_4y - T_5y\|^2[1+\|T_1T_2x - T_5x\|^2]}{1+\|T_1T_2x - T_3T_4y\|^2} + \alpha_2 \frac{\|T_1T_2x - T_5x\|^2[1+\|T_3T_4y - T_5x\|^2]}{1+\|T_5x - T_5y\|^2}$

$$+\alpha_3 \frac{\|T_3T_4y - T_5y\|^2[1+\|T_1T_2x - T_5y\|^2]}{1+\|T_5x - T_5y\|^2} + \alpha_4\|T_1T_2x - T_3T_4y\|^2$$

$$+\alpha_5\|T_1T_2x - T_5y\|^2 + \alpha_6\|T_3T_4y - T_5y\|^2$$

Where $0 \leq \alpha_4 < 1 \quad \alpha_i \geq 0, \alpha_1 + \alpha_3 + \alpha_6 < 1$.

(B) $T_5T_2 = T_2T_5, T_5T_4 = T_4T_5, T_1T_2 = T_2T_1, T_3T_4 = T_4T_3$.

(C) T_5, T_1T_2 is a compatible pair.

(D) T_1, T_2 are continuous.

(E) $\exists$ an asymptotically T_5 − regular sequence with respect to both T_1T_2 and T_3T_4 then $T_i \;\; i = 1,2,3,4,5$ have a unique common fixed point in X.

Proof - From condition (A) we have for any positive integers m, n

$$\|T_5x_n - T_5x_m\|^2$$

$$\leq \alpha_1 \frac{\|T_3T_4x_m - T_5x_m\|^2[1+\|T_1T_2x_n - T_5x_n\|^2]}{1+\|T_1T_2x_n - T_3T_4x_m\|^2}$$

$$+\alpha_2 \frac{\|T_1T_2x_n - T_5x_n\|^2[1+\|T_3T_4x_m - T_5x_n\|^2]}{1+\|T_5x_n - T_5x_m\|^2}$$

$$+\alpha_3 \frac{\|T_3T_4x_m - T_5x_m\|^2[1+\|T_1T_2x_n - T_5x_m\|^2]}{1+\|T_5x_n - T_5x_m\|^2} + \alpha_4\|T_1T_2x_n - T_3T_4x_m\|^2$$

$$+\alpha_5\|T_1T_2x_n - T_5x_m\|^2 + \alpha_6\|T_3T_4x_m - T_5x_m\|^2$$

Taking $\lim_{n,m\to\infty} \|T_5x_n - T_5x_m\|^2 \le \alpha_4 \lim_{n,m\to\infty} \|T_5x_n - T_5x_m\|^2$

Since $\|T_1T_2x_n - z\|^2 \le \|T_1T_2x_n - T_5x_n\|^2 + \|T_5x_n - z\|^2 \to 0$ as $n \to \infty$

so $T_1T_2x_n \to z$ and similarly, $T_3T_4x_n \to z$.

Applying conditions:

(C) T_5, T_1T_2 is a compatible pair and

(D) T_1T_2 are continuous and Lemma (2.6)

$(T_1T_2)^2x_n \to T_1T_2z$ and $T_5(T_1T_2)x_n \to T_1T_2z$

Further condition (A) gives

$$\|T_5(T_1T_2)x_n - T_5x_n\|^2$$

$$\le \alpha_1 \frac{\|T_3T_4x_n - T_5x_n\|^2[1 + \|(T_1T_2)^2x_n - T_5(T_1T_2)x_n\|^2]}{1 + \|(T_1T_2)^2x_n - T_3T_4x_n\|^2}$$

$$+ \alpha_2 \frac{\|(T_1T_2)^2x_n - T_5(T_1T_2)x_n\|^2[1 + \|T_3T_4x_n - T_5T_1T_2x_n\|^2]}{1 + \|T_5T_1T_2x_n - T_5x_n\|^2}$$

$$+\alpha_3 \frac{\|T_3T_4x_n - T_5x_m\|^2[1 + \|(T_1T_2)^2x_n - T_5x_n\|^2]}{1 + \|T_5T_1T_2x_n - T_5x_n\|^2}$$

$$+\alpha_4\|(T_1T_2)^2x_n - T_3T_4x_n\|^2$$

$$+\alpha_5\|(T_1T_2)^2x_n - T_5T_1T_2x_n\|^2 + \alpha_6\|T_3T_4x_n - T_5x_n\|^2$$

Taking $\lim_{n\to\infty}$ we have $\|T_1T_2z - z\|^2 \le \alpha_4 \|T_1T_2z - z\|^2$ a contradiction. Hence

$T_1T_2z = z$ ----------------(3.3.1)

From (E)

We obtain, $\|z - T_3T_4z\|^2 \le \|z - T_1T_2z\|^2 = 0$

$i.e., T_3T_4z = z$ ---------------(3.3.2)

Again by (A), (3.3.1) and (3.3.2) we have

$$\|T_5(T_1T_2)x_n - T_5z\|^2$$

$$\leq \alpha_1 \frac{\|T_3T_4z - T_5z\|^2[1 + \|(T_1T_2)^2x_n - T_5(T_1T_2)x_n\|^2]}{1 + \|(T_1T_2)^2x_n - T_3T_4z\|^2}$$

$$+ \alpha_2 \frac{\|(T_1T_2)^2x_n - T_5(T_1T_2)x_n\|^2[1 + \|T_3T_4z - T_5T_1T_2x_n\|^2]}{1 + \|T_5T_1T_2x_n - T_5z\|^2}$$

$$+\alpha_3 \frac{\|T_3T_4z - T_5z\|^2[1 + \|(T_1T_2)^2x_n - T_5z\|^2]}{1 + \|T_5T_1T_2x_n - T_5z\|^2} + \alpha_4\|(T_1T_2)^2x_n - T_3T_4z\|^2$$

$$+\alpha_5\|(T_1T_2)^2x_n - T_5T_1T_2x_n\|^2 + \alpha_6\|T_3T_4z - T_5z\|^2$$

Taking $\lim_{n\to\infty}$ we have $\|T_1T_2z - T_5z\|^2 \leq (\alpha_1 + \alpha_3 + \alpha_6)\|T_1T_2z - T_5z\|^2$

$\Rightarrow T_1T_2z = T_5z$ as $\sqrt{\alpha_1 + \alpha_3 + \alpha_6} < 1$

$T_1T_2z = T_5z = z = T_3T_4z$ --------------(3.3.3)

We show that $T_2z = z,$ if not then using (A), (B) and (3.3.3) we get

$$\|T_2z - z\|^2 = \|T_2T_5z - T_5z\|^2 = \|T_5(T_2)z - T_5z\|^2$$

$$\leq \alpha_1 \frac{\|T_3T_4z - T_5z\|^2[1 + \|T_1T_2(T_2z) - T_5(T_2z)\|^2]}{1 + \|T_1T_2(T_2z) - T_3T_4z\|^2}$$

$$+ \alpha_2 \frac{\|T_1T_2(T_2z) - T_5(T_2z)\|^2[1 + \|T_3T_4z - T_5(T_2z)\|^2]}{1 + \|T_5(T_2z) - T_5z\|^2}$$

$$+\alpha_3 \frac{\|T_3T_4z - T_5z\|^2[1 + \|T_1T_2(T_2z) - T_5z\|^2]}{1 + \|T_5(T_2z) - T_5z\|^2}$$

$$+\alpha_4\|T_1T_2(T_2z) - T_3T_4z\|^2$$

$$+\alpha_5\|T_1T_2(T_2z) - T_5(T_2z)\|^2 + \alpha_6\|T_3T_4z - T_5z\|^2$$

Taking $\lim_{n\to\infty}$ we have $\|T_2z - z\|^2 \leq \alpha_4\|T_2z - z\|^2$ a contradiction.

$T_2z = z$ and then from (3.3.3) $T_1z = z$

Now, if possible, let $z \neq T_4z$ then using (A), (B) and (3.3.3) we get

$$\|T_4z - z\|^2 = \|T_4T_5z - T_5z\|^2 = \|T_5z - T_5(T_4z)\|^2$$

$$\leq \alpha_1 \frac{\|T_3T_4(T_4z) - T_5(T_4z)\|^2[1 + \|T_1T_2z - T_5z\|^2]}{1 + \|T_1T_2z - T_3T_4(T_4z)\|^2}$$

$$+\alpha_2 \frac{\|T_1T_2z - T_5z\|^2[1 + \|T_3T_4(T_4z) - T_5z\|^2]}{1 + \|T_5z - T_5(T_4z)\|^2}$$

$$+\alpha_3 \frac{\|T_3T_4(T_4z) - T_5(T_4z)\|^2[1 + \|T_1T_2z - T_5(T_4z)\|^2]}{1 + \|T_5z - T_5(T_4z)\|^2}$$

$$+\alpha_4\|T_1T_2z - T_3T_4(T_4z)\|^2$$

$$+\alpha_5\|T_1T_2z - T_5z\|^2 + \alpha_6\|T_3T_4(T_4z) - T_5(T_4z)\|^2$$

Taking $\lim_{n\to\infty}$ we have $\|T_4z - z\|^2 \leq \alpha_4 \|z - T_4z\|^2$ a contradiction.

$T_4z = z$ and then from (3.3.3)

$T_3z = z$

combining above results we have $T_1z = T_2z = T_3z = T_4z = T_5z = z$

This uniqueness follows direct from condition (A).

This completes the proof of theorem.

CONCLUSION

In this paper, we proved existence and uniqueness of some fixed-point theorems in Hilbert space for following different mappings:

1. The existence and uniqueness of the fixed-point for contraction type mapping.
2. The existence and uniqueness of the fixed-point of asymptotically T-regular for five self-mapping.
3. The existence of the fixed-point of Expansive type mapping.

FUTURE WORK

In future we can extend and elaborate fixed point theorems in various spaces for different type of mappings.

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