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On the Solution of Faltung Type Multiplicative Integral Equations

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Abstract : In this study, we defined the Faltung type Volterra integral equation and the Faltung type Volterra integro-differential equation in the sense of multiplicative calculus. The solutions of the Faltung type multiplicative Volterra integral equation and the Faltung type multiplicative Volterra integro-differential equation of the first kind are determined by using the multiplicative Sumudu transform. The approach for solving these equations using the multiplicative Sumudu transform is explained, with several examples that illustrate the process.

***Keywords* - Non-Newtonian calculus, Multiplicative calculus, Integral transform, Multiplicative Volterra integral equation, Multiplicative Volterra integro-differential equation, Multiplicative Sumudu transform.**

I. INTRODUCTION AND PRELIMINARIES

The non-Newtonian calculus, which was presented by Grossman and Katz [13], is a new structure made up of the branches of the geometric, bigeometric, harmonic, biharmonic, quadratic, and biquadratic calculus. It has numerous applications in science, engineering, and mathematics. Among the topics studied are interest rates, biology, blood viscosity, the theory of economic elasticity, including image processing and artificial intelligence, computer science, differential equations and functional analysis. A new kind of derivative and integral was established by Grossman and Katz [13] by replacing addition and subtraction with multiplication and division. Stanley [23] dubbed this new branch of calculus, which was established in this manner, multiplicative calculus. Multiplicative calculus offers new vantage points for use in the sciences and engineering. Aniszewska [5] used the multiplicative version of the Runge-Kutta method for solving multiplicative differential equations. Bashirov, Mısırlı and Özyapıcı [7] demonstrated some applications and usefulness of multiplicative calculus for the attention of researchers in the branch of analysis. Bhat et al. [9] defined multiplicative the Fourier transform and found the solution of multiplicative differential equations by applying multiplicative Fourier transform. Bhat et al. [10] defined the multiplicative Sumudu transform and solved some multiplicative differential equations by using multiplicative Sumudu transform. Güngör and Durmaz [16] defined multiplicative Volterra integral equations and find the solution of these equations by using successive approximation method. Also, they investigated the relationship of the multiplicative integral equations with the multiplicative differential equations. For more details about non-Newtonian calculus, see in [8, 11-16, 21, 22, 25, 26].

The integral transforms have recently been the focus of the studies, because the integral transforms provide simple and minimal computations for solving complicated problems in engineering and science. The Laplace transform is the most widely used of several integral transforms used to solve integral equations. Asiru [6] applied the Sumudu transform to solve integral equations of the convolution (Faltung) type. Song and Kim [27] examined convolution type Volterra integral equations by utilizing the Elzaki transform. Aggarwal et al. [1-4] used Aboodh, Kamal, Mahgoub and Shehu transformations to solve linear Volterra integral equations with an integral in the form of Faltung. Güngör [17, 18] used Kaharrat-Toma and Kashuri-Fundo transforms to solve convolution type linear Volterra integral equations. Mansour et al. [20] revealed how to use the SEE transform to solve Faltung type Volterra integro-differential equation of the first kind. For relevant terminology on integral equations, the reader should consult [19, 24].

In this work, the notion of multiplicative integral is used to define the Faltung type multiplicative Volterra integral equation and the Faltung type multiplicative Volterra integro-differential equation. The multiplicative Sumudu transform is used to find the exact solution of first and second kind Faltung type multiplicative Volterra integral equation and also Faltung type multiplicative Volterra integro-differential equation of first kind. The method of solving these equations using the multiplicative Sumudu transform is described, along with a few examples. Moreover, the multiplicative Sumudu transform is used to examine the solution of multiplicative linear differential equations with initial conditions, by transforming them into multiplicative Volterra integral equations.

Now, we will provide some essential details:

Definition 1. [7] Let g be a function whose domain is $\mathbb{R}$ the set of real numbers and whose range is a subset of $\mathbb{R}$. The multiplicative derivative of the g at t is defined as the limit

$$\frac{d^*g(t)}{dt} = g^*(t) = \lim_{h\to 0}\left(\frac{g(t+h)}{g(t)}\right)^{\frac{1}{h}}.$$

Briefly, the limit is also called $*$-derivative of g at t. If g is a positive function on an open set $A \subseteq \mathbb{R}$ and its classical derivative $g'(t)$ exists, then its multiplicative derivative also exists and

$$g^*(t) = e^{\left[\frac{g'(t)}{g(t)}\right]} = e^{(\ln\circ g)'(t)}$$

where $\ln\circ g(t) = \ln g(t)$. Moreover, if g is multiplicative differentiable and $g^*(t) \neq 0$, then its classical derivative exists and $g'(t) = g(t)\ln g^*(t)$. If n-th derivative $g^{(n)}(t)$ exists, then its n-th multiplicative derivative $g^{*(n)}(t)$ also exists and $g^{*(n)}(t) = e^{(\ln\circ g)^{(n)}(t)}$, $n = 0,1,2,\dots$.

Definition 2. [23] Let g be a positive function and continuous on the interval $[a,b]$, then it is multiplicative integrable or briefly $*$-integrable on $[a,b]$ and

$$*\int_a^b g(t)^{dt} = e^{\int_a^b \ln(g(t))dt}.$$

Theorem 1. [7] If g and h are integrable functions on $[a,b]$ in the sense of multiplicative, then

(1) $*\int_a^b \left(g(t)^{\lambda}\right)^{dt} = \left(*\int_a^b g(t)^{dt}\right)^{\lambda}$

(2) $*\int_a^b \left(g(t)h(t)\right)^{dt} = *\int_a^b g(t)^{dt} * \int_a^b h(t)^{dt}$

(3) $*\int_a^b \left(\frac{g(t)}{h(t)}\right)^{dt} = \frac{*\int_a^b g(t)^{dt}}{*\int_a^b h(t)^{dt}}$

(4) $*\int_a^b g(t)^{dt} = *\int_a^c g(t)^{dt} * \int_c^b g(t)^{dt}$

where $\lambda \in \mathbb{R}$ and $a \leq c \leq b$.

Definition 3. [10] Let $f(t)$ be a positive definite function given on interval $[0,\infty)$. Then, multiplicative Sumudu transform of $f(t)$ is defined as

$$\mathcal{S}_m[f(t)] = F_m(u) = \left(*\int_0^\infty f(t)^{e^{-\frac{t}{u}}dt}\right)^{\frac{1}{u}} = *\int_0^\infty f(t)^{\frac{1}{u}e^{-\frac{t}{u}}dt} = e^{\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\ln f(t)dt} = e^{\mathcal{S}[\ln f(t)]}$$

Some functions' multiplicative Sumudu transforms are as follows:

$f(t)$	$\mathcal{S}_m[f(t)] = F_m(u)$
1	1
e^t	e^u
e^{t^n}	$e^{n!u^n}$
$e^{e^{at}}$	$e^{\frac{1}{1-au}}$
$e^{\sin(at)}$	$e^{\frac{au}{1+a^2u^2}}$
$e^{\cos(at)}$	$e^{\frac{1}{1+a^2u^2}}$
$e^{\sinh(at)}$	$e^{\frac{au}{1-a^2u^2}}$
$e^{\cos(at)}$	$e^{\frac{1}{1-a^2u^2}}$

Theorem 2. [10] Multiplicative Sumudu transform is multiplicatively linear. In other words, if f_1 and f_2 are two given functions which have multiplicative Sumudu transform exist, then

$$\mathcal{S}_m\left[f_1^{\lambda_1}f_2^{\lambda_2}\right] = \mathcal{S}_m[f_1]^{\lambda_1}\mathcal{S}_m[f_2]^{\lambda_2}$$

where λ_1, λ_2 are arbitrary exponents.

Definition 4. [10] If $F_m(u)$ is the multiplicative Sumudu transform of the function f, i.e. $\mathcal{S}_m[f(t)] = F_m(u)$ then $\mathcal{S}_m^{-1}[F_m(u)]$ is called as the inverse multiplicative Sumudu transform of F_m.

Theorem 3. *(Multiplicative Faltung (Convolution) Property)* If $\mathcal{S}_m^{-1}[F_m(u)] = f(t)$ and $\mathcal{S}^{-1}[G(u)] = g(t)$, then

$$\mathcal{S}_m^{-1}\left[F_m(u)^{uG(u)}\right] = *\int_0^t f(z)^{g(t-z)dz}.$$

Proof. Applying multiplicative Sumudu transform to multiplicative integral $*\int_0^t f(z)^{g(t-z)^{dz}}$, we find

$$\mathcal{S}_m\left[*\int_0^t f(z)^{g(t-z)^{dz}}\right] = \mathcal{S}_m\left[e^{\int_0^t g(t-z)\ln f(z)dz}\right]$$
$$= e^{\mathcal{S}\left[\ln e^{\int_0^t g(t-z)\ln f(z)dz}\right]}$$
$$= e^{\mathcal{S}\left[\int_0^t g(t-z)\ln f(z)dz\right]}.$$

From the Faltung (convolution) property of Sumudu transform, we obtain

$$\mathcal{S}_m\left[*\int_0^t f(z)^{g(t-z)^{dz}}\right] = e^{u\mathcal{S}[\ln f(t)]\mathcal{S}[g(t)]}$$
$$= \left[e^{\mathcal{S}[\ln f(t)]}\right]^{u\mathcal{S}[g(t)]}$$
$$= [F_m(u)]^{uG(u)}.$$

Theorem 4. Let f is continuous on the interval $[0, \mathrm{A}]$ also suppose that there exist positive real numbers k, α and t_0 such that $|f(t)| \le ke^{e^{\alpha t}}$ for $t > t_0$ and let f^* be a piecewise continuous function on the interval $[0, \mathrm{A}]$. Then multiplicative Sumudu transform of multiplicative derivative is

$$\mathcal{S}_m[f^*(t)] = \frac{1}{f(0)^{\frac{1}{u}}} F_m(u)^{\frac{1}{u}}$$

for $u < \frac{1}{\alpha}$.

Proof. It is obtain that

$$\mathcal{S}_m[f^*(t)] = *\int_0^\infty f^*(t)^{\frac{1}{u}e^{-\frac{t}{u}}dt}$$
$$= e^{\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\ln f^*(t)dt}$$
$$= e^{\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\ln e^{\frac{f'(t)}{f(t)}}dt}$$
$$= e^{\frac{1}{u}\int_0^\infty \frac{f'(t)}{f(t)}e^{-\frac{t}{u}}dt}$$
$$= e^{\lim_{A\to\infty}\frac{1}{u}\int_0^A \frac{f'(t)}{f(t)}e^{\frac{-t}{u}}dt}$$

by using the definition of the multiplicative Sumudu transform. Since f^* is a piecewise continuous function on $[0, A]$ and hence f^* is continuous on every finite interval $(0, A)$ except possibly at a finite number of points $\beta_0, \beta_1, \dots, \beta_n$ in $(0, A)$. We can write the integral as follows by using these points as endpoints of the domain of integration

$$\frac{1}{u}\int_0^A \frac{f'(t)}{f(t)}e^{-\frac{t}{u}}dt = \frac{1}{u}\int_0^{\beta_1} \frac{f'(t)}{f(t)}e^{-\frac{t}{u}}dt + \frac{1}{u}\int_{\beta_1}^{\beta_2} \frac{f'(t)}{f(t)}e^{-\frac{t}{u}}dt + \cdots + \frac{1}{u}\int_{\beta_n}^{A} \frac{f'(t)}{f(t)}e^{-\frac{t}{u}}dt.$$

Using the integration by parts method separately to each term on the right-hand side of this expression, we get

$$\frac{1}{u}\int_0^A \frac{f'(t)}{f(t)}e^{-\frac{t}{u}}dt = \frac{1}{u}\left(e^{-\frac{t}{u}}\ln f(t)\Big|_0^{\beta_1} + e^{-\frac{t}{u}}\ln f(t)\Big|_{\beta_1}^{\beta_2} + \cdots + e^{-\frac{t}{u}}\ln f(t)\Big|_{\beta_n}^{A}\right) + \frac{1}{u^2}\left(\int_0^{\beta_1} e^{-\frac{t}{u}}\ln f(t)\,dt + \int_{\beta_1}^{\beta_2} e^{-\frac{t}{u}}\ln f(t)\,dt + \cdots + \int_{\beta_n}^{A} e^{-\frac{t}{u}}\ln f(t)\,dt\right).$$

Since $f(t)$ is continuous, the above expression can be written as

$$\frac{1}{u}\int_0^A \frac{f'(t)}{f(t)}e^{-\frac{t}{u}}dt = \frac{1}{u}e^{-A/u}\ln f(A) - \frac{1}{u}\ln f(0) + \frac{1}{u^2}\int_0^A e^{-\frac{t}{u}}\ln f(t)\,dt.$$

Hence we find

$$e^{\frac{1}{u}\int_0^A \frac{f'(t)}{f(t)}e^{-\frac{t}{u}}dt} = e^{\frac{1}{u}e^{-A/u}\ln f(A) - \frac{1}{u}\ln f(0) + \frac{1}{u^2}\int_0^A e^{-\frac{t}{u}}\ln f(t)dt}.$$

As $A \to \infty$, $\frac{1}{u}e^{-A/u}\ln f(A) \to 0$ and so $e^{\frac{1}{u}e^{-A/u}\ln f(A)} \to 1$ for $u < \frac{1}{\alpha}$. Therefore, we obtain

$$\mathcal{S}_m[f^*(t)] = e^{-\frac{1}{u}\ln f(0) + \frac{1}{u^2}\int_0^\infty e^{-\frac{t}{u}}\ln f(t)dt}$$
$$= \left(e^{\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\ln f(t)dt}\right)^{\frac{1}{u}} \frac{1}{e^{\ln f(0)^{\frac{1}{u}}}}$$
$$= F_m(u)^{\frac{1}{u}}\left[\frac{1}{f(0)}\right]^{\frac{1}{u}}$$

for $u < \frac{1}{\alpha}$. This completes the proof.

Corollary 1. Let $f, f^* \dots f^{*(n-1)}$ be continuous function and $f^{*(n)}$ be a piecewise continuous function on the interval $0 \le t \le A$ also suppose that there is positive real numbers k, α and t_0 such that

$$|f(t)| \le ke^{e^{\alpha t}}, |f^*(t)| \le ke^{e^{\alpha t}}, \dots, \left|f^{*(n-1)}(t)\right| \le ke^{e^{\alpha t}}.$$

for $t > t_0$. Then multiplicative Sumudu transform of $f^{*(n)}(t)$ exists and can be calculated by the formula

$$\mathcal{S}_m[f^{*(n)}(t)] = \frac{F_m(u)^{\left(\frac{1}{u}\right)^n}}{f(0)^{\left(\frac{1}{u}\right)^n} f^*(0)^{\left(\frac{1}{u}\right)^{n-1}} \dots f^{*(n-1)}(0)^{\frac{1}{u}}}$$

for $u < \frac{1}{\alpha}$.

Theorem 5. If $\mathcal{S}_m[f(t)] = F_m(u)$, then $\mathcal{S}_m[f(t)^t] = F_m^*(u)^{u^2} F_m(u)^u$.
Proof. By using the multiplicative Leibniz formula [16], we have

$$\begin{aligned} F_m^*(u) = \frac{d^* F_m(u)}{du} &= \frac{d^*}{du}\left(*\int_0^\infty f(t)^{\frac{1}{u}e^{-\frac{t}{u}}dt}\right) \\ &= \frac{d^*}{du}\left(e^{\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\ln f(t)dt}\right) \\ &= e^{\frac{\frac{d}{du}\left(e^{\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\ln f(t)dt}\right)}{e^{\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\ln f(t)dt}}} \\ &= e^{\int_0^\infty \frac{1}{u^3}e^{\frac{-t}{u}}(t\ln f(t))dt - \int_0^\infty \frac{1}{u^2}e^{\frac{-t}{u}}\ln f(t)dt} \\ &= \left(e^{\frac{1}{u}\int_0^\infty e^{\frac{-t}{u}}(\ln f(t)^t)dt}\right)^{\frac{1}{u^2}} \left(e^{\frac{1}{u}\int_0^\infty e^{\frac{-t}{u}}\ln f(t)dt}\right)^{-\frac{1}{u}} \\ &= \mathcal{S}_m[f(t)^t]^{\frac{1}{u^2}} F_m(u)^{-\frac{1}{u}}. \end{aligned}$$

If this equation is adjusted, the desired equality $\mathcal{S}_m[f(t)^t] = F_m^*(u)^{u^2} F_m(u)^u$ is found.

Definition 5. [16] If the multiplicative integral exists, an equation with an unknown function under one or more signs of multiplicative integration is called a multiplicative integral equation (MIE). The linear multiplicative Volterra integral equation of the second kind (LMVIESK) is constructed as

$$y(t) = f(t) * \int_0^t y(z)^{k(t,z)^{dz}}$$

where the unknown function $y(t)$ that will be determined, $k(t,z)$ is kernel of the equation. The first kind linear multiplicative Volterra integral equation (LMVIEFK) is given as

$$f(t) =* \int_0^t y(z)^{k(t,z)^{dz}}.$$

II. Faltung Type Multiplicative Volterra Integral Equations

This section presents the concept of Faltung type multiplicative Volterra integral equations and discusses the solutions of these equations by use of the multiplicative Sumudu transform.
We focus on the Faltung (convolution) type kernel $k(t,z)$, which is represented by the difference $(t-z)$. The Faltung type LMVIESK has the form

$$y(t) = f(t) * \int_0^t y(z)^{k(t-z)^{dz}}$$

and Faltung type LMVIEFK has the formula

$$f(t) =* \int_0^t y(z)^{k(t-z)^{dz}}.$$

Theorem 6. The solution of Faltung type MVIEFK

$$f(t) =* \int_0^t y(z)^{k(t-z)^{dz}} \tag{1}$$

is given by

$$y(t) = \mathcal{S}_m^{-1}[Y_m(u)] = \mathcal{S}_m^{-1}\left[\mathcal{S}_m[f(t)]^{\frac{1}{u\mathcal{S}[k(t)]}}\right]$$

where k is the kernel and $\mathcal{S}_m\{y(t)\} = Y_m(u)$.

Proof. If we apply the multiplicative Sumudu transformation to either side of (1), we get

$$\mathcal{S}_m[f(t)] = \mathcal{S}_m\left[*\int_0^t y(z)^{k(t-z)^{dz}}\right].$$

Utilizing the multiplicative Faltung theorem of the multiplicative Sumudu transform, we find

$$\mathcal{S}_m[f(t)] = \mathcal{S}_m[y(t)]^{u\mathcal{S}[k(t)]}$$

$$\mathcal{S}_m[y(t)] = \mathcal{S}_m[f(t)]^{\frac{1}{u\mathcal{S}[k(t)]}} \tag{2}$$

Having applied the inverse multiplicative Sumudu transform on either side of (2), we obtain

$$y(t) = \mathcal{S}_m^{-1}\left[\mathcal{S}_m[f(t)]^{\frac{1}{u\mathcal{S}[k(t)]}}\right]$$

which represents the desired solution.

Theorem 7. The solution of Faltung type MVIESK

$$y(t) = f(t) * \int_0^t y(z)^{k(t-z)^{dz}} \tag{3}$$

is given by

$$y(t) = \mathcal{S}_m^{-1}[Y_m(u)] = \mathcal{S}_m^{-1}\left[\mathcal{S}_m[f(t)]^{\frac{1+u\mathcal{S}[k(t)]}{u\mathcal{S}[k(t)]}}\right]$$

where k is the kernel and $\mathcal{S}_m\{y(t)\} = Y_m(u)$.

Proof. We can write

$$\mathcal{S}_m[y(t)] = \mathcal{S}_m\left[f(t) * \int_0^t y(z)^{k(t-z)^{dz}}\right]$$

$$\mathcal{S}_m[y(t)] = \mathcal{S}_m[f(t)]\mathcal{S}_m\left[* \int_0^t y(z)^{k(t-z)^{dz}}\right].$$

by taking multiplicative Sumudu transform to either side of (3). We find the following expression

$$\mathcal{S}_m[y(t)] = \mathcal{S}_m[f(t)]\mathcal{S}_m[y(t)]^{\frac{1}{u\mathcal{S}[k(t)]}}$$

$$\mathcal{S}_m[y(t)] = \mathcal{S}_m[f(t)]^{\frac{1+u\mathcal{S}[k(t)]}{u\mathcal{S}[k(t)]}} \tag{4}$$

by using the multiplicative Faltung theorem of the multiplicative Sumudu transform. Having applied the inverse multiplicative Sumudu transform on either side of (4), we obtain the solution as

$$y(x) = \mathcal{S}_m^{-1}\left[\mathcal{S}_m[f(t)]^{\frac{1+u\mathcal{S}[k(t)]}{u\mathcal{S}[k(t)]}}\right].$$

The approach for solving Faltung type MVIE by using the multiplicative Sumudu transform is explained with the help of a few examples that are shown below.

Example 1. Using the multiplicative Sumudu transform method, find the solution of Faltung type LMVIEFK

$$e^{t^2} = * \int_0^t y(z)^{e^{(t-z)^{dz}}}.$$

Let us write $\mathcal{S}_m[y(t)] = Y_m(u)$. Apply the multiplicative Sumudu transform

$$\mathcal{S}_m\left[e^{t^2}\right] = \mathcal{S}_m\left[* \int_0^t y(z)^{e^{(t-z)^{dz}}}\right]$$

Now, by implementing Faltung theorem for multiplicative Sumudu transform, it is found as

$$e^{2!u^2} = \mathcal{S}_m[y(t)]^{u\mathcal{S}[e^t]}$$

$$e^{2u^2} = \mathcal{S}_m[y(t)]^{u\cdot\frac{1}{1-u}}$$

Hence, we find

$$\mathcal{S}_m[y(t)] = Y_m(u) = e^{2u-2u^2}$$

We obtain by using the inverse multiplicative Sumudu transform as

$$\begin{aligned} y(t) &= \mathcal{S}_m^{-1}\left[e^{2u-2u^2}\right] \\ &= \mathcal{S}_m^{-1}\left[e^{2u}\right]\mathcal{S}_m^{-1}\left[e^{-2u^2}\right] \\ &= e^{2t}e^{-t^2}. \end{aligned}$$

As a result, we find the solution as

$$y(t) = e^{2t-t^2}.$$

Example 2. Use the multiplicative Sumudu transform method to solve Faltung type LMVIESK

$$y(t) = e^{\sin t} * \int_0^t y(z)^{2\cos(t-z)^{dz}}.$$

Let us write $\mathcal{S}_m[y(t)] = Y_m(u)$. Having applied the multiplicative Sumudu transform

$$\begin{aligned} \mathcal{S}_m[y(t)] &= \mathcal{S}_m\left[e^{\sin t} * \int_0^t y(z)^{2\cos(t-z)^{dz}}\right] \\ &= \mathcal{S}_m\left[e^{\sin t}\right]\mathcal{S}_m\left[* \int_0^t y(z)^{2\cos(t-z)^{dz}}\right] \\ &= \mathcal{S}_m\left[e^{\sin t}\right]\mathcal{S}_m\left[* \int_0^t y(z)^{\cos(t-z)^{dz}}\right]^2. \end{aligned}$$

Utilizing Faltung theorem for multiplicative Sumudu transform, we have

$$Y_m(u) = \mathcal{S}_m[e^{\sin t}]\mathcal{S}_m[y(t)]^{2u\mathcal{S}[\cos t]}$$
$$= e^{\frac{u}{1+u^2}}(Y_m(u))^{2u\frac{1}{1+u^2}}.$$

Hence, we write

$$\mathcal{S}_m[y(t)] = Y_m(u) = e^{\frac{u}{(u-1)^2}}$$

Operating inverse multiplicative Sumudu transform, we obtain

$$y(t) = \mathcal{S}_m^{-1}\left[e^{\frac{u}{(u-1)^2}}\right] = \mathcal{S}_m^{-1}\left[\left(e^{\frac{1}{(u-1)^2}}\right)^{u^2}\left(e^{\frac{1}{1-u}}\right)^{u}\right] = e^{te^t}.$$

Consequently, we arrive at the answer as

$$y(t) = e^{te^t}.$$

Now, we will give an example of how to solve a linear multiplicative differential equation with initial condition by transforming this equation into a multiplicative integral transform with the aid of the multiplicative Sumudu transform.

Example 3. Take the initial value problem

$$\begin{cases} u^{**}(t)u(t) = 1 \\ u(0) = e, u^{*}(0) = 1 \end{cases} \tag{5}$$

This is equivalent to multiplicative Volterra equation

$$y(t) = e^{-1} * \int_0^t y(z)^{(z-t)^{dz}}.$$

If we apply multiplicative Sumudu transform on either side, we obtain

$$\mathcal{S}_m[y(t)] = \mathcal{S}_m\left[e^{-1} * \int_0^t y(z)^{(z-t)^{dz}}\right]$$
$$= \mathcal{S}_m[e^{-1}]\mathcal{S}_m\left[\int_0^t y(z)^{(z-t)^{dz}}\right]$$

Let us write $\mathcal{S}_m[y(x)] = Y_m(u)$. Considering the multiplicative Faltung theorem for multiplicative Sumudu, we have

$$Y_m(u) = e^{-1}\mathcal{S}_m[y(t)]^{-u\mathcal{S}[t]}$$
$$= e^{-1}Y_m(u)^{-u^2}$$

Hence, we find

$$\mathcal{S}_m[y(t)] = Y_m(u) = e^{-\frac{1}{u^2+1}}$$

Then having applied the inverse multiplicative Sumudu transform, we get

$$y(t) = \mathcal{S}_m^{-1}\left[e^{-\frac{1}{u^2+1}}\right]$$
$$= \mathcal{S}_m^{-1}\left[e^{\frac{1}{u^2+1}}\right]^{-1}$$
$$= e^{-\cos t}$$

Consequently, we have the solution as

$$y(t) = e^{-\cos t}$$

Since equation (5) is equivalent to the second-order differential equation

$$u''(t)u(t) - \left(u'(t)\right)^2 + (u(t))^2 \ln u(t) = 0, u(0) = e, u'(0) = 0$$

its solution is also $u(t) = e^{-\cos t}$.

III. Faltung Type Multiplicative Volterra Integro-Differential Equations

This section introduces the idea of multiplicative Volterra integro-differential equations, and we deal with the solution of Faltung type multiplicative Volterra integro-differential equations of the first kind by using the multiplicative Sumudu transform.
The multiplicative Volterra integro-differential equation is constructed as

$$y^{*(n)}(t) = f(t) * \int_0^t y(z)^{k(t,z)^{dz}}$$

where $y^{*(n)}(t) = \frac{d^{*(n)}y}{dt^n}$. The first kind of multiplicative Volterra integro-differential equation is defined as

$$* \int_0^t y(z)^{k_1(t,z)^{dz}} .* \int_0^t y^{*(n)}(z)^{k_2(t,z)^{dz}} = f(t), \qquad k_2(t,z) \neq 0$$

where initial conditions are prescribed. We will focus on equations where the kernels $k_1(t,z)$ and $k_2(t,z)$ are difference kernels, i.e., each kernel depends on the difference $t - z$. Therefore, the Faltung type multiplicative Volterra integro-differential equation of the first kind has form

$$* \int_0^t y(z)^{k_1(t-z)^{dz}} .* \int_0^t y^{*(n)}(z)^{k_2(t-z)^{dz}} = f(t), \qquad k_2(t,z) \neq 0$$

Theorem 8. The solution of Faltung type multiplicative Volterra integro-differential equation of the first kind

$$*\int_0^t y(z)^{k_1(t-z)^{dz}} . * \int_0^t y^{*(n)}(z)^{k_2(t-z)^{dz}} = f(t), \qquad k_2(t,z) \neq 0 \tag{6}$$

is given by

$$y(t) = \mathcal{S}_m^{-1}\left[\left(y(0)^{\frac{\mathcal{S}[k_2(t)]}{u^n\mathcal{S}[k_1(t)]+\mathcal{S}[k_2(t)]}} y^*(0)^{\frac{u\mathcal{S}[k_2(t)]}{u^n\mathcal{S}[k_1(t)]+\mathcal{S}[k_2(t)]}} \dots y^{*(n-1)}(0)^{\frac{u^{n-1}\mathcal{S}[k_2(t)]}{u^n\mathcal{S}[k_1(t)]+\mathcal{S}[k_2(t)]}}\right)(\mathcal{S}_m[f(t)])^{\frac{u^{n-1}}{u^n\mathcal{S}[k_1(t)]+\mathcal{S}[k_2(t)]}}\right]$$

where $u^n\mathcal{S}[k_1(t)] + \mathcal{S}[k_2(t)] \neq 0$.

Proof. Taking multiplicative Sumudu transformation of either side of (6) gives

$$\mathcal{S}_m\left[*\int_0^t y(z)^{k_1(t-z)^{dz}}\right] . \mathcal{S}_m\left[*\int_0^t y^{*(n)}(z)^{k_2(t-z)^{dz}}\right] = \mathcal{S}_m[f(t)].$$

Utilizing Faltung theorem for multiplicative Sumudu transform, we have

$$\mathcal{S}_m[y(t)]^{u\mathcal{S}[k_1(t)]} . \mathcal{S}_m[y^{*(n)}(t)]^{u\mathcal{S}[k_2(t)]} = \mathcal{S}_m[f(t)].$$

From the property of multiplicative Sumudu transform of multiplicative derivatives of functions, we find

$$\mathcal{S}_m[y(t)]^{u\mathcal{S}[k_1(t)]} . \left(\frac{\mathcal{S}_m[y(t)]^{\left(\frac{1}{u}\right)^n}}{y(0)^{\left(\frac{1}{u}\right)^n} y^*(0)^{\left(\frac{1}{u}\right)^{n-1}} \dots y^{*(n-1)}(0)^{\frac{1}{u}}}\right)^{u\mathcal{S}[k_2(t)]} = \mathcal{S}_m[f(t)]$$

$$\mathcal{S}_m[y(t)]^{u\mathcal{S}[k_1(t)]} . \frac{\mathcal{S}_m[y(t)]^{\frac{\mathcal{S}[k_2(t)]}{u^{n-1}}}}{y(0)^{\frac{\mathcal{S}[k_2(t)]}{u^{n-1}}} y^*(0)^{\frac{\mathcal{S}[k_2(t)]}{u^{n-2}}} \dots y^{*(n-1)}(0)^{\mathcal{S}[k_2(t)]}} = \mathcal{S}_m[f(t)]$$

$$\mathcal{S}_m[y(t)]^{\left(\frac{u^n\mathcal{S}[k_1(t)]+\mathcal{S}[k_2(t)]}{u^{n-1}}\right)} = \left(y(0)^{\frac{\mathcal{S}[k_2(t)]}{u^{n-1}}} y^*(0)^{\frac{\mathcal{S}[k_2(t)]}{u^{n-2}}} \dots y^{*(n-1)}(0)^{\mathcal{S}[k_2(t)]}\right)\mathcal{S}_m[f(t)]$$

Using the initial conditions provided and solving for $\mathcal{S}_m[y(t)]$, we get

$$\mathcal{S}_m[y(t)] = \left(y(0)^{\frac{\mathcal{S}[k_2(t)]}{u^n\mathcal{S}[k_1(t)]+\mathcal{S}[k_2(t)]}} y^*(0)^{\frac{u\mathcal{S}[k_2(t)]}{u^n\mathcal{S}[k_1(t)]+\mathcal{S}[k_2(t)]}} \dots y^{*(n-1)}(0)^{\frac{u^{n-1}\mathcal{S}[k_2(t)]}{u^n\mathcal{S}[k_1(t)]+\mathcal{S}[k_2(t)]}}\right)(\mathcal{S}_m[f(t)])^{\frac{u^{n-1}}{u^n\mathcal{S}[k_1(t)]+\mathcal{S}[k_2(t)]}} \tag{7}$$

provided that $u^n\mathcal{S}[k_1(t)] + \mathcal{S}[k_2(t)] \neq 0$. Having applied the inverse multiplicative Sumudu transform of either side of (7), the exact solution is readily obtained.

The approach for solving these equations by using the multiplicative Sumudu transform is explained with the help of a few examples that are shown below.

Example 4. Use the multiplicative Sumudu transform method to solve Faltung type multiplicative Volterra integro-differential equation

$$e^{3t-3\mathrm{sin}t} = *\int_0^t y(z)^{(t-z)^{dz}} . * \int_0^t y^*(z)^{(t-z)^{2\,dz}}$$

with $y(0) = 1$.

Let's taken $\mathcal{S}_m[y(x)] = Y_m(u)$. Having applied the multiplicative Sumudu transform, we find

$$\mathcal{S}_m[e^{3t-3sint}] = \mathcal{S}_m\left[*\int_0^t y(z)^{(t-z)^{dz}} . * \int_0^t y^*(z)^{(t-z)^{2\,dz}}\right]$$

$$\mathcal{S}_m[e^t]^3 \ \mathcal{S}_m[e^{sint}]^{-3} = \mathcal{S}_m\left[*\int_0^t y(z)^{(t-z)^{dz}}\right]\mathcal{S}_m\left[*\int_0^t y^*(z)^{(t-z)^{2\,dz}}\right]$$

$$e^{3u} \ e^{\frac{-3u}{1+u^2}} = \mathcal{S}_m\left[*\int_0^t y(z)^{(t-z)^{dz}}\right]\mathcal{S}_m\left[*\int_0^t y^*(z)^{(t-z)^{2\,dz}}\right]$$

$$e^{\frac{u^3}{1+u^2}} = \mathcal{S}_m\left[*\int_0^t y(z)^{(t-z)^{dz}}\right]\mathcal{S}_m\left[*\int_0^t y^*(z)^{(t-z)^{2\,dz}}\right]$$

Utilizing Faltung theorem for multiplicative Sumudu transform and the property of multiplicative Sumudu transformation of multiplicative derivative of functions, we have

$$e^{\frac{3u^3}{1+u^2}} = \mathcal{S}_m[y(t)]^{u\mathcal{S}[t]}\mathcal{S}_m[y^*(t)]^{u\mathcal{S}[t^2]}$$

$$= Y_m(u)^{u^2}\left(\frac{Y_m(u)^{\frac{1}{u}}}{y(0)^{\frac{1}{u}}}\right)^{2u^3}$$

$$= Y_m(u)^{3u^2}.$$

Hence, we get

$$Y_m(u) = e^{\frac{u}{1+u^2}}.$$

Implementing the inverse multiplicative Sumudu transform, it is found as

$$y(t) = \mathcal{S}_m^{-1}\left[e^{\frac{u}{1+u^2}}\right] = e^{\sin t}.$$

Consequently, we arrive at the answer as

$$y(t) = e^{\sin t}.$$

Example 5. Use the multiplicative Sumudu transform method to solve Faltung type multiplicative Volterra integro-differential equation

$$e^{1+\sin t-\cos t} = * \int_0^t y(z)^{\cos(t-z)^{dz}} .* \int_0^t y^{*(3)}(z)^{\sin(t-z)^{dz}}$$

with $y(0) = e, y^*(0) = e, y^{**}(0) = e^{-1}$.

Let's taken $\mathcal{S}_m[y(x)] = Y_m(u)$. Having applied the multiplicative Sumudu transform, we find

$$\mathcal{S}_m[e^{1+\sin t-\cos t}] = \mathcal{S}_m\left[* \int_0^t y(z)^{\cos(t-z)^{dz}} .* \int_0^t y^{*(3)}(z)^{\sin(t-z)^{dz}}\right]$$

$$e^{1+\frac{u}{1+u^2}-\frac{1}{1+u^2}} = \mathcal{S}_m\left[* \int_0^t y(z)^{\cos(t-z)^{dz}}\right]\mathcal{S}_m\left[* \int_0^t y^{*(3)}(z)^{\sin(t-z)^{dz}}\right]$$

Utilizing Faltung theorem for multiplicative Sumudu transform and the property of multiplicative Sumudu transformation of multiplicative derivative of functions, we have

$$e^{1+\frac{u}{1+u^2}-\frac{1}{1+u^2}} = \mathcal{S}_m[y(t)]^{u\mathcal{S}[\cos t]}\mathcal{S}_m[y^{*(3)}(t)]^{u\mathcal{S}[\sin t]}$$

$$= Y_m(u)^{\frac{u}{1+u^2}}\left(\frac{Y_m(u)^{\frac{1}{u^3}}}{y(0)^{\left(\frac{1}{u}\right)^3} y^*(0)^{\left(\frac{1}{u}\right)^2} y^{**}(0)^{\left(\frac{1}{u}\right)}}\right)^{\frac{u^2}{1+u^2}}$$

$$= \frac{Y_m(u)^{\frac{1}{u}}}{e^{\frac{1}{u(u^2+1)}}e^{\frac{1}{u^2+1}}e^{-\frac{u}{u^2+1}}}.$$

Hence, we get

$$Y_m(u) = e^{\frac{u^3+u+1}{1+u^2}} = e^u e^{\frac{1}{1+u^2}}.$$

Implementing the inverse multiplicative Sumudu transform, it is found as

$$y(t) = \mathcal{S}_m^{-1}\left[e^u e^{\frac{1}{1+u^2}}\right] = e^t e^{\cos t}.$$

Consequently, we arrive at the answer as

$$y(t) = e^{t+\cos t}.$$

IV. CONCLUSIONS

In this paper, the concept of multiplicative integral is used to define the Faltung type multiplicative Volterra integral equations and the Faltung type multiplicative Volterra integro-differential equations. The Faltung type multiplicative Volterra integral equation of the first and second kinds, as well as the Faltung type multiplicative Volterra integro-differential equation of the first kind, are solved using the multiplicative Sumudu transformation. With the aid of various illustrative examples, the method for resolving these equations using the multiplicative Sumudu transform is shown. The given applications show that the accurate solutions of these integral equations are achieved with little computing effort and time. Also, a numerical example is given for solving multiplicative ordinary differential equations by the multiplicative Sumudu transform after converting them to multiplicative Volterra integral equations. So, we see from this example that we can obtain the solutions of some ordinary differential equations with the aid of the relation between multiplicative calculus and classic calculus.

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