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On Chromatic Number of Some classes of Graceful Graphs

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Abstract: In this paper, we find results for chromatic number of some classes of graceful graphs, namely wheel graphs, book graphs, Jahangir graphs, prism graphs, and web graphs.

Keywords: Labeling of a graph, Vertex coloring, chromatic number, graceful graphs.

1. INTRODUCTION

In graph theory, graph coloring is a special case of graph labeling. It is an assignment of colors to elements of a graph. As a graph has vertices and edges, so, we may assign colors to vertices or edges. If a graph is planar, then regions of such a graph are well-defined, and so, colors may be assigned to regions of a planar graph. It is known that a graceful labeling is a special graph labeling of a graph on m edges in which the vertices are labeled with a subset of distinct non-negative integers from 0 to m and the edges are labeled with the absolute difference between vertex values. If the resulting edge numbers run from 1 to m (both inclusive), then the labeling is said to be graceful labeling ([4]). In this paper, we will discuss only vertex coloring and find vertex chromatic number of some classes of graceful graphs.

2. PRELIMINARIES

All graph-theoretic notions and results used here, but not defined or explained, are fairly standard by now (and can be found in [3], [6]). However, forconvenience, we recall some of the notions used in the sequel.

By a graph G, we mean an ordered pair (V, E), where V is a non-empty set, whose elements are called vertices and E is a set, containing unordered pair of vertices. Elements of E are called edges. The vertices, to which an edge is associated, are called end vertices of that edge. Conventionally, a graph is represented by a diagram, in which vertices are denoted by points (or circles) and edges are denoted by line segments joining their end vertices. Two vertices are said to be adjacent, if they are end vertices of some edge in the graph.

If the end vertices of an edge are the same, then such an edge is called a self-loop. If any two edges have the same end vertices, then the edges are said to be parallel edges. A graph, having no self-loops and parallel edges, is called a simple graph. The number of edges, incident on a vertex (with self-loop counted twice) is called the degree of that vertex.

A graph G'= (V', E') is said to be a subgraph of a graph G= (V, E) if $V' \subseteq V$ and $E' \subseteq E$, such that each edge of E' is incident with the vertices in V'.A simple graph is said to be complete if there is an edge between every pair of vertices. A simple graph G= (V, E) is said to be bipartite if its vertex set V can be partitioned into two subsets V_1 and V_2 such that any two vertices of V_i are not adjacent for *i*=1, 2. A bipartite graph is said to be a complete bipartite graph if each vertex of V_1 is adjacent to every vertex of V_2.If V_1 and V_2 contain m and n vertices respectively, then the complete bipartite graph is denoted by $k_{m,n}$([3]).

By a path in a graph, we mean a finite sequence of vertices and edges, beginning and ending with vertices such that no edge and vertex appear more than once. If beginning and ending vertices of a path are the same, then such a path is called a cycle. A graph is called connected if there is a path between its every pair of vertices. If a graph is not connected, then it is called disconnected. A maximal connected subgraph of a graph is called a component of the graph. A connected graph, having no cycles, is called a tree.

For a graph G= (V, E), a vertex labeling is a function of V to a set of labels; a graph with such a function, is called a vertex-labeled graph. If the set of labels is replaced by a set of colors, then we get a vertex-labeled graph, more specifically, vertex colored graph.

By vertex coloring, we mean assigning colors to the vertices of a graph such that no adjacent vertices have the same color. The minimum number of colors required for coloring of vertices of a graph G, is called the vertex chromatic number (or simply chromatic number) of G and it is denoted by $\chi(G)$. Then clearly, $\chi(G)$ is a natural number. If chromatic number of a graph is k, then we say that the graph is k-chromatic.

Some observations:

1. A graph having no edge is 1-chromatic.
2. Chromatic number of a cycle of even length is 2.
3. Chromatic number of a cycle of odd length is 3.
4. Chromatic number of a tree is 2.
5. Chromatic number of a complete graph of n vertices is n.
6. Chromatic number of a bipartite graph with at least one edge, is 2.

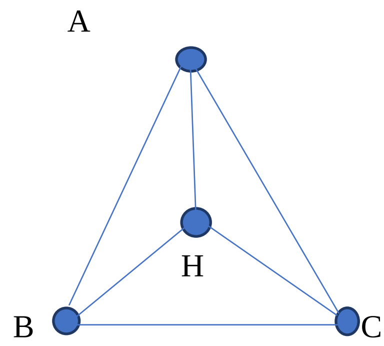
Remark: If a graph has a self-loop, then we cannot assign different colors to the end vertices and so, we consider only those graphs If a graph has parallel edges, then vertex coloring is the same, whether there is one edge between a pair of vertices or more edges, so we consider those graphs, which have no parallel edges. As vertex coloring of one component of a graph is unaffected by vertex coloring of any other component, so we consider only connected graphs.

3. CHROMATIC NUMBER OF SOME CLASSES OF GRACEFUL GRAPHS

Definition 3.1([4],[9]):A graph, which has a graceful labeling is called a *graceful graph*. By **a** graceful labeling of a graph with *m* edges, we mean, labeling of its vertices with some subset of the integers from 0 to *m* inclusive, such that no two vertices have the same label, and each edge is uniquely identified by the absolute difference between its endpoints, such that this magnitude lies between 1 and *m* inclusive.

Definition 3.2([6]): A *wheel graph* W_n , n≥ 4, is a graph with *n* vertices such that a single vertex (called hub vertex) is adjacent to every vertex in an $(n-1)$-cycle. The vertices in the cycle are called rim vertices and the edges between hub and rim vertices are called spokes.

For example, the following graph is W_4, in which H is the hub vertex, A, B, C are rim verices and the edges AH, BH, CH are spokes.



Theorem 3.1: The Chromatic number of a wheel graph W_n , where n≥ 4 is

(i) 4, if n is even and
(ii) 3, n is odd.

Proof: We know that W_n contains a cycle of length $n-1$.

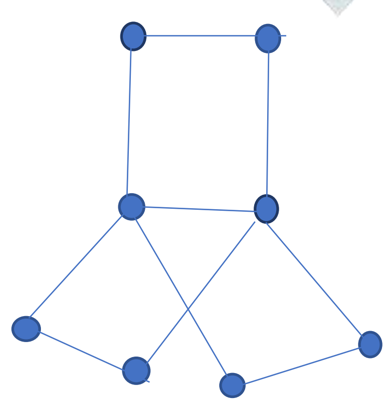
If n is even, W_n contains a cycle of odd length. As a cycle of odd length is 3-chromatic, i.e., the minimum number of colors for coloring of the rim vertices of $W_n$is 3 and all the rim vertices are adjacent to the hub vertex, so we need one more color. Hence W_n is 4-chromatic.

If n is odd, W_n contains a cycle of even length. As a cycle of even length is 2-chromatic, i.e., the minimum number of colors for coloring of the rim vertices of W_n is 2 and all the rim vertices are adjacent to the hub vertex, so we need one more color. Hence W_n is 3-chromatic.

Before defining a book graph, it is needed to define the Cartesian product of two graphs and star graphs. The Cartesian graph product G = $G_1 \boxdot G_2$, ([1], p. 104) of two graphs G_1 and G_2 with disjoint point V_1 sets and V_2 and edge sets E_1 and E_2 is the graph with vertex set $V_1 \times V_2$and $u = (u_1, u_2)$ and $v = (v_1, v_2)$ adjacent whenever ($u_1 = v_1$ and u_2 and v_2 are adjacent) or ($u_2 = v_2$ and u_1 and v_1 are adjacent).

A star graph S_n is a complete bipartite graph $k_{1,n-1}$. Clearly, it is a tree.

Definition 3.3([2], [8]): The n-book graph is defined as the graph Cartesian product $B_n = S_{n+1} \boxdot P_2$, where $S_{k+1}$is a star graph and P_2 is the path graph on two vertices. The following graph is B_3

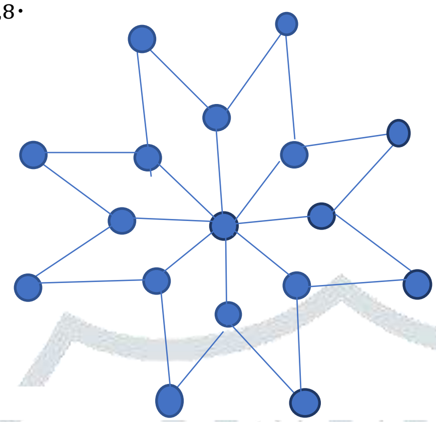


Theorem 3.2: The Chromatic number of a book graph B_n is 2.

Proof: First, we note that the book graph B_n for $n \geq 3$, has n cycles of 4 length such that exactly one edge is common in any two cycles. So, every book graph is 2-chromatic.

Definition 3.4([9]): The Jahangir graph $J_{n,m}$, n ≥ 1, m ≥ 3, is a graph with $nm + 1$ vertices, consisting of the cycle $C_{nm}$with an additional central vertex say u which is adjacent to cyclically labeled vertices v_1,v_2, v_3 . . . ,v_m such that d(v_i, v_{i+1}) = n, 1 ≤ i ≤ m − 1 in C_{nm}.

e.g., the following graph is $J_{2,8}$.



Theorem 3.3: Any Jahangir graph $J_{n,m}$is 2-chromatic, if n is even and 3-chromatic is n is odd.

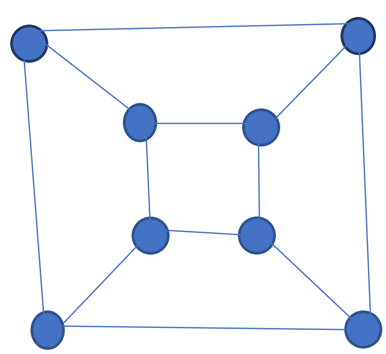
Proof: We know that the Jahangir graph $J_{n,m}$ has a vertex in the centre, say u and there are *m* vertices, adjacent to it. Let the *m* vertices be v_1,v_2, v_3 ,. . ., v_m. As any two vertices of these m vertices are not adjacent, so we can assign the same color to these *m* vertices. By definition, there is a path of length n between v_i and v_{i+1}, for 1 ≤ *i* ≤ *m* – 1. Now, we consider the following two cases:

Case1. If n is even, then there is a path of even length between v_i, v_{i+1}, $1 \leq i \leq m - 1$. So, there is a cycle of length *n*+2, starting and ending with u, i.e., a cycle of even length, containing v_i, v_{i+1}, 1 ≤ *i*≤ *m*– 1. Hence $J_{n,m}$ is 2-chromatic.

Case 2: If n is odd, then there is a path of odd length between v_i and v_{i+1}, for 1 ≤ i ≤ m – 1. So, there is a cycle of length *n*+2, starting and ending with u, i.e., a cycle of odd length, containing $v_i and\ v_{i+1}$, 1 ≤ *i*≤ *m*– 1. Hence $J_{n,m}$ is 3-chromatic.

Definition 3.5([6]): A prism graphY_n is a graph, corresponding to a skeleton of an n-prism. An n-prism graph has 2n vertices and 3n edges.It is isomorphic to graph Cartesian product $P_2 \boxdot C_n$, where P_2 is a path graph with two vertices and C_n is a cycle of length n.

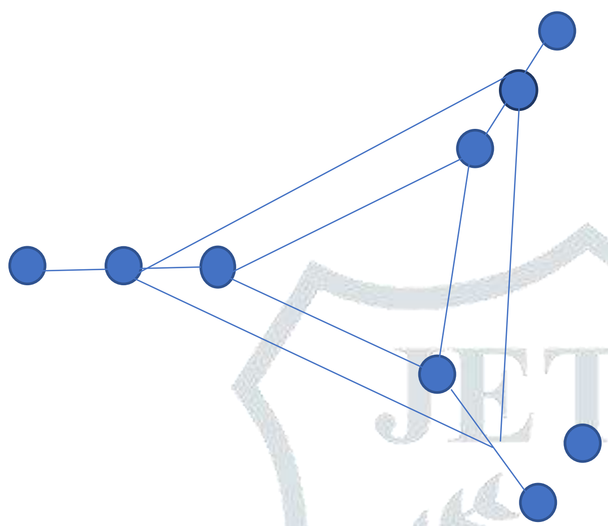
e.g., the following graph is Y_4.



Theorem 3.4: A prism graph Y_n is 2-chromatic or 3-chromatic.

Proof: Clearly, Y_n contains two copies of C_n. If n is even, then Y_n has two cycles of even length and so, it is 2-chromatic. If n is odd, then Y_n has two cycles of odd length and so, it is 3-chromatic.

Definition 3.6([7]): A web graph $Y_{m,n}$ is a stacked prism graph, i.e., $Y_{m,n} = C_m \boxdot P_n$, where C_m is a cycle graph of m vertices and P_n is a path graph with n vertices.



From the definition of $Y_{m,n}$, the proof of the following theorem clearly follows.

Theorem 3.5: A web graph $Y_{m,n}$ is 2-chromatic if m is even and it is 3-chromatic if m is odd.

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