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STABILITIES OF A ADDITIVE FUNCTIONAL EQUATION ORIGINATING FROM SUM OF ASCENDING AND DESCENDING *N* NATURAL NUMBERS IN VARIOUS BANACH SPACES

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Abstract : In this paper, the authors provide the generalized Ulam - Hyers stability of a additive functional equation which is originating from sum of ascending and descending *N* natural numbers in various Banach Spaces.

Index Terms - **Natural number series, Additive functional equations, generalized Hyers-Ulam stability, Banach space, Quasi beta Banach Space, Intuitionistic Fuzzy Banach Space, Fixed point.**

I. INTRODUCTION

During the past eight decades, many authors have extensively studied the stability problems of several functional equations in [2,19,20,26,29,30]. The generalized terminology Ulam-Hyers stability comes from these chronological backgrounds. These terminologies are also applied to the case of other functional equations. More detailed definitions of such terms [1,17,18,21,22,24,25,31-35].

The famous Cauchy additive functional equation is

$$A\left(\sum_{R=1}^{2} u_R\right)=\sum_{R=1}^{2} A(u_R). \tag{1.1}$$

Its stability in various settings were inspected in [3,20,22,28.31,32]. Several other types of additive functional equations in various normed spaces were discussed by Aczel, Dhombres [1], Arunkumar [3-13]. Bodaghi [15-16], Lee [23], Rassias [27-28] .

Theorem 1.1 [18,25] (The alternative of fixed point) Let (X, d) be a complete generalized metric space. Let $J: X \to Y$ be a strictly contractive mapping with Lipschitz constant $L < 1$.Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all non negative integers n. or there exists positive integers n_0 such that

- [FP1] $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$.
- [FP2] The sequence $\{J^n x\}$ converges to a fixed point y^* of J.
- [FP3] y^* is the unique fixed point of J in the set $Y = \{y \in X / d(J^{n_0} x, y < \infty)\}$.
- [FP4] $d(y, y^*) \leq \left(\frac{1}{1-L}\right) d(y, Jy)$ for all $y \epsilon X$.

I.I SUM OF *n* NATURAL NUMBERS

The sum of *n* natural numbers formula is used to find 1 + 2 + 3 + 4 +..... up to *n* terms. This is arranged in an arithmetic sequence. Hence we use the formula of the sum of *n* terms in the arithmetic progression for deriving the formula for the sum of natural numbers. Sum of *n* natural numbers can be defined as a form of arithmetic progression where the sum of *n* terms are arranged in a sequence with the first term being 1, n being the number of terms along with the nth term. The sum of *n* natural numbers is represented as *[n(n+1)]/2*.

In this paper, the authors provide the general solution and generalized Ulam - Hyers stability of a additive functional equation which is originating from sum of ascending and descending *N* natural numbers

$$A\left(\sum_{R=1}^{N} R u_R\right)+A\left(\sum_{R=1}^{N}(N-R+1)u_R\right)=(N+1)\sum_{R=1}^{N}A(u_R). \tag{1.2}$$

in various Banach Spaces with the help of two different methods.

II. General Solution

In this part, we consult about the general solution of functional equation (1.2) by taking E and S be real vector spaces.

Theorem 2.1. If $A:E\to S$ be a function agreeable the functional equation (1.1) if and only if $A:E\to S$ be a function agreeable the functional equation (1.2) for all $u_1,\cdots,u_N\in E$.

Proof. By data, if $A:E\to S$ be a function agreeable the functional equation (1.1). Altering, (u_1,u_2) by $(0,0),(u,-u),(u,u)$ $(u,2u)$ in (1.1) and for any positive integer θ, one can achieve

$$A(0)=0; A(-u)=-A(u); A(2u)=2A(u); A(3u)=3A(u);\ A(\theta u)=\theta A(u);\ \forall u\in E. \tag{2.1}$$

Substituting (u_1,u_2) by $(u_1,u_2+u_3+\cdots+u_N)$ in (1.1) using (2.1) as well as (1.1), we attain

$$A(u_1+u_2+u_3+\cdots+u_N)=A(u_1)+A(u_2)+A(u_3)+\cdots+A(u_N);\ \forall u\in E. \tag{2.2}$$

Interchanging $(u_1,u_2,u_3,\cdots,u_N)$ by $(1u_1,2u_2,3u_3,\cdots,Nu_N)$ in (1.1) and using (2.1), we land

$$A(1u_1+2u_2+3u_3+\cdots+Nu_N)=1A(u_1)+2A(u_2)+3A(u_3)+\cdots+NA(u_N);\ \forall u\in E. \tag{2.3}$$

Again, Interchanging $(u_1,u_2,u_3,\cdots,u_N)$ by $(Nu_N,(N-1)u_{N-1},\cdots,1u_1)$ in (1.1) and using (2.1), we arrive

$$A(Nu_N+(N-1)u_{N-1}+\cdots+1u_1)=NA(u_N)+(N-1)A(u_{N-1})+\cdots+1A(u_1);\ \forall u\in E. \tag{2.4}$$

Adding (2.3) and (2.4), we see $A:E\to S$ agreeable the functional equation (1.2) for all $u_1,\cdots,u_N\in E$.

Conversely, by data, if $A:E\to S$ be a function agreeable the functional equation (1.2). Altering, $(u_1,u_2,u_3,\cdots,u_N)$ by $(0,0,0,\cdots,0),(u,-u,0,\cdots,0),(u,u,0,\cdots,0)$ $(u,2u,0,\cdots,0)$ in (1.2) and for any positive integer θ, one can achieve

$$A(0)=0; A(-u)=-A(u); A(2u)=2A(u); A(3u)=3A(u);\ A(\theta u)=\theta A(u); A\left(\frac{u}{\theta}\right)=\frac{1}{\theta}A(u);\ \forall u\in E. \tag{2.5}$$

Substituting $(u_1,u_2,u_3,\cdots,u_N)$ by $\left(u_1,\frac{u_2}{2},0,\cdots,0\right)$ in (1.2) using (2.5), we see $A:E\to S$ agreeable the functional equation (1.1) for all $u_1,u_2\in E$.

III STABILITY RESULTS: BANACH SPACE

In this part, we study the generalized Ulam – Hyers stability in Banach space using direct and fixed point methods. In order to prove the stability results, we take that T be a normed space and M be a Banach space.

III .1 HYERS DIRECT ANALYSIS.

Theorem 3.1: Let $A:T\longrightarrow M$ be a function fulfilling the inequality

$$\left\|A\left(\sum_{R=1}^{N}Ru_R\right)+A\left(\sum_{R=1}^{N}(N-R+1)u_R\right)-(N+1)\sum_{R=1}^{N}A(u_R)\right\|\le\Theta(u_1,\cdots,u_N);\ \forall u_1,\cdots,u_N\in T \tag{3.1}$$

where $\Theta:T^N\longrightarrow[0,\infty)$ be a function with the condition

$$\lim_{C\longrightarrow\infty}\frac{\Theta\left(M^{CD}u_1,\cdots,M^{CD}u_N\right)}{M^{CD}}=0;\ M=\frac{N(N+1)}{2}; D=\pm1;\ \forall u_1,\cdots,u_N\in T. \tag{3.2}$$

Then there exists a unique additive mapping $Y:T\longrightarrow M$ which satisfying the functional equation (1.2) and the inequality

$$\|A(u)-Y(u)\|\le\frac{1}{2M}\sum_{J=\frac{1-D}{2}}^{\infty}\frac{\Theta\left(M^{JD}u,\cdots,M^{JC}u\right)}{M^{JD}};\ \ Y(u)=\lim_{C\longrightarrow\infty}\frac{A\left(M^{CD}u\right)}{M^{CD}};\ \forall u\in T. \tag{3.3}$$

Proof. Substituting $(u_1,u_2,u_3,\cdots,u_N)$ by $(u,u,u,\cdots,u)$ in (3.1), one can arrive that

$$\left\|A(1u+2u+3u+\cdots+Nu)+A(Nu+(N-1)u+\cdots+1u)-(N+1)\sum_{R=1}^{N}A(u)\right\|\le\Theta(u,\cdots,u);\ \forall u\in T;$$

which implies

$$\|2A((1+2+3+\cdots+N)u)-N(N+1)A(u)\|\le\Theta(u,\cdots,u);\ \forall u\in T.$$

By definition of M in (3.2), which gives

$$\left\|A\left(\left(\frac{N(N+1)}{2}\right)u\right)-\frac{N(N+1)}{2}A(u)\right\|\le\frac{1}{2}\times\Theta(u,\cdots,u)\Rightarrow\|A(Mu)-MA(u)\|\le\frac{1}{2}\times\Theta(u,\cdots,u);\ \forall u\in T \tag{3.4}$$

It follows from (3.4), that

$$\left\|\frac{1}{M}A(Mu)-A(u)\right\|\le\frac{1}{2M}\times\Theta(u,\cdots,u);\ \forall u\in T. \tag{3.5}$$

Once can verify for any positive integer C, (3.5) can be generalized as

$$\left\|\frac{1}{M^C}A\left(M^C u\right)-A(u)\right\| \le \frac{1}{2M}\times\sum_{J=0}^{C-1}\frac{\Theta\left(M^J u,\cdots,M^J u\right)}{M^J};\ \forall u\in T. \tag{3.6}$$

Replacing $u = M^{C_1} u$ and divided by M^{C_1} in (3.6), one can have that

$$\left\|\frac{1}{M^{C+C_1}}A\left(M^{C+C_1} u\right)-\frac{1}{M^{C_1}}A\left(M^{C_1}u\right)\right\| \le \frac{1}{2M}\times\sum_{J=0}^{C-1}\frac{\Theta\left(M^{J+C_1}u,\cdots,M^{J+C_1}u\right)}{M^{J+C_1}};\ \forall u\in T. \tag{3.7}$$

Letting $C_1\to\infty$ in (3.7), one can see that the sequence $\left\{\frac{1}{M^C}A\left(M^C u\right)\right\}$ is a Cauchy sequence and converges to $Y(u)\in M$. So, define a mapping $Y:T\longrightarrow M$ by

$$Y(u)=\lim_{C\longrightarrow\infty}\frac{A\left(M^C u\right)}{M^C};\ \forall u\in T. \tag{3.8}$$

Approaching limit C tends to infinity in (3.7) and using (3.8), one can attain that

$$\left\|A(u)-Y(u)\right\| \le \frac{1}{2M}\times\sum_{J=0}^{\infty}\frac{\Theta\left(M^J u,\cdots,M^J u\right)}{M^J};\ \forall u\in T. \tag{3.9}$$

Now, to prove the existence of $Y(u)$ satisfies the functional equation (1.2), changing $(u_1,\cdots,u_N)$ by $\left(M^C u_1,\cdots,M^C u_N\right)$ and divided by M^C in (3.1), one can obtain that

$$\frac{1}{M^C}\left\|A\left(\sum_{R=1}^{N}M^C Ru_R\right)+A\left(\sum_{R=1}^{N}M^C\left(N-R+1\right)u_R\right)-(N+1)\sum_{R=1}^{N}A\left(M^C u_R\right)\right\| \le \frac{1}{M^C}\times\Theta\left(M^C u_1,\cdots,M^C u_N\right);\ \forall u_1,\cdots,u_N\in T. \tag{3.10}$$

Approaching limit C tends to infinity in (3.10) and using (3.8), one can see that $Y(u)$ satisfies the functional equation (1.2). It is easy to verify that the existence of $Y(u)$ is unique, it follows form (3.8), (3.9) and for any positive number C_1, we have

$$\left\|Y(u)-Y'(u)\right\|=\left\|\frac{Y\left(M^{C_1}u\right)}{M^{C_1}}-\frac{Y'\left(M^{C_1}u\right)}{M^{C_1}}\right\|\le\left\|\frac{Y\left(M^{C_1}u\right)}{M^{C_1}}-\frac{A\left(M^{C_1}u\right)}{M^{C_1}}\right\|+\left\|\frac{Y'\left(M^{C_1}u\right)}{M^{C_1}}-\frac{A'\left(M^{C_1}u\right)}{M^{C_1}}\right\|=\frac{1}{M}\sum_{J=0}^{\infty}\frac{\Theta\left(M^{J+C_1}u,\cdots,M^{J+C_1}u\right)}{M^{J+C_1}};$$

for all $u\in T$. Taking limit C_1 tends to infinity in the above inequality, one can see the desired result. So, (3.3) holds for $D=1$.

Alternatively, interchanging $u=\frac{u}{M}$ in (3.14), one can arrive that

$$\left\|A(u)-M\,A\left(\frac{u}{M}\right)\right\| \le \frac{1}{2}\times\Theta\left(\frac{u}{M},\cdots,\frac{u}{M}\right);\ \forall u\in T. \tag{3.12}$$

Once can verify for any positive integer C, (3.12) can be generalized as

$$\left\|A(u)-M^C A\left(\frac{u}{M^C}\right)\right\| \le \frac{1}{2M}\times\sum_{J=1}^{C}M^J\,\Theta\left(\frac{u}{M^J},\cdots,\frac{u}{M^J}\right);\ \forall u\in T. \tag{3.13}$$

The rest of the proof is analogous to that of preceding case. So, (3.3) holds for $D=-1$. Hence the proof is complete.

Example 3.2: Let $A:T\longrightarrow M$ be a function fulfilling the inequality

$$\left\|A\left(\sum_{R=1}^{N}Ru_R\right)+A\left(\sum_{R=1}^{N}(N-R+1)u_R\right)-(N+1)\sum_{R=1}^{N}A(u_R)\right\| \le \Omega;\ \forall u_1,\cdots,u_N\in T;\Omega>0. \tag{3.14}$$

Then there exists a unique additive mapping $Y:T\longrightarrow M$ which satisfying the functional equation (1.2) and the inequality

$$\left\|A(u)-Y(u)\right\| \le \frac{\Omega}{2|M-1|};\ \forall u\in T. \tag{3.15}$$

Corollary 3.3: Let $A:T\longrightarrow M$ be a function fulfilling the inequality

$$\left\|A\left(\sum_{R=1}^{N}Ru_R\right)+A\left(\sum_{R=1}^{N}(N-R+1)u_R\right)-(N+1)\sum_{R=1}^{N}A(u_R)\right\| \le \Omega\sum_{R=1}^{N}\left\|u_R\right\|^H;\ \forall u_1,\cdots,u_N\in T;\Omega>0;H\neq1. \tag{3.16}$$

Then there exists a unique additive mapping $Y:T\longrightarrow M$ which satisfying the functional equation (1.2) and the inequality

$$\left\|A(u)-Y(u)\right\| \le \frac{\Omega\left\|u\right\|^H}{2\left|M-M^H\right|};\forall u\in T. \tag{3.17}$$

Corollary 3.4: Let $A:T\longrightarrow M$ be a function fulfilling the inequality

$$\left\|A\left(\sum_{R=1}^{N}Ru_R\right)+A\left(\sum_{R=1}^{N}(N-R+1)u_R\right)-(N+1)\sum_{R=1}^{N}A(u_R)\right\| \le \Omega\sum_{R=1}^{N}\left\|u_R\right\|^{H_R};\ \forall u_1,\cdots,u_N\in T;\Omega>0;H_R\neq1. \tag{3.18}$$

Then there exists a unique additive mapping $Y:T\longrightarrow M$ which satisfying the functional equation (1.2) and the inequality

$$\left\|A(u)-Y(u)\right\| \le \frac{\Omega}{2}\sum_{R=1}^{N}\frac{\left\|u\right\|^{H_R}}{\left|M-M^{H_R}\right|};\forall u\in T. \tag{3.19}$$

Corollary 3.5: Let $A:T\longrightarrow M$ be a function fulfilling the inequality

$$\left\|A\left(\sum_{R=1}^{N}Ru_R\right)+A\left(\sum_{R=1}^{N}(N-R+1)u_R\right)-(N+1)\sum_{R=1}^{N}A(u_R)\right\|\le\Omega\prod_{R=1}^{N}\|u_R\|^{H};\ \forall u_1,\cdots,u_N\in T;\Omega>0;NH\ne1. \quad (3.20)$$

Then there exists a unique additive mapping $Y:T\longrightarrow M$ which satisfying the functional equation (1.2) and the inequality

$$\|A(u)-Y(u)\|\le\frac{\Omega\|u\|^{NH}}{2\left|M-M^{NH}\right|};\forall u\in T. \quad (3.21)$$

Corollary 3.6: Let $A:T\longrightarrow M$ be a function fulfilling the inequality

$$\left\|A\left(\sum_{R=1}^{N}Ru_R\right)+A\left(\sum_{R=1}^{N}(N-R+1)u_R\right)-(N+1)\sum_{R=1}^{N}A(u_R)\right\|\le\Omega\prod_{R=1}^{N}\|u_R\|^{H_R};\ \forall u_1,\cdots,u_N\in T;\Omega>0;\sum_{R=1}^{N}H_R\ne1. \quad (3.22)$$

Then there exists a unique additive mapping $Y:T\longrightarrow M$ which satisfying the functional equation (1.2) and the inequality

$$\|A(u)-Y(u)\|\le\frac{\Omega\|u\|^{\sum_{R=1}^{N}H_R}}{2\left|M-M^{\sum_{R=1}^{N}H_R}\right|};\forall u\in T. \quad (3.23)$$

Corollary 3.7: Let $A:T\longrightarrow M$ be a function fulfilling the inequality

$$\left\|A\left(\sum_{R=1}^{N}Ru_R\right)+A\left(\sum_{R=1}^{N}(N-R+1)u_R\right)-(N+1)\sum_{R=1}^{N}A(u_R)\right\|\le\Omega\left(\sum_{R=1}^{N}\|u_R\|^{NH}+\prod_{R=1}^{N}\|u_R\|^{H}\right);\ \forall u_1,\cdots,u_N\in T;\Omega>0,NH\ne1. \quad (3.24)$$

Then there exists a unique additive mapping $Y:T\longrightarrow M$ which satisfying the functional equation (1.2) and the inequality

$$\|A(u)-Y(u)\|\le\frac{\Omega\,(N+1)\|u\|^{NH}}{2\left|M-M^{NH}\right|};\forall u\in T. \quad (3.25)$$

Corollary 3.8: Let $A:T\longrightarrow M$ be a function fulfilling the inequality

$$\left\|A\left(\sum_{R=1}^{N}Ru_R\right)+A\left(\sum_{R=1}^{N}(N-R+1)u_R\right)-(N+1)\sum_{R=1}^{N}A(u_R)\right\|\le\Omega\left(\sum_{R=1}^{N}\|u_R\|^{H_R}+\prod_{R=1}^{N}\|u_R\|^{H_R}\right);\ \forall u_1,\cdots,u_N\in T;\Omega>0;\sum_{R=1}^{N}H_R\ne1. \quad (3.26)$$

Then there exists a unique additive mapping $Y:T\longrightarrow M$ which satisfying the functional equation (1.2) and the inequality

$$\|A(u)-Y(u)\|\le\frac{\Omega}{2}\left(\sum_{R=1}^{N}\frac{\|u\|^{H_R}}{\left|M-M^{H_R}\right|}+\frac{\|u\|^{\sum_{R=1}^{N}H_R}}{\left|M-M^{\sum_{R=1}^{N}H_R}\right|}\right);\forall u\in T. \quad (3.27)$$

III .2 RADUS FIXED ANALYSIS.

Theorem 3.9: Let $A:T\longrightarrow M$ be a function fulfilling the inequality (3.1) where $\Theta:T^N\longrightarrow[0,\infty)$ be a function with the condition

$$\lim_{C\longrightarrow\infty}\frac{\Theta\left(\mathrm{E}_e^C u_1,\cdots,\mathrm{E}_e^C u_N\right)}{\mathrm{E}_e^C}=0;\mathrm{E}_0=M;\mathrm{E}_1=\frac{1}{M};\text{with } e=0\text{ or }1;\ \forall u_1,\cdots,u_N\in T. \quad (3.28)$$

If there exists $L=L(e)$ such that the functions $\Theta(u,\cdots,u)$ has the properties

$$\Theta(u,\cdots,u)=\frac{1}{2}\times\Theta\left(\frac{u}{M},\cdots,\frac{u}{M}\right)\ \text{ and }\ \frac{1}{\mathrm{E}_e}\Theta(\mathrm{E}_e u,\cdots,\mathrm{E}_e u)=L\,\Theta(u,\cdots,u);\ \forall u\in T. \quad (3.29)$$

Then there exists a unique additive mapping $Y:T\longrightarrow M$ which satisfying the functional equation (1.2) and the inequality

$$\|A(u)-Y(u)\|\le\left[\frac{L^{1-e}}{(1-L)}\right]\Theta(u,\cdots,u);\ Y(u)=\lim_{C\longrightarrow\infty}\frac{A\left(\mathrm{E}_e^C u\right)}{\mathrm{E}_e^C};\ \forall u\in T. \quad (3.30)$$

Proof. Assume the set $B=\left\{\ a:T\longrightarrow M/a(0)=0\ \right\}$ and introduce the generalized metric on the B as $d(a,b)=\inf\left\{\ K\in(0,\infty)/\|a(u)-b(u)\|\le\Theta(u,\cdots,u);\ u\in T\ \right\}$. It is easy to see that (B,d) is complete. Suppose assume a function $Z:B\longrightarrow B$ by $Z(u)=\dfrac{a(\mathrm{E}_e u)}{\mathrm{E}_e};\ \forall u\in T$. For any $a,b\in B$, by [25] it is easy to verify that $d(Za,Zb)\le LK$ which implies $d(Za,Zb)\le L\,d(a,b)$. That is Z is a strictly contractive mapping on B with Lipschitz constant L.

By definition of B, Z, (3.29) and (3.5) for $e=0$ it comes to

$$\left\|\frac{1}{M}A(Mu)-A(u)\right\|\le L\Theta(u,\cdots,u)\Rightarrow d(ZA,A)\le L^{1-e};\ \forall u\in T. \quad (3.31)$$

By definition of B, Z, (3.29) and (3.12) for $e=1$ it comes to

$$\left\| A(u) - M\,A\left(\frac{u}{M}\right) \right\| \le \Theta(u,\cdots,u) \Rightarrow d(A, Z\,A) \le L^{1-e};\ \forall u \in T. \quad (3.32)$$

From (3.31) and(3.32), we land that

$$d(A, Z\,A) \le L^{1-e}. \quad (3.33)$$

By Theorem 1.1, the proof holds and this completes the proof.

Corollary 3.10: Let $A : T \longrightarrow M$ be a function fulfilling the inequalities (3.14), (3.16), (3.20), (3.24). Then there exists a unique additive mapping $Y : T \longrightarrow M$ which satisfying the functional equation (1.2) and the inequalities (3.15), (3.17), (3.21), (3.25).

Proof. If we take the RHS of (3.1) as (3.14), (3.16), (3.20), (3.24) with replacing $(u_1,\cdots,u_N) = (\mathrm{E}_e^C u_1,\cdots,\mathrm{E}_e^C u_N)$ and dividing by E_e^C one can see that (3.28) holds. By definition of E_e, (3.29) and (3.30), the proof holds.

IV STABILITY RESULTS: QUASI-BETA BANACH SPACE

In this part, we study the generalized Ulam – Hyers stability in Quasi-Beta Banach space using direct and fixed point methods. In order to prove the stability results, we take that T be a linear space and M be a Quasi-Beta Banach space. For basic facts concerning Quasi-Beta Banach space and some preliminary results one can refere [17,31,35].

IV .1 HYERS DIRECT ANALYSIS.

Theorem 4.1: Let $A : T \longrightarrow M$ be a function fulfilling the inequality (3.1) where $\Theta : T^N \longrightarrow [0,\infty)$ be a function with the condition

$$\lim_{C \longrightarrow \infty} \frac{\Theta(M^{CD}u_1,\cdots,M^{CD}u_N)}{M^{\beta CD}} = 0;\ M = \frac{N(N+1)}{2}; D = \pm 1;\ \forall u_1,\cdots,u_N \in T. \quad (4.1)$$

Then there exists a unique additive mapping $Y : T \longrightarrow M$ which satisfying the functional equation (1.2) and the inequality

$$\|A(u) - Y(u)\| \le \frac{1}{2^\beta M^\beta} \sum_{J=\frac{1-D}{2}}^{\infty} \frac{K^J\Theta(M^{JD}u,\cdots,M^{JC}u)}{M^{\beta JD}};\ Y(u) = \lim_{C \longrightarrow \infty} \frac{A(M^{CD}u)}{M^{CD}};\ \forall u \in T. \quad (4.2)$$

Proof. By definition of M in (3.2), which gives

$$\left\| A\left(\left(\frac{N(N+1)}{2}\right)u\right) - \frac{N(N+1)}{2}A(u) \right\| \le \frac{1}{2^\beta} \times \Theta(u,\cdots,u) \Rightarrow \|A(M\,u) - MA(u)\| \le \frac{1}{2^\beta} \times \Theta(u,\cdots,u);\ \forall u \in T. \quad (4.3)$$

It follows from (4.4), that

$$\left\| \frac{1}{M}A(M\,u) - A(u) \right\| \le \frac{1}{2^\beta M^\beta} \times \Theta(u,\cdots,u);\ \forall u \in T. \quad (4.4)$$

Once can verify for any positive integer C, (4.5) can be generalized as

$$\left\| \frac{1}{M^C}A(M^C u) - A(u) \right\| \le \frac{1}{2^\beta M^\beta} \times \sum_{J=0}^{C-1} \frac{K^J\Theta(M^J u,\cdots,M^J u)}{M^{\beta J}};\ \forall u \in T. \quad (4.5)$$

Corollary 4.2: Let $A : T \longrightarrow M$ be a function fulfilling the inequality

$$\left\| A\left(\sum_{R=1}^{N} R u_R\right) + A\left(\sum_{R=1}^{N}(N-R+1)u_R\right) - (N+1)\sum_{R=1}^{N} A(u_R) \right\| \le \begin{cases} \Omega; \\ \Omega\sum_{R=1}^{N}\|u_R\|^H; H \ne 1; \\ \Omega\prod_{R=1}^{N}\|u_R\|^H; NH \ne 1; \\ \Omega\left(\sum_{R=1}^{N}\|u_R\|^{NH} + \prod_{R=1}^{N}\|u_R\|^H\right); NH \ne 1; \end{cases} \quad \forall\ u_1,\cdots,u_N \in T; \Omega > 0; \quad (4.6)$$

Then there exists a unique additive mapping $Y : T \longrightarrow M$ which satisfying the functional equation (1.2) and the inequality

$$\|A(u) - Y(u)\| \le \begin{cases} \dfrac{\Omega}{2^\beta |M^\beta - K|}; \\ \dfrac{\Omega\|u\|^H}{2^\beta |M^\beta - KM^H|}; \\ \dfrac{\Omega\|u\|^{NH}}{2^\beta |M^\beta - KM^{NH}|}; \\ \dfrac{\Omega\,(N+1)\|u\|^{NH}}{2^\beta |M^\beta - KM^{NH}|}; \end{cases} \quad \forall u \in T. \quad (4.7)$$

IV .2 RADUS FIXED ANALYSIS.

Theorem 4.3: Let $A : T \longrightarrow M$ be a function fulfilling the inequality (3.1) where $\Theta : T^N \longrightarrow [0,\infty)$ be a function with the condition

$$\lim_{C\longrightarrow\infty}\frac{\Theta\left(\mathrm{E}_e^C u_1,\cdots,\mathrm{E}_e^C u_N\right)}{\mathrm{E}_e^C}=0;\mathrm{E}_0=M;\mathrm{E}_1=\frac{1}{M};\text{with } e=0\,\text{or}\,1;\ \forall u_1,\cdots,u_N\in T. \tag{4.8}$$

If there exists $L=L(e)$ such that the functions $\Theta(u,\cdots,u)$ has the properties

$$\Theta(u,\cdots,u)=\frac{1}{2^\beta}\times\Theta\left(\frac{u}{M},\cdots,\frac{u}{M}\right)\ \text{ and }\ \frac{1}{\mathrm{E}_e}\Theta\left(\mathrm{E}_e u,\cdots,\mathrm{E}_e u\right)=L\ \Theta(u,\cdots,u);\ \forall u\in T. \tag{4.9}$$

Then there exists a unique additive mapping $Y:T\longrightarrow M$ which satisfying the functional equation (1.2) and the inequality

$$\left\|A(u)-Y(u)\right\|\le\left[\frac{L^{1-e}}{(1-L)}\right]\Theta(u,\cdots,u);\ Y(u)=\lim_{C\longrightarrow\infty}\frac{A\left(\mathrm{E}_e^C u\right)}{\mathrm{E}_e^C};\ \forall u\in T. \tag{4.10}$$

Corollary 4.4: Let $A:T\longrightarrow M$ be a function fulfilling the inequalities (4.6). Then there exists a unique additive mapping $Y:T\longrightarrow M$ which satisfying the functional equation (1.2) and the inequalities (3.15), (3.17), (3.21), (3.25).

V STABILITY RESULTS: INTUITIONISTIC FUZZY BANACH SPACE

In this part, we study the generalized Ulam – Hyers stability in Banach space using direct and fixed point methods. In order to prove the stability results, we take that T be a linear space, (Z,α,β) is an intuitionistic fuzzy normed space and (M,α',β') be a Banach space. For basic facts concerning Intuitionistic Fuzzy Banach Space and some preliminary results one can refere [11,14,16,24,32-34].

V.1 HYERS DIRECT ANALYSIS.

Theorem 5.1 Let $A:T\longrightarrow M$ be a function fulfilling the inequality

$$\left.\begin{aligned}\alpha\left(A\left(\sum_{R=1}^N Ru_R\right)+A\left(\sum_{R=1}^N(N-R+1)u_R\right)-(N+1)\sum_{R=1}^N A(u_R),I\right)&\le\alpha'\left(\Theta(u_1,\cdots,u_N),I\right)\\ \beta\left(A\left(\sum_{R=1}^N Ru_R\right)+A\left(\sum_{R=1}^N(N-R+1)u_R\right)-(N+1)\sum_{R=1}^N A(u_R),I\right)&\le\beta'\left(\Theta(u_1,\cdots,u_N),I\right)\end{aligned}\right\};\forall u_1,\cdots,u_N\in T;I>0 \tag{5.1}$$

where $\Theta:T^N\longrightarrow[0,\infty)$ be a function with the conditions

$$\left.\begin{aligned}\alpha'\left(\Theta\left(M^{CD}u_1,\cdots,M^{CD}u_N\right),I\right)&\ge\alpha'\left(\wp^{CD}\Theta(u_1,\cdots,u_N),I\right)\\ \beta'\left(\Theta\left(M^{CD}u_1,\cdots,M^{CD}u_N\right),I\right)&\le\beta'\left(\wp^{CD}\Theta(u_1,\cdots,u_N),I\right)\end{aligned}\right\};\forall u_1,\cdots,u_N\in T;I>0;D=\pm1 \tag{5.2}$$

and

$$\left.\begin{aligned}\lim_{C\longrightarrow\infty}\alpha'\left(\Theta\left(M^{CD}u_1,\cdots,M^{CD}u_N\right),M^{CD}I\right)&=1\\ \lim_{C\longrightarrow\infty}\beta'\left(\Theta\left(M^{CD}u_1,\cdots,M^{CD}u_N\right),M^{CD}I\right)&=0\end{aligned}\right\};\forall u_1,\cdots,u_N\in T;I>0;M=\frac{N(N+1)}{2};D=\pm1. \tag{5.3}$$

Then there exists a unique additive mapping $Y:T\longrightarrow M$ which satisfying the functional equation (1.2) and the inequality

$$\left.\begin{aligned}\alpha\left(A(u)-Y(u),I\right)&\le\alpha'\left(\Theta(u,\cdots,u),2I\,|M-\wp|\right)\\ \beta\left(A(u)-Y(u),I\right)&\le\beta'\left(\Theta(u,\cdots,u),2I\,|M-\wp|\right)\end{aligned}\right\};\forall u\in T;I>0 \tag{5.4}$$

where

$$\left.\begin{aligned}\lim_{C\longrightarrow\infty}\alpha\left(\frac{A\left(M^{CD}u\right)}{M^{CD}}-Y(u),I\right)&=1\\ \lim_{C\longrightarrow\infty}\beta\left(\frac{A\left(M^{CD}u\right)}{M^{CD}}-Y(u),I\right)&=0\end{aligned}\right\};\forall u\in T;I>0. \tag{5.5}$$

Proof. Substituting $(u_1,u_2,u_3,\cdots,u_N)$ by $(u,u,u,\cdots,u)$ in (5.1), one can arrive that

$$\left.\begin{aligned}\alpha\left(2A((1+2+3+\cdots+N)u)-N(N+1)A(u),I\right)&\le\alpha'\left(\Theta(u,\cdots,u),I\right)\\ \beta\left(2A((1+2+3+\cdots+N)u)-N(N+1)A(u),I\right)&\le\beta'\left(\Theta(u,\cdots,u),I\right)\end{aligned}\right\};\forall u\in T;I>0. \tag{5.6}$$

By definition of M in (5.6), which gives

$$\left.\begin{aligned}\alpha\left(A\left(\left(\frac{N(N+1)}{2}\right)u\right)-\frac{N(N+1)}{2}A(u),\frac{I}{2}\right)&\le\alpha'\left(\Theta(u,\cdots,u),I\right)\\ \beta\left(A\left(\left(\frac{N(N+1)}{2}\right)u\right)-\frac{N(N+1)}{2}A(u),\frac{I}{2}\right)&\le\beta'\left(\Theta(u,\cdots,u),I\right)\end{aligned}\right\};\forall u\in T;I>0. \tag{5.7}$$

It follows from (5.7) that

$$\left.\begin{array}{l}\alpha\left(A(M\,u)-M\,A(u),\frac{I}{2}\right)\le\alpha'\left(\Theta(u,\cdots,u),I\right)\\ \beta\left(A(M\,u)-M\,A(u),\frac{I}{2}\right)\le\beta'\left(\Theta(u,\cdots,u),I\right)\end{array}\right\};\forall u\in T;I>0. \tag{5.8}$$

Also, from (5.8) and (IFN4),(IFN10) of [11], which implies

$$\left.\begin{array}{l}\alpha\left(\frac{1}{M}A(M\,u)-A(u),\frac{I}{2M}\right)\le\alpha'\left(\Theta(u,\cdots,u),I\right)\\ \beta\left(\frac{1}{M}A(M\,u)-A(u),\frac{I}{2M}\right)\le\beta'\left(\Theta(u,\cdots,u),I\right)\end{array}\right\};\forall u\in T;I>0. \tag{5.9}$$

Changing u by $M^C u$ in (5.9) using (5.2), (IFN4), (IFN10) of [11], and again interchanging I by $I\,\wp^C$, we see that

$$\left.\begin{array}{l}\alpha\left(\frac{1}{M^{C+1}}A(M^{C+1}u)-\frac{1}{M^C}A(M^C u),\frac{I\,\wp^C}{2M^{C+1}}\right)\le\alpha'\left(\Theta(u,\cdots,u),I\right)\\ \beta\left(\frac{1}{M^{C+1}}A(M^{C+1}u)-\frac{1}{M^C}A(M^C u),\frac{I\,\wp^C}{2M^{C+1}}\right)\le\beta'\left(\Theta(u,\cdots,u),I\right)\end{array}\right\};\forall u\in T;I>0. \tag{5.10}$$

From (5.10) and (IFN5), (IFN11) of [11], one can arrive

$$\left.\begin{array}{l}\alpha\left(\frac{1}{M^C}A(M^C u)-M(u),I\right)=\alpha\left(\sum_{J=0}^{C-1}\frac{1}{M^{J+1}}A(M^{J+1}u)-\frac{1}{M^J}A(M^J u),\sum_{J=0}^{C-1}\frac{I\,\wp^J}{2M^{J+1}}\right)\le\prod_{J=0}^{C-1}\alpha'\left(\Theta(u,\cdots,u),I\right)\\ \beta\left(\frac{1}{M^C}A(M^C u)-M(u),I\right)=\beta\left(\sum_{J=0}^{C-1}\frac{1}{M^{J+1}}A(M^{J+1}u)-\frac{1}{M^J}A(M^J u),\sum_{J=0}^{C-1}\frac{I\,\wp^J}{2M^{J+1}}\right)\le\prod_{J=0}^{C-1}\beta'\left(\Theta(u,\cdots,u),I\right)\end{array}\right\};\forall u\in T;I>0 \tag{5.11}$$

Interchanging u by $M^{C_1}u$ in (5.11) using (5.2), (IFN4), (IFN10) of [11], and again interchanging I by $I\,\wp^{C_1}$ in the resulting inequality, one can obtain

$$\left.\begin{array}{l}\alpha\left(\frac{1}{M^C M^{C_1}}A(M^C M^{C_1}u)-\frac{1}{M^{C_1}}A(M^{C_1}u),I\right)\le\alpha'\left(\Theta(u,\cdots,u),\frac{I}{\frac{1}{2M}\sum_{J=C_1}^{C-1}\frac{\wp^J}{M^J}}\right)\\ \beta\left(\frac{1}{M^C M^{C_1}}A(M^C M^{C_1}u)-\frac{1}{M^{C_1}}A(M^{C_1}u),I\right)\le\beta'\left(\Theta(u,\cdots,u),\frac{I}{\frac{1}{2M}\sum_{J=C_1}^{C-1}\frac{\wp^J}{M^J}}\right)\end{array}\right\};\forall u\in T;I>0. \tag{5.12}$$

The Cauchy criterion for convergence in IFBS shows that the sequence $\left\{\frac{1}{M^C}A(M^C u)\right\}$ is Cauchy sequence in M with $0<\wp<1$. So, by definition of Cauchy in IFBS, we have

$$\left.\begin{array}{l}\lim_{C\longrightarrow\infty}\alpha\left(\frac{A(M^C u)}{M^C}-Y(u),I\right)=1\\ \lim_{C\longrightarrow\infty}\beta\left(\frac{A(M^C u)}{M^C}-Y(u),I\right)=0\end{array}\right\};\forall u\in T;I>0;Y:T\longrightarrow M. \tag{5.13}$$

Taking $C_1=0$ and $c\to\infty$ in (5.12) and using (5.13), we get that

$$\left.\begin{array}{l}\alpha\left(A(u)-Y(u),I\right)\le\alpha'\left(\Theta(u,\cdots,u),2I\,(M-\wp)\right)\\ \beta\left(A(u)-Y(u),I\right)\le\beta'\left(\Theta(u,\cdots,u),2I\,(M-\wp)\right)\end{array}\right\};\forall u\in T;I>0 \tag{5.14}$$

Now, to prove the existence of $Y(u)$ satisfies the functional equation (1.2), changing $(u_1,\cdots,u_N)$ by $(M^C u_1,\cdots,M^C u_N)$ and divided by M^C in (3.1), one can obtain

$$\left.\begin{array}{l}\alpha\left(\frac{1}{M^C}A\left(\sum_{R=1}^{N}M^C Ru_R\right)+\frac{1}{M^C}A\left(\sum_{R=1}^{N}(N-R+1)M^C u_R\right)-\frac{1}{M^C}(N+1)\sum_{R=1}^{N}A\left(M^C u_R\right),I\right)\\ \qquad\qquad \leq\alpha'\left(\Theta\left(M^C u_1,\cdots,M^C u_N\right),M^C I\right)\\ \beta\left(\frac{1}{M^C}A\left(\sum_{R=1}^{N}M^C Ru_R\right)+\frac{1}{M^C}A\left(\sum_{R=1}^{N}(N-R+1)M^C u_R\right)-\frac{1}{M^C}(N+1)\sum_{R=1}^{N}A\left(M^C u_R\right),I\right)\\ \qquad\qquad \leq\beta'\left(\Theta\left(M^C u_1,\cdots,M^C u_N\right),M^C I\right)\end{array}\right\};\forall u_1,\cdots,u_N\in T;I>0. \quad (5.15)$$

Now,

$$\begin{array}{l}\alpha\left(Y\left(\sum_{R=1}^{N}Ru_R\right)+Y\left(\sum_{R=1}^{N}(N-R+1)u_R\right)-(N+1)\sum_{R=1}^{N}Y(u_R),I\right)\\ \geq\alpha\left(Y\left(\sum_{R=1}^{N}Ru_R\right)-\frac{1}{M^C}A\left(\sum_{R=1}^{N}M^C Ru_R\right),\frac{I}{4}\right)*\alpha\left(Y\left(\sum_{R=1}^{N}(N-R+1)u_R\right)-\frac{1}{M^C}A\left(\sum_{R=1}^{N}(N-R+1)M^C u_R\right),\frac{I}{4}\right)*\\ \quad\alpha\left(-(N+1)\sum_{R=1}^{N}Y(u_R)+\frac{1}{M^C}(N+1)\sum_{R=1}^{N}A\left(M^C u_R\right),\frac{I}{4}\right)*\\ \quad\alpha\left(\frac{1}{M^C}A\left(\sum_{R=1}^{N}M^C Ru_R\right)+\frac{1}{M^C}A\left(\sum_{R=1}^{N}(N-R+1)M^C u_R\right)-\frac{1}{M^C}(N+1)\sum_{R=1}^{N}A\left(M^C u_R\right),\frac{I}{4}\right)\\ \beta\left(Y\left(\sum_{R=1}^{N}Ru_R\right)+Y\left(\sum_{R=1}^{N}(N-R+1)u_R\right)-(N+1)\sum_{R=1}^{N}Y(u_R),I\right)\\ \leq\beta\left(Y\left(\sum_{R=1}^{N}Ru_R\right)-\frac{1}{M^C}A\left(\sum_{R=1}^{N}M^C Ru_R\right),\frac{I}{4}\right)\diamond\beta\left(Y\left(\sum_{R=1}^{N}(N-R+1)u_R\right)-\frac{1}{M^C}A\left(\sum_{R=1}^{N}(N-R+1)M^C u_R\right),\frac{I}{4}\right)\diamond\\ \quad\beta\left(-(N+1)\sum_{R=1}^{N}Y(u_R)+\frac{1}{M^C}(N+1)\sum_{R=1}^{N}A\left(M^C u_R\right),\frac{I}{4}\right)\diamond\\ \quad\beta\left(\frac{1}{M^C}A\left(\sum_{R=1}^{N}M^C Ru_R\right)+\frac{1}{M^C}A\left(\sum_{R=1}^{N}(N-R+1)M^C u_R\right)-\frac{1}{M^C}(N+1)\sum_{R=1}^{N}A\left(M^C u_R\right),\frac{I}{4}\right);\forall u_1,\cdots,u_N\in T;I>0.\end{array} \quad (5.16)$$

Approaching limit C tends to infinity in (5.16) and using (5.15), (5.13), (5.3), one can see that $Y(u)$ satisfies the functional equation (1.2). It is easy to verify that the existence of $Y(u)$ is unique, it follows form (5.13), (5.14) and for any positive number C_1 , we have

$$\begin{array}{l}\alpha\left(Y(u)-Y'(u),I\right)\geq\alpha\left(Y\left(M^{C_1}u\right)-A\left(M^{C_1}u\right),\frac{IM^{C_1}}{2}\right)*\alpha\left(Y'\left(M^{C_1}u\right)-A'\left(M^{C_1}u\right),\frac{IM^{C_1}}{2}\right)\\ \qquad =\alpha'\left(\Theta(u,\cdots,u),\frac{I\,M^{C_1}(M-\wp)}{\wp^{C_1}}\right) \qquad ;\forall u\in T;I>0.\\ \beta\left(Y(u)-Y'(u),I\right)\geq\beta\left(Y\left(M^{C_1}u\right)-A\left(M^{C_1}u\right),\frac{IM^{C_1}}{2}\right)\diamond\beta\left(Y'\left(M^{C_1}u\right)-A'\left(M^{C_1}u\right),\frac{IM^{C_1}}{2}\right)\\ \qquad =\beta'\left(\Theta(u,\cdots,u),\frac{I\,M^{C_1}(M-\wp)}{\wp^{C_1}}\right)\end{array}$$

Taking limit C_1 tends to infinity in the above inequality and using (5.2), (5.3), (5.13), one can see the desired result. So, Theorem holds for $D=1$.

Alternatively, interchanging $u=\frac{u}{M}$ in (5.8), one can arrive that

$$\left.\begin{array}{l}\alpha\left(A(Mu)-MA\left(\frac{u}{M}\right),\frac{I}{2}\right)\leq\alpha'\left(\Theta\left(\frac{u}{M},\cdots,\frac{u}{M}\right),I\right)\\ \beta\left(A(Mu)-MA\left(\frac{u}{M}\right),\frac{I}{2}\right)\leq\beta'\left(\Theta\left(\frac{u}{M},\cdots,\frac{u}{M}\right),I\right)\end{array}\right\};\forall u\in T;I>0. \quad (5.17)$$

The rest of the proof is analogous to that of preceding case. So, Theorem holds for $D=-1$. Hence the proof is complete.

Corollary 5.2: Let $A:T\longrightarrow M$ be a function fulfilling the inequality

$$\left.\begin{array}{l}\alpha\left(A\left(\sum_{R=1}^{N}Ru_R\right)+A\left(\sum_{R=1}^{N}(N-R+1)u_R\right)-(N+1)\sum_{R=1}^{N}A(u_R),I\right)\leq\begin{cases}\alpha'(\Omega,I)\\ \alpha'\left(\Omega\sum_{R=1}^{N}\|u_R\|^H,I\right);H\neq 1;\\ \alpha'\left(\Omega\prod_{R=1}^{N}\|u_R\|^H,I\right);NH\neq 1;\end{cases}\\ \beta\left(A\left(\sum_{R=1}^{N}Ru_R\right)+A\left(\sum_{R=1}^{N}(N-R+1)u_R\right)-(N+1)\sum_{R=1}^{N}A(u_R),I\right)\leq\begin{cases}\beta'(\Omega,I)\\ \beta'\left(\Omega\sum_{R=1}^{N}\|u_R\|^H,I\right);H\neq 1;\\ \beta'\left(\Omega\prod_{R=1}^{N}\|u_R\|^H,I\right);NH\neq 1;\end{cases}\end{array}\right\};\forall u_1,\cdots,u_N\in T;I>0. \quad (5.18)$$

Then there exists a unique additive mapping $Y:T\longrightarrow M$ which satisfying the functional equation (1.2) and the inequality

$$\left.\begin{array}{l}\alpha\left(A(u)-Y(u),I\right)\leq\begin{cases}\alpha'\left(\Omega,2I\,|M-1|\right)\\ \alpha'\left(\Omega N\|u\|^H,2I\,|M-M^H|\right)\\ \alpha'\left(\Omega\|u\|^{NH},2I\,|M-M^{NH}|\right)\end{cases}\\ \beta\left(A(u)-Y(u),I\right)\leq\begin{cases}\beta'\left(\Omega,2I\,|M-1|\right)\\ \beta'\left(\Omega N\|u\|^H,2I\,|M-M^H|\right)\\ \beta'\left(\Omega\|u\|^{NH},2I\,|M-M^{NH}|\right)\end{cases}\end{array}\right\};\forall u\in T;I>0. \quad (5.19)$$

V .2 RADUS FIXED ANALYSIS.

Theorem 5.3: Let $A:T\longrightarrow M$ be a function fulfilling the inequality (5.1) where $\Theta:T^N\longrightarrow[0,\infty)$ be a function with the condition

$$\left.\begin{array}{l}\lim_{C\longrightarrow\infty}\alpha'\left(\Theta\left(M^C u_1,\cdots,M^C u_N\right),M^C I\right)=1\\ \lim_{C\longrightarrow\infty}\beta'\left(\Theta\left(M^C u_1,\cdots,M^C u_N\right),M^C I\right)=0\end{array}\right\};;\mathrm{E}_0=M;\mathrm{E}_1=\frac{1}{M};\text{with } e=0\text{or}1;\forall u_1,\cdots,u_N\in T;I>0. \quad (5.20)$$

If there exists $L=L(e)$ such that the functions $\Theta(u,\cdots,u)$ has the properties

$$\left.\begin{array}{l}\alpha'\left(\Theta(u,\cdots,u),I\right)=\alpha'\left(\frac{1}{2}\times\Theta\left(\frac{u}{M},\cdots,\frac{u}{M}\right),I\right)\\ \beta'\left(\Theta(u,\cdots,u),I\right)=\beta'\left(\frac{1}{2}\times\Theta\left(\frac{u}{M},\cdots,\frac{u}{M}\right),I\right)\end{array}\right\}\text{ and }\left.\begin{array}{l}\alpha'\left(\frac{1}{\mathrm{E}_e}\Theta\left(\mathrm{E}_e u,\cdots,\mathrm{E}_e u\right),I\right)=\alpha'\left(L\,\Theta(u,\cdots,u),I\right)\\ \beta'\left(\frac{1}{\mathrm{E}_e}\Theta\left(\mathrm{E}_e u,\cdots,\mathrm{E}_e u\right),I\right)=\beta'\left(L\,\Theta(u,\cdots,u),I\right)\end{array}\right\};\forall u\in T;I>0. \quad (5.21)$$

Then there exists a unique additive mapping $Y:T\longrightarrow M$ which satisfying the functional equation (1.2) and the inequality

$$\left.\begin{array}{l}\alpha'\left(A(u)-Y(u),I\right)=\alpha'\left(\left[\frac{L^{1-e}}{(1-L)}\right]\Theta(u,\cdots,u),I\right)\\ \beta'\left(A(u)-Y(u),I\right)=\beta'\left(\left[\frac{L^{1-e}}{(1-L)}\right]\Theta(u,\cdots,u),I\right)\end{array}\right\};Y(u)=\lim_{C\longrightarrow\infty}\frac{A\left(\mathrm{E}_e^C u\right)}{\mathrm{E}_e^C};\ \forall u\in T;I>0. \quad (5.22)$$

Proof. The follows from similar tracing Theorem 3.9 with the help of Theorem 5.1.

Corollary 5.4: Let $A:T\longrightarrow M$ be a function fulfilling the inequalities (5.18). Then there exists a unique additive mapping $Y:T\longrightarrow M$ which satisfying the functional equation (1.2) and the inequalities (5.19).

VI. Applications

The functional equation (1.2) have the additive solution $A(u)=u$. By Theorem 3.1, 4.1, 5.1, it follows from (1.2) and number series formula, that

$$(1u+2u+\cdots+Nu)+(Nu+(N-1)u+\cdots+1u)=(N+1)(u+u+\cdots+u)$$

$$\left(\frac{N(N+1)}{2}\right)u+\left(\frac{N(N+1)}{2}\right)u=(N+1)(u+u+\cdots+u)$$

$$N(N+1)u=N(N+1)u$$

So, the functional equation (1.2) is originating from sum of ascending and descending N natural numbers with additive solution

Natural numbers are the set of numbers that start from the numeral 1 and can extend up to infinity. The alphabet N is used as a symbol to address natural numbers. The natural number set does not include – negative numbers, fractional numbers, and decimal numbers. The properties of natural numbers are – closure property, distributive property, associative property, and commutative property. These properties make the natural number set unique. Natural numbers can be used in everyday activities. The two predominant daily applications of natural numbers are ordering and counting.

- ✓ In Counting, we have to count the specific amount of objects by assigning the first object to the natural number 1. The next object will be assigned the number 2 and so on until all the objects are counted. Counting is also known as enumeration
- ✓ In Ordering is also known as ranking the objects. For orders, we have to first select the object with an extreme value (example – tallest, smallest, etc.) and we will assign this object with the natural number 1. The next object with the second-highest or extreme value will be assigned with the number 2 and so on the ranking will continue

III. Acknowledgment

We dedicate this work to all mathematicians who are working in this field.

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