SOME CONTRIBUTIONS IN THE CONSTRUCTION OF QUASI SYMMETRIC DESIGNS WITH FIXED BLOCKS INTERSECTION PATTERN

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ABSTRACT : Some construction methods of η-multiple balanced incomplete block designs, where η = 2,3,4 with quasi - symmetric structure with no repeated blocks and fixed three intersection numbers x = 0 , y=1 and z =2 between the blocks are proposed with illustrations.

KEYWORDS : **Balanced Incomplete Block Design , Symmetric Design , Quasi-Symmetric Design , Block Intersection , Galois Field , Difference Set, Non-isomorphic Solution.**

1. INTRODUCTION

In statistical design of experiments the importance of BIB designs for varietal trial was realized when in 1936 Yates discussed these designs in the context of biological and agriculture experiments. In the early 1930's Prof. R.A. Fisher and F. Yates gave the concept of design of experiments. BIB design play important role in design of experiments especially in field of experiments. Many construction methods of BIB design were given by Prof. Fisher [8] ,Yates [33] and Bose [1]. These BIB designs ensure that treatments are compared with equal precision. A fundamental method of constructing 2- designs (and also the first systematic construction method) is due to Bose [1]. It is known as the method of differences. Suppose an abelian group (G, +) with a subset $B \subseteq G$, $|B| = k$, consider the k(k-1) ordered differences of its elements. We get a collection of m subsets $B_1, B_2, \ldots B_m$ of G, a set of initial blocks if among the mk(k-1) differences arising from these m blocks, each non-zero element of G occurs exactly a constant number λ times. Several series of BIB designs have been constructed by Sprott (1954, 1956) ,using the method of differences. Teirlinck [31] has proved that non – trival t – designs without repeated blocks exist for all t. In the early 1970's Richard Wilson did the most significant improvement in the design theory. Prof. Fisher (1940), Cox (1940) and Q.M. Hussain (1945) discussed the problem of all possible non-isomorphic solutions.

A balanced incomplete block design is an arrangement of v treatments in b blocks, of size k where each treatment replicated r times , and every pair of treatment appears together in λ blocks. A BIB design is symmetric iff v = b and r = k . It is also denoted as 2- (v, k, λ) .

Quasi Symmetric Design - Let S be a finite set of v objects (points) , and γ be a finite family of distinct k subsets of S (blocks). Then the pair D = { S , γ } is called a block design (or 2-design) with parameters (v, b, r, k, λ). For $0 \le x < k$, where an intersection number of D is x , if there exist B , B' ∈ γ such that $|B \cap B'| = x$. A 2-design *D* is **quasi-symmetric design** with two numbers of intersection x and y and $0 \le x < y < k$ if every two distinct blocks intersect in either x or *y* points.

The term "non-isomorphic" means "not having the same form". Two balanced incomplete block designs D_1 and D_2 with parameters v, b, r, k, λ are said to be isomorphic if there exists a partition of the set of treatments of D_1 into that of D_2 such that under this partition the set of blocks of D_1 goes into the set of blocks of D_2. Otherwise they are said to be non-isomorphic.

Quasi-symmetric designs and its classification has been important in the study of design theory over the last several years. Sane and Shrikhande [23] gave many important results for quasi-symmetric designs. Ray et al.(see [28]) proved that for a 0-design with t- intersection number $b \le \binom{v}{t}$. Pawale [16] proved that for a fixed block size k , there exist finitely many parametrically feasible t – designs with t – numbers of intersection and λ > 1. Bhat and Shrikhande [3-5] developed some techniques for generating non-isomorphic solutions of balanced incomplete block designs belonging to the series of symmetric designs for suitable values of t and method of testing non-isomorphism of solutions of balanced incomplete block designs (BIBD) belonging to the series of symmetric designs with the parameters (4t + 3, 2t + 1, t) and to the series with parameters (4t + 4, 8t + 6, 4t + 3,2t + 2, 2t + 1) when t is even. Singhi [30] developed a technique to get a large number of non-isomorphic solutions of a (4t + 3, 2t + 1, t).

Our main objective in this paper is to study the η-multiple balanced incomplete block designs with no repeated blocks which is also have quasi-symmetric structure with fixed three types of intersection numbers x = 0 , y = 1 and z = 2 between the blocks and all possible non isomorphic solutions.

2 . CONSTRUCTION OF η - MULTIPLE BALANCED INCOMPLETE BLOCK DESIGNS

Method : Let D_1 be a balanced incomplete block design with parameters v = 20t + 1, b = t (20t+1) , r = 5t , k = 5 , λ = 1 ,where $v(=20t + 1) = p^n$ is a prime or prime power and x is a primitive element of $GF(p^n)$. Design D_1 have initial blocks set $(x^0, x^{4t}, x^{8t}, x^{12t}, x^{16t})$, $(x^2, x^{4t+2}, x^{8t+2}, x^{12t+2}, x^{16t+2})$, … , $(x^{2t-2}, x^{6t-2}, x^{10t-2}, x^{14t-2}, x^{18t-2})$. Multiply above initial blocks set by primitive element x and get another design D_2 with initial blocks set $(x^1, x^{4t+1}, x^{8t+1}, x^{12t+1}, x^{16t+1})$, $(x^3, x^{4t+3}, x^{8t+3}$,

x^{12t+3}, x^{16t+3}) ,… ,(x^{2t-1}, x^{6t-1}, x^{10t-1}, x^{14t-1},x^{18t-1}).Now multiply the last initial block of D_2 by primitive element x and make it the first initial block of another design and get design D_3 with initial blocks set (x^{2t}, x^{6t}, x^{10t}, x^{14t} ,x^{18t}) , (x^{2t+2}, x^{6t+2}, x^{10t+2}, x^{14t+2},x^{18t+2}) , … , (x^{4t-2}, x^{8t-2}, x^{12t-2}, x^{16t-2},x^{20t-2}) and then multiply design D_3 by primitive element x and get next design D_4 with initial blocks set (x^{2t+1}, x^{6t+1}, x^{10t+1}, x^{14t+1} ,x^{18t+1}) , (x^{2t+3}, x^{6t+3}, x^{10t+3}, x^{14t+3},x^{18t+3}) , … ,(x^{4t-1}, x^{8t-1}, x^{12t-1}, x^{16t-1},x^{20t-1}). Thus, we get 3 new designs **D_2 , D_3 ,D_4**, now arrange **D_1 , D_2 , D_3 ,D_4** these 4 designs in form of **D* = [D_1 : D_2: D_3 : D_4]** and then by developing D^* we get 4-multiple solution of balanced incomplete block designs with no repeated blocks with parameters **v= 20t + 1, b*= 4t(20t + 1)=4b , r*=20t =4r , k , λ*=4= 4λ.**

Theorem 2.1 The existence of the series of balanced incomplete block design D_1 with parameters v = 20t + 1, b = t (20t+1) , r = 5t , k = 5 , λ = 1 ,where v(=20t + 1) = p^n is a prime or prime power and x is a primitive element of GF(p^n) , implies the existence of D* a 4-multiple balanced incomplete block design with no repeated blocks with parameters v= 20t + 1, b*= 4t(20t + 1)=4b , r*=20t =4r , k , λ*=4= 4λ .

Proof :-Consider a BIB design D_1 with given parameters v = 20t + 1,b = t (20t+1), r = 5t, k = 5 , λ = 1. (1)
Let v = p^n ,where p is a prime. Since x is a primitive element and all the non zero elements of GF(p^n) can be shown as
x^0 = 1 , x , x^2,………………………………. x^{20t-1} ,

Then ,

$$x^{v-1} = x^{20t} = 1 \text{ and } x^{10t} = -1$$

bLet x^{4t}-1 = x^s , x^{8t}-1 = x^u. Then x^{4t} + 1 = x^{u-s} ,
Consider the initial blocks set

$$D_1 = (x^0, x^{4t}, x^{8t}, x^{12t}, x^{16t}),(x^2, x^{4t+2}, x^{8t+2}, x^{12t+2}, x^{16t+2}),\ldots,(x^{2t-2}, x^{6t-2}, x^{10t-2}, x^{14t-2}, x^{18t-2}) \quad (2)$$

The differences from the first initial block can be written as

$$x^s, x^{4t+s}, x^{8t+s}, x^{12t+s}, x^{16t+s}, x^{10t+s}, x^{14t+s}, x^{18t+s}, x^{2t+s}, x^{6t+s}$$
$$x^u, x^{4t+u}, x^{8t+u}, x^{12t+u}, x^{16t+u}, x^{10t+u}, x^{14t+u}, x^{18t+u}, x^{2t+u}, x^{6t+u}. \quad (3)$$

The differences arising from other initial blocks in (2) are obtained from (3) by multiplication with x^2, x^4, …, x^{2t-2} respectively. Thus , among the totality of these differences , every non-zero element of GF (p^n) occurs just once in (3) .
Now, multiply (2) by primitive element x of GF (p^n) ,we get design D_2 with initial blocks set is

$$D_2 = (x^1, x^{4t+1}, x^{8t+1}, x^{12t+1}, x^{16t+1}), (x^3, x^{4t+3}, x^{8t+3}, x^{12t+3}, x^{16t+3}), \ldots,$$
$$(x^{2t-1}, x^{6t-1}, x^{10t-1}, x^{14t-1}, x^{18t-1}) \quad (4)$$

The differences from the first initial block of (4) can be written as

$$x^{s+1}, x^{4t+s+1}, x^{8t+s+1}, x^{12t+s+1}, x^{16t+s+1}, x^{10t+s+1}, x^{14t+s+1}, x^{18t+s+1}, x^{2t+s+1}, x^{6t+s+1}$$
$$x^{u+1}, x^{4t+u+1}, x^{8t+u+1}, x^{12t+u+1}, x^{16t+u+1}, x^{10t+u+1}, x^{14t+u+1}, x^{18t+u+1}, x^{2t+u+1}, x^{6t+u+1}. \quad (5)$$

The differences arising from other initial blocks in (4) are obtained from (5) by multiplication with x^2, x^4, …, x^{2t-2} respectively. Thus , among the totality of these differences , every non-zero element of GF (p^n) occurs just once in (5) .
Now, multiply the last initial block of (4) by primitive element x and make it the first initial block of next design ,we get design D_3 with initial blocks set

$$D_3 = (x^{2t}, x^{6t}, x^{10t}, x^{14t}, x^{18t}), (x^{2t+2}, x^{6t+2}, x^{10t+2}, x^{14t+2}, x^{18t+2}), \ldots,$$
$$(x^{4t-2}, x^{8t-2}, x^{12t-2}, x^{16t-2}, x^{20t-2}) \quad (6)$$

The differences from the first initial block of (6) can be written as

$$x^{2t+s}, x^{6t+s}, x^{10t+s}, x^{14t+s}, x^{18t+s}, x^{12t+s}, x^{16t+s}, x^s, x^{4t+s}, x^{8t+s}$$
$$x^{2t+u}, x^{6t+u}, x^{10t+u}, x^{14t+u}, x^{18t+u}, x^{12t+u}, x^{16t+u}, x^u, x^{4t+u}, x^{8t+u}. \quad (7)$$

The differences arising from other initial blocks in (6) are obtained from (7) by multiplication with x^2, x^4, …, x^{2t-2} respectively. Thus , among the totality of these differences , every non-zero element of GF (p^n) occurs just once in (7) .
Next , multiply (6) by primitive element x of GF (p^n) ,we get design D_4 with initial blocks set is

$$D_4 = (x^{2t+1}, x^{6t+1}, x^{10t+1}, x^{14t+1}, x^{18t+1}), (x^{2t+3}, x^{6t+3}, x^{10t+3}, x^{14t+3}, x^{18t+3}), \ldots,$$
$$(x^{4t-1}, x^{8t-1}, x^{12t-1}, x^{16t-1}, x^{20t-1}) \quad (8)$$

The differences from the first initial block of (8) can be written as

$$x^{2t+s+1}, x^{6t+s+1}, x^{10t+s+1}, x^{14t+s+1}, x^{18t+s+1}, x^{12t+s+1}, x^{16t+s+1}, x^{s+1}, x^{4t+s+1}, x^{8t+s+1}$$
$$x^{2t+u+1}, x^{6t+u+1}, x^{10t+u+1}, x^{14t+u+1}, x^{18t+u+1}, x^{12t+u+1}, x^{16t+u+1}, x^{u+1}, x^{4t+u+1}, x^{8t+u+1}. \quad (9)$$

The differences arising from other initial blocks in (8) are obtained from (9) by multiplication with x^2, x^4, …, x^{2t-2} respectively. Thus , among the totality of these differences , every non-zero element of GF (p^n) occurs just once in (9) .
Thus, from (2) to (9) we get 3 new designs **D_2 , D_3 ,D_4**, arrange **D_1 , D_2 , D_3 ,D_4** these 4 designs in form of
D* = [D_1 : D_2: D_3 : D_4] . Hence , by developing D* we get the complete solution of 4-multiple balanced incomplete block design with no repeated blocks with parameters
v= 20t + 1, b*= 4t(20t + 1)=4b , r*=20t =4r , k , λ*=4= 4λ . (10)
This complete the proof.

Corollary 2.2 Let D* a 4-multiple balanced incomplete block design with no repeated blocks with parameters $v = 20t + 1$, $b^* = 4t(20t + 1) = 4b$, $r^* = 20t = 4r$, k, $\lambda^* = 4 = 4\lambda$ has 1 non isomorphic solution.

Theorem 2.3 The existence of balanced incomplete block design D_1 with parameters $v = 20t + 1$, $b = t(20t+1)$, $r = 5t$, $k = 5$, $\lambda = 1$, implies the existence of D_{ij}^* 2-multiple balanced incomplete block designs with no repeated blocks with parameters **(v, 2b, 2r , k ,2λ)** and also have 6 non isomorphic solutions D_{ij}^*, where i, j = 1,2,3,4 ($i \neq j$).

Proof : - From above theorem 2.1 we obtained 4 designs $\mathbf{D_1}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_4}$,each design have parameters (v, b, r, k, λ). Now , arrange these 4 designs in form of $D_{ij}^* = [\mathbf{D_i} : \mathbf{D_j}]$ where i , j = 1,2,3,4 ($i \neq j$) then by developing D_{ij}^* we get the solution of 2-multiple balanced incomplete block designs with no repeated blocks with parameters **(v, 2b, 2r , k ,2λ)**. Thus, each 2-multiple balanced incomplete block design does not have any repeated block so it has non isomorphic solution. Hence, 2-multiple balanced incomplete block designs have 6 non isomorphic solutions $\mathbf{D_{ij}^*}$.

Theorem 2.4 The existence of balanced incomplete block design D_1 with parameters $v = 20t + 1$, $b = t(20t+1)$, $r = 5t$, $k = 5$, $\lambda = 1$, implies the existence of D_{ijk}^* 3-multiple balanced incomplete block designs with no repeated blocks with parameters **(v, 3b, 3r , k , 3λ)** and it also have 4 non isomorphic solutions D_{ijk}^*, where i, j, k = 1,2 ,3,4 ($i \neq j \neq k$) .

Proof : - According to above theorem 2.1 we obtained 4 designs $\mathbf{D_1}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_4}$, each design with parameters (v, b, r, k, λ). Now , arrange these 4 designs in form of $D_{ijk}^* = [\mathbf{D_i} : \mathbf{D_j} : \mathbf{D_k}]$ where i, j, k = 1,2,3,4 ($i \neq j \neq k$) then by developing D_{ijk}^* we get the solution of 3-multiple balanced incomplete block design with no repeated blocks with parameters **(v, 3b ,3r, k ,3λ)**. Thus, each 3-multiple balanced incomplete block design does not have any repeated block so it has non isomorphic solution. Hence , 3-multiple balanced incomplete block designs have 4 non isomorphic solutions D_{ijk}^*.

Example 2.5 Consider the BIB design D_1 with the parameters **v = 41 , b = 82 , r = 10, k = 5 , λ = 1.** Since primitive element of GF (41) is **x = 7** and the solution of the design is given by the initial blocks set $(7^0,7^8,7^{16},7^{24},7^{32})$, $(7^2,7^{10},7^{18},7^{26},7^{34})$ that is (1, 37, 16, 18, 10) , (8,9,5,21,39) mod 41 provides $\mathbf{D_1}$, Second initial blocks set is $(7^1,7^9,7^{13},7^{25},7^{33})$, $(7^3,7^{11},7^{20},7^{28},7^{35})$ that is (7,13,30,3,29) , (15, 22, 35, 24, 27) mod 41 provides $\mathbf{D_2}$, third initial block sets are $(7^4,7^{12},7^{20},7^{28},7^{36})$, $(7^6,7^{14},7^{22},7^{30},7^{38})$ that is (23, 31, 40, 4, 25), (20,2,33,32,36) mod 41 provides $\mathbf{D_3}$ and forth initial blocks set is $(7^5,7^{13},7^{21},7^{29},7^{37})$, $(7^7,7^{15}\ 7^{23}, 7^{31}, 7^{39})$ that is (38,12,34,28,11) , (17,14,26,19,6) mod 41 provides $\mathbf{D_4}$. A 4- multiple balanced incomplete block design D* as given below

D_1

1	37	16	18	10
2	38	17	19	11
3	39	18	20	12
4	40	19	21	13
5	41	20	22	14
6	1	21	23	15
7	2	22	24	16
8	3	23	25	17
9	4	24	26	18
10	5	25	27	19
11	6	26	28	20
12	7	27	29	21
13	8	28	30	22
14	9	29	31	23
15	10	30	32	24
16	11	31	33	25
17	12	32	34	26
18	13	33	35	27
19	14	34	36	28
20	15	35	37	29
21	16	36	38	30
22	17	37	39	31
23	18	38	40	32
24	19	39	41	33

D_2

7	13	30	3	29
8	14	31	4	30
9	15	32	5	31
10	16	33	6	32
11	17	34	7	33
12	18	35	8	34
13	19	36	9	35
14	20	37	10	36
15	21	38	11	37
16	22	39	12	38
17	23	40	13	39
18	24	41	14	40
19	25	1	15	41
20	26	2	16	1
21	27	3	17	2
22	28	4	18	3
23	29	5	19	4
24	30	6	20	5
25	31	7	21	6
26	32	8	22	7
27	33	9	23	8
28	34	10	24	9
29	35	11	25	10
30	36	12	26	11

D_3

23	31	40	4	25
24	32	41	5	26
25	33	1	6	27
26	34	2	7	28
27	35	3	8	29
28	36	4	9	30
29	37	5	10	31
30	38	6	11	32
31	39	7	12	33
32	40	8	13	34
33	41	9	14	35
34	1	10	15	36
35	2	11	16	37
36	3	12	17	38
37	4	13	18	39
38	5	14	19	40
39	6	15	20	41
40	7	16	21	1
41	8	17	22	2
1	9	18	23	3
2	10	19	24	4
3	11	20	25	5
4	12	21	26	6
5	13	22	27	7

D_4

38	12	34	28	11
39	13	35	29	12
40	14	36	30	13
41	15	37	31	14
1	16	38	32	15
2	17	39	33	16
3	18	40	34	17
4	19	41	35	18
5	20	1	36	19
6	21	2	37	20
7	22	3	38	21
8	23	4	39	22
9	24	5	40	23
10	25	6	41	24
11	26	7	1	25
12	27	8	2	26
13	28	9	3	27
14	29	10	4	28
15	30	11	5	29
16	31	12	6	30
17	32	13	7	31
18	33	14	8	32
19	34	15	9	33
20	35	16	10	34

25	20	40	1	34
26	21	41	2	35
27	22	1	3	36
28	23	2	4	37
29	24	3	5	38
30	25	4	6	39
31	26	5	7	40
32	27	6	8	41
33	28	7	9	1
34	29	8	10	2
35	30	9	11	3
36	31	10	12	4
37	32	11	13	5
38	33	12	14	6
39	34	13	15	7
40	35	14	16	8
41	36	15	17	9
8	9	5	21	39
9	10	6	22	40
10	11	7	23	41
11	12	8	24	1
12	13	9	25	2
13	14	10	26	3
14	15	11	27	4
15	16	12	28	5
16	17	13	29	6
17	18	14	30	7
18	19	15	31	8
19	20	16	32	9
20	21	17	33	10
21	22	18	34	11
22	23	19	35	12
23	24	20	36	13
24	25	21	37	14
25	26	22	38	15
26	27	23	39	16
27	28	24	40	17
28	29	25	41	18
29	30	26	1	19
30	31	27	2	20
31	32	28	3	21
32	33	29	4	22
33	34	30	5	23
34	35	31	6	24
35	36	32	7	25

31	37	13	27	12
32	38	14	28	13
33	39	15	29	14
34	40	16	30	15
35	41	17	31	16
36	1	18	32	17
37	2	19	33	18
38	3	20	34	19
39	4	21	35	20
40	5	22	36	21
41	6	23	37	22
1	7	24	38	23
2	8	25	39	24
3	9	26	40	25
4	10	27	41	26
5	11	28	1	27
6	12	29	2	28
15	22	35	24	27
16	23	36	25	28
17	24	37	26	29
18	25	38	27	30
19	26	39	28	31
20	27	40	29	32
21	28	41	30	33
22	29	1	31	34
23	30	2	32	35
24	31	3	33	36
25	32	4	34	37
26	33	5	35	38
27	34	6	36	39
28	35	7	37	40
29	36	8	38	41
30	37	9	39	1
31	38	10	40	2
32	39	11	41	3
33	40	12	1	4
34	41	13	2	5
35	1	14	3	6
36	2	15	4	7
37	3	16	5	8
38	4	17	6	9
39	5	18	7	10
40	6	19	8	11
41	7	20	9	12
1	8	21	10	13

6	14	23	28	8
7	15	24	29	9
8	16	25	30	10
9	17	26	31	11
10	18	27	32	12
11	19	28	33	13
12	20	29	34	14
13	21	30	35	15
14	22	31	36	16
15	23	32	37	17
16	24	33	38	18
17	25	34	39	19
18	26	35	40	20
19	27	36	41	21
20	28	37	1	22
21	29	38	2	23
22	30	39	3	24
20	2	33	32	36
21	3	34	33	37
22	4	35	34	38
23	5	36	35	39
24	6	37	36	40
25	7	38	37	41
26	8	39	38	1
27	9	40	39	2
28	10	41	40	3
29	11	1	41	4
30	12	2	1	5
31	13	3	2	6
32	14	4	3	7
33	15	5	4	8
34	16	6	5	9
35	17	7	6	10
36	18	8	7	11
37	19	9	8	12
38	20	10	9	13
39	21	11	10	14
40	22	12	11	15
41	23	13	12	16
1	24	14	13	17
2	25	15	14	18
3	26	16	15	19
4	27	17	16	20
5	28	18	17	21
6	29	19	18	22

21	36	17	11	35
22	37	18	12	36
23	38	19	13	37
24	39	20	14	38
25	40	21	15	39
26	41	22	16	40
27	1	23	17	41
28	2	24	18	1
29	3	25	19	2
30	4	26	20	3
31	5	27	21	4
32	6	28	22	5
33	7	29	23	6
34	8	30	24	7
35	9	31	25	8
36	10	32	26	9
37	11	33	27	10
17	14	26	19	6
18	15	27	20	7
19	16	28	21	8
20	17	29	22	9
21	18	30	23	10
22	19	31	24	11
23	20	32	25	12
24	21	33	26	13
25	22	34	27	14
26	23	35	28	15
27	24	36	29	16
28	25	37	30	17
29	26	38	31	18
30	27	39	32	19
31	28	40	33	20
32	29	41	34	21
33	30	1	35	22
34	31	2	36	23
35	32	3	37	24
36	33	4	38	25
37	34	5	39	26
38	35	6	40	27
39	36	7	41	28
40	37	8	1	29
41	38	9	2	30
1	39	10	3	31
2	40	11	4	32
3	41	12	5	33

36	37	33	8	26
37	38	34	9	27
38	39	35	10	28
39	40	36	11	29
40	41	37	12	30
41	1	38	13	31
1	2	39	14	32
2	3	40	15	33
3	4	41	16	34
4	5	1	17	35
5	6	2	18	36
6	7	3	19	37
7	8	4	20	38

2	9	22	11	14
3	10	23	12	15
4	11	24	13	16
5	12	25	14	17
6	13	26	15	18
7	14	27	16	19
8	15	28	17	20
9	16	29	18	21
10	17	30	19	22
11	18	31	20	23
12	19	32	21	24
13	20	33	22	25
14	21	34	23	26

7	30	20	19	23
8	31	21	20	24
9	32	22	21	25
10	33	23	22	26
11	34	24	23	27
12	35	25	24	28
13	36	26	25	29
14	37	27	26	30
15	38	28	27	31
16	39	29	28	32
17	40	30	29	33
18	41	31	30	34
19	1	32	31	35

4	1	13	6	34
5	2	14	7	35
6	3	15	8	36
7	4	16	9	37
8	5	17	10	38
9	6	18	11	39
10	7	19	12	40
11	8	20	13	41
12	9	21	14	1
13	10	22	15	2
14	11	23	16	3
15	12	24	17	4
16	13	25	18	5

Thus, arrange these 4 designs in form of **$D^* = [\, D_1 : D_2 : D_3 : D_4 \,]$** by developing **D*** we get 4-multiple solution of balanced incomplete block designs with no repeated blocks with parameters **$v = 41$, $b^* = 328$, $r^* = 40$, $k = 5$, $\lambda^* = 4$** and 1 non isomorphic solution .

3. QUASI SYMMETRIC DESIGNS WITH INTERSECTION NUMBERS 0,1 and 2

Theorem 3.1 : Let D_{ij}^*, D_{ijk}^* and D^* be quasi-symmetric designs with parameter set (v, ηb, ηr, k, $\eta\lambda$; x , y, z) have three intersection numbers $x = 0$, $y = 1$ and $z = 2$, where $\eta = 2,3,4$. Then corresponding to a block there are

$$\alpha_1 = \frac{k^2(\eta - 1) + k\,\{(X + z)(1 - \eta r) - \eta + \eta r\} + Xz(\eta b - 1)}{(y - X)\,(y - z)}$$

with one type of intersection number ,

$$\alpha_2 = \frac{k^2(\eta - 1) + k\,\{(X + y)(1 - \eta r) - \eta + \eta r\} + Xy(\eta b - 1)}{(z - X)\,(z - y)}$$

with second type of intersection number , and

$$\eta b - 1 - \alpha_1 - \alpha_2 = \frac{k^2(\eta - 1) + k\{(y + z)(1 - \eta r) - \eta + \eta r\} + yz(\eta b - 1)}{(y - X)(z - X)}$$

with third type of intersection number and where α_1 , α_2 and ($\eta b - 1 - \alpha_1 - \alpha_2$) be the number of blocks intersecting B_* in y , z and x number of points.

Proof :- Jain and Banerjee (see [11]) Theorem 3.1.

Example 3.2 If t= 2, in equation (10), then the quasi-symmetric design D with parameter set (41, 82, 10 , 5 ,1) and D* is also a quasi-symmetric design with parameters $v = 41$, $b^* = 328$, $r^* = 40$, $k = 5$, $\lambda^* = 4$, $x = 0$, $y = 1$ and $z = 2$ and for D* from theorem 3.1 , $\alpha_1 = 135$, $\alpha_2 = 30$ and $(b^* - 1 - \alpha_1 - \alpha_2) = 162$.

Corollary 3.3 Let the designs D_{ij}^*, D_{ijk}^* and D^* with parameter set (v, ηb, ηr, k, $\eta\lambda$; x , y, z) are quasi-symmetric design have three intersection numbers $x = 0$, $y = 1$ and $z = 2$. Then these designs holds the following relations :

1. $v\eta r = \eta bk$,
2. $\eta\lambda(v-1) = \eta r(k-1)$,
3. $(\eta r)^2(k-1) + \eta^2 r\lambda = \eta^2 bk\lambda$,
4. $\alpha_1 y + \alpha_2 z + (\eta b - 1 - \alpha_1 - \alpha_2)x = k(\eta r - 1)$,
5. $\alpha_1 y(y-1) + \alpha_2 z(z-1) + (\eta b - 1 - \alpha_1 - \alpha_2)x(x-1) = k(k-1)(\eta\lambda - 1)$.

4. RESULTS

The following tables provide a list of parameters which can be obtained by using theorems 2.1, 2.2, 2.3, 2.4 and theorem 3.1.

Table :1

2-Multiple BIB Designs D_{ij}*(η=2)

S.No.	t	V	b	r	k	λ	b*= ηb	r* = ηr	λ* =ηλ	Non-isomorphic Solutions	b* -1 – α_1 – α_2	α_1	α_2
1	2	41	82	10	5	1	164	20	2	D_{12}*,D_{13}*,D_{14}*, D_{23}*,D_{24}*,D_{34}*	78	75	10
2	3	61	183	15	5	1	366	30	2		230	125	10
3	5	101	505	25	5	1	1010	50	2		774	225	10
4	7	141	987	35	5	1	1974	70	2		1638	325	10

Table :2

3-Multiple BIB Designs D_{ijk}*(η=3)

S.No.	t	v	b	r	k	λ	b*= ηb	r* = ηr	λ* =ηλ	Non-isomorphic Solutions	b* -1 – α_1 – α_2	α_1	α_2
1	2	41	82	10	5	1	246	30	3	D_{123}*,D_{124}*,D_{134}* , D_{234}*	120	105	20
2	3	61	183	15	5	1	549	45	3		348	180	20
3	5	101	505	25	5	1	1515	75	3		1164	330	20
4	7	141	987	35	5	1	2961	105	3		2460	480	20

Table :3

4-Multiple BIB Designs D*(η=4)

S.No.	T	v	b	r	k	λ	b*= ηb	r* = ηr	λ* =ηλ	Non-isomorphic Solution	b* -1 – α_1 – α_2	α_1	α_2
1	2	41	82	10	5	1	328	40	4	D*	162	135	30
2	3	61	183	15	5	1	732	60	4		466	235	30
3	5	101	505	25	5	1	2020	100	4		1554	435	30
4	7	141	987	35	5	1	3948	140	4		3282	635	30

4. ACKNOWLEDGEMENTS

The authors are thankful to the referees for their comments and valuable suggestions.

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