OPERATION APPROACHES ON gs - OPEN SETS IN TOPOLOGICAL SPACES

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Abstract: The concept of an operation κ on the family of gs-open sets $GSO(X,\tau)$, in a topological space (X,τ) is introduced. The concepts of κ-open sets, κ-regular and κ-closed are introduced using the operation κ and their related topological properties are studied.

Key words: κ-interior, κ-open set, κ-regular set,κ-closed set, gs_κclosure, gs-closure$_\kappa$, $regular_\kappa$operation and $open_\kappa$operation.

1. Introduction

In 1963, Levine [4] introduced the concept of semi-open set. Following this the notion of generalized semi closed sets was introduced by Arya and Nour [1] in 1990.

Kasahara [3] defined the concept of an operation on a topological space and introduced the concept of a α-closed graphs of functionsin 1979. Following his work, Jankovic [2], developed the concept of α-closed sets and investigated functions with α-closed graphs in 1983.Ogata [5] defined and investigated the concept of operation-open sets.

In this paper, we shall introduce operation κ on generalized semi open sets. Section 3 of this paper deals with the definition and properties of κ-interior,κ-open set, κ-regular set,κ-closed set,gs_κclosure, gs-closure$_\kappa$, $regular_\kappa$operation and $open_\kappa$operation.

2. Preliminary

Definition 2.1

Let (X,τ) be a topological space. A subset A of a space (X,τ) is called **generalized semi closed (gs-closed)** set if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X,τ).

Definition 2.2

Let (X,τ) be a topological space.A subset A of a space $(X,\tau)$is called **generalized semi open (gs-open) set** if $X\backslash A$ is gs-closed. The collection of all gs-open sets is denoted by $GSO(X,\tau)$. Clearly $\tau \subseteq GSO(X,\tau)$.

Remark 2.3

Every closed set is gs-closed but the converse not true.

Definition 2.4 [2]

Let (X,τ) be a topological space. An ***operation*** $\gamma: \tau \to P(X)$ is a mapping from τ into the power set of X such that $V \subseteq V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V.

Definition 2.5 [5]

A subset A of a space (X,τ) will be called a ***γ- open set*** of (X,τ) if for each $x \in A$, there exists an open set U such that $x \in U$ and $U^\gamma \subset A$. τ_γ will denote the set of all γ-open sets. Clearly we have $\tau \supset \tau_\gamma$.

Definition 2.6 [5]

A subset B of (X,τ) is said to be ***γ- closed***in (X,τ) if $X\backslash B$ is γ-open in(X,τ).

Definition 2.7 [5]

A point $x \in X$ is in the ***γ- closure*** of a set $A \subseteq X$ if $U^\gamma \cap A \neq \phi$ for each open set U of x. The γ closure of a set A is denoted by $Cl_\gamma(A)$.

Definition 2.8[5]

An operation $\gamma: \tau \to P(X)$ is a mapping from τ into the power set $P(X)$. τ_γ- $Cl(A) = \cap\{F: A \subseteq F, X\backslash F \in \tau_\gamma\}$. Where τ_γ denotes the set of all γ-open sets in (X,τ).

3. κ - Open sets.

Definition 3.1

Let (X,τ) be a topological space. A mapping $\kappa: GSO(X,\tau) \to P(X)$ the family of generalized semi open sets $GSO(X,\tau)$ to the power set of X such that $V \subseteq V^\kappa$ for every $V \in GSO(X,\tau)$ where V^κ denotes the value of V under the operation κ .

Example 3.2

Let $X=\{a,b,c\}$ and $\tau=\{X,\phi,\{a\},\{b\},\{a,b\},\{a,c\}\}$. Then $\kappa: GSO(X,\tau)\to P(X)$ defined by $A^{\kappa}=\begin{cases}A & if\ b\in A\\ cl(A) & if\ b\notin A\end{cases}$ is a κ-operation on (X,τ) as $A\subseteq A^{\kappa}$, for every $A\in GSO(X,\tau)=\{X,\phi,\{a\},\{b\},\{c\},\{a,b\},\{a,c\}\}$.

Definition 3.3

A subset A of a space (X,τ) will be called a ***κ-open set*** of (X,τ) if for each $x\in A$, there exists a gs-open neighbourhood U of x and $U^{\kappa}\subseteq A$.

$\boldsymbol{\kappa O(X,\tau)}$ will denote the set of all κ-open sets.

Example 3.4

Let (X,τ) and κ be defined as in Example(3.2). Then the κ-open sets are $\{X,\phi,\{b\},\{a,b\},\{b,c\}\}$.

Theorem 3.5

If κ is an operation on $GSO(X,\tau)$, then the following results are true.

(a) Every κ- $open\ set$ of (X,τ) is gs- $open$ in (X,τ).

(i.e) $\kappa O(X,\tau)\subseteq GSO(X,\tau)$.

(b) Every γ- $open\ set$of (X,τ) is κ- $open$.

(c) Arbitrary union of κ- $open\ sets$ in (X,τ) is also aκ- $open\ set$.

Proof

(a) Consider a κ- open set$A$in (X,τ) and a point $x\in A$. By Definition(3.3), there exists a gs-open neighbourhood B of x such that $B^{\kappa}\subseteq A$. By Definition(3.1), $B\subseteq B^{\kappa}$ and hence $x\in B\subseteq B^{\kappa}\subseteq A$. (i.e,)$x\in B\subseteq A$ implying that A is a gs-open set. Thus $\kappa O(X,\tau)\subseteq GSO(X,\tau)$.

(b) Consider a γ-open set C of (X,τ) with $x\in C$. By Definition(2.7), there exists an open set U such that $x\in U\subseteq U^{\kappa}\subseteq C$. Since every open set is gs-open, C is κ-open.

(c) Consider $\{B_{\alpha}:\alpha\in J\}$, a collection of κ-open sets in (X,τ). Let $x\in B=\cup B_{\alpha}$. Hence $x\in B_{\alpha}$for some $\alpha$and since B_{α} is κ-open, there exists a gs-open neighborhood U of x such that $U^{\kappa}\subseteq B_{\alpha}\subseteq B$. Therefore B is κ-open.

Example 3.6

This example shows that a gs- $open$ set need not be κ- $open$.

Let $X=\{a,b,c\}$, $\tau=\{x,\phi,\{a\},\{a,b\}\}$ and $gs\text{-}open=\{x,\phi,\{a\},\{b\},\{a,b\},\{a,c\}\}$. Let κ be an operation on $GSO(X,\tau)$ such that $\kappa(A)=\begin{cases}A & ,\ if\ b\in A\\ \{a,c\}, & if\ b\notin A\end{cases}$.

Then the κ- $open$ sets are $\{X,\phi,\{b\},\{a,b\},\{a,c\}\}$. Here $\{a\}$ is gs - $open$ but not κ- $open$.

Example 3.7

This example shows that a κ- $open$ set need not be γ- $open$.

Let $X=\{a,b,c\}$, $\tau=\{x,\phi,\{a\},\{b\},\{a,b\}\}$ and $gs\text{–}open=\{x,\phi,\{a\},\{b\},\{a,b\},\{a,c\}\}$. Let κ be an operation on $GSO(X,\tau)$ such that $\kappa(A)=gcl(A)$. Then the κ- $open$ sets are $\{X,\phi,\{b\},\{a,c\}\}$. And γ be an operation on τ. The γ- $open$ sets are $\{X,\phi,\{b\}\}$. Here $\{a,c\}$ is κ- $open$ but not γ- $open$.

Remark 3.8

Intersection of any two κ-open sets need not be κ-open.

Counter Example 3.9

The following example shows that intersection of κ- $open$ sets need not be κ- $open$.

Let $X=\{a,b,c\}$, $\tau=\{x,\phi,\{a\},\{a,b\}\}$ and $GSO(X,\tau)=\{x,\phi,\{a\},\{b\},\{a,b\},\{a,c\}\}$. Let κ be an operation on $GSO(X,\tau)$ such that $\kappa(A)=\begin{cases}A & ,\ if\ b\in A\\ \{a,c\}, & if\ b\notin A\end{cases}$.

Then the κ- $open$sets are $\{X,\phi,\{b\},\{a,b\},\{a,c\}\}$.

Intersection of the κ- $open$sets $\{a,b\}$ and $\{a,c\}$ is not a κ- $open$.

Definition 3.10

A κ-operation $\kappa: GSO(X,\tau)\to P(X)$ is called $\boldsymbol{regular_{\kappa}}$**operation** given $x\in X$ and for each pair of gs-open neighbourhoods A and B of x, there exists a gs-open neighbourhood $C$of x such that $A^{K}\cap B^{K}\supseteq C^{K}$.

Definition 3.11

A topological space (X,τ) is called $\boldsymbol{\kappa\text{-}regular}$ if for given $x\in X$ and each gs-open neighbourhood U of X, there exists a gs-open neighbourhood V of x such that $V^{\kappa}\subseteq U$.

Theorem 3.12

Let (X,τ) be a topological space and κ an operation on $GSO(X,\tau)$. Then the following results are equivalent.

(a) $GSO(X,\tau)=\kappa O(X,\tau)$.

(b) (X,τ) is a κ-regular space.

(c) Given $x\in X$ and every gs-open set B of (X,τ) containing x there exists a κ-open set W of (X,τ) such that $x\in W$ and $W\subseteq B$.

Proof

(a) ⇒ (b)

Let x in X and V, a gs-open neighbourhood of x. By (a), V is κ-open in (X,τ). By Definition(3.3), there exists a gs-open neighbourhood U of $x$such that $U^{\kappa} \subseteq V$. Hence by Definition(3.11), (X,τ) is κ-regular.

(b) ⇒ (c)

Consider $x \in X$ and a gs-open neighbourhood B of x. By(b),(X,τ) is a κ-regular space. Hence by Definition (3.11), there exists a gs-open neighbourhood W of x such that $W^{\kappa} \subseteq B$. By Definition(3.1), $W \subseteq W^{\kappa}$. Hence $x \in W \subseteq W^{\kappa} \subseteq B$.

Claim:W is κ-open. Let $y \in W$. Implies $y \in X$ and W, be the gs-open neighbourhood of y. Then By (b), there exists a gs-open neighbourhood U of x such that $U^{\kappa} \subseteq W$. By Definition(3.3), W is κ-open. Hence, there exists a κ-open set W such that $x \in W \subseteq B$, proving (c).

(c) ⇒ (a)

In Theorem(3.5)(i), it is proved that $\kappa O(X,\tau) \subseteq GO(X,\tau)$. It is left to prove $GSO(X,\tau) \subseteq \kappa O(X,\tau)$. Let A be a gs-open set in (X,τ) and $x \in A$. Then $x \in X$ and By (c), there exists a κ-open set W of (X,τ) such that $x \in W \subseteq A$. (1) Since W is a κ-open set there exists a gs-open set V such that $x \in V^{\kappa} \subseteq W$. (2)

(1) and (2) implies $x \in V^{\kappa} \subseteq A$. Implies A is κ-open. Therefore,$GSO(X,\tau) \subseteq \kappa O(X,\tau)$. Hence, $GSO(X,\tau) = \kappa O(X,\tau)$.

In general the intersection of two κ-open sets is not κ-open. The following theorem proves that if κ is a regular$_\kappa$ –operation then the intersection of κ-open sets is κ-open.

Theorem 3.13

Let κ be a$regular_{\kappa}$operation on $GSO(X,\tau)$. If A and B are κ-open sets in (X,τ) then $A \cap B$ is κ-open.

Proof

Let A and B be κ-open sets in (X,τ). Consider $C = A \cap B$. Let $x \in C$ implies $x \in A$ and $x \in B$. Since A and B are κ-open sets, there exists a gs-open neighbourhoods U and V of x such that $U^{\kappa} \subseteq A$ and $V^{\kappa} \subseteq B$. Since the operation κ is $regular_{\kappa}$, by Definition (3.10), there exists a gs-open neighbourhood C of x such that $C^{\kappa} \subseteq U^{\kappa} \cap V^{\kappa} \subseteq A \cap B$. Therefore $A \cap B$ is κ-open.

Remark 3.14

$\kappa O(X,\tau)$ forms a topology wheneverκ is $regular_{\kappa}$.

Definition 3.15

A subset A of a topological space (X,τ) is called ***κ- closed*** whenever $X - A$ is κ-open.

Example 3.16

From the Example(3.2) and Example(3.4), the κ -closed sets are $\{X, \phi, \{a\}, \{c\}, \{a,c\}\}$.

Definition 3.17

Let κ be an operation on $GSO(X,\tau)$. A point $x \in X$ is said to be a ***κ- closure point*** of the set Aif $U^{\kappa} \cap A \neq \phi$ for each gs-open neighbourhood U of x.

$$\boldsymbol{gs\ Cl_{\kappa}(A)} = \{x \in X / U^{\kappa} \cap A \neq \phi,\ \forall U, gs\text{-}open\ neighborhood\ of\ x\}$$

Example 3.18

Let X, τ and κ be defined as in Example(3.2). Let $A = \{b,c\}$ then $Cl_{\kappa}(A) = \{a,b,c\} = X$.

Remark 3.19

Let κ be an operation on $GSO(X,\tau)$. Then $gs\ Cl(A) \subseteq gs\ Cl_{\kappa}(A)$.

Proof

Let $x \in gscl(A)$. Implies $A \cap V \neq \phi$, for every gs-open neighbourhood V of x. Now, $V \subseteq V^{\kappa}$ implies $A \cap V^{\kappa} \neq \phi$. By Definition (3.17), $x \in Cl_{\kappa}(A)$. Hence, $gsCl(A) \subseteq gs\ Cl_{\kappa}(A)$.

Definition 3.20

Let κ be an operation on $GSO(X,\tau)$. Then $\boldsymbol{gs_{\kappa}Cl(A)}$is defined as the intersection of all κ-closed sets containing A.

$\boldsymbol{gs_{\kappa}Cl(A)} = \cap \{F \subseteq X/\ A \subseteq F\ and\ X \backslash F \in \kappa O(X,\tau)\}$

Theorem 3.21

Let (X,τ) be a topological space and A a subset of X and κ be an operation on $GSO(X,\tau)$. Then for a given $y \in X$, $y \in gs_{\kappa}Cl(A)$ if and only if $V \cap A \neq \phi$ for every $V \in \kappa O(X,\tau)$ such that $y \in V$.

Proof

Define: $F = \{y \in X / V \cap A \neq \phi$ for every $V \in \kappa O(X,\tau)$and $y \in V\}$. It is to be proved that $gs_{\kappa}Cl(A) = F$.

Take $x \notin F$. By the construction of F, there exists a κ-open set V containing x such that $V \cap A = \phi$. Then $X \backslash V$ is κ-closed and $A \subseteq X \backslash V$. Taking $gs_{\kappa}Cl(A)$ on both sides, $gs_{\kappa}Cl(A) \subseteq gs_{\kappa}Cl(X \backslash V) = X \backslash V$. Since $x \in V$, $x \notin X \backslash V$ implie$x \notin gs_{\kappa}Cl(A)$. Hence $gs_{\kappa}Cl(A) \subseteq F$.

Take $x \notin gs_{\kappa}Cl(A) = \cap \{E/A \subseteq E\ and\ X \backslash E \in \kappa O(X,\tau)\}$. Then there exists κ-closed set E such that $A \subseteq E$, but $x \notin E$ implies $x \in X \backslash E \in \kappa O(X,\tau)$ and $(X \backslash E) \cap A = \phi$ implies $x \notin E$. Therefore $E \subseteq gs_{\kappa}Cl(A)$.Hence $gs_{\kappa}Cl(A) = E$.

Theorem 3.22

Let (X,τ) be a topological space. Let A and B be subsets of X and κ be an operation on $GSO(X,\tau)$. The statements below are true.

(a) The set $gs_\kappa Cl(A)$ is κ-closed and $A \subseteq gs_\kappa Cl(A)$.
(b) A is κ-closed if and only if $A = gs_\kappa Cl(A)$.
(c) If $A \subseteq B$ then $gs_\kappa Cl(A) \subseteq gs_\kappa Cl(A)$.
(d) $gs_\kappa Cl(A) \cup gs_\kappa Cl(B) \subseteq gs_\kappa Cl(A \cup B)$.
(e) If κ is $regular_\kappa$, then $gs_\kappa Cl(A) \cup gs_\kappa Cl(B) = gs_\kappa Cl(A \cup B)$.
(f) $gs_\kappa Cl(A \cap B) \subseteq gs_\kappa Cl(A) \cap gs_\kappa Cl(B)$.
(g) $gs_\kappa Cl(gs_\kappa Cl(A)) = gs_\kappa Cl(A)$.

Proof

(a) Let $A \subseteq X$. Consider $gs_\kappa Cl(A) = B$, say. Claim: B is κ-closed.To prove:$X - B$ is κ-open. Now B = $gs_\kappa Cl(A)$ =∩ {all κ- closed sets containg A}.
$X\backslash B = B^C = \left(gs_\kappa Cl(A)\right)^C$ =∪ {*all κ- open sets contained in A*}= a κ-open set (by Theorem (3.5) (c).
$A \subseteq gs_\kappa Cl(A)$ follows directly from the Definition(3.20).

(b) A is κ-closed if and only if $A = gs_\kappa Cl(A)$

Necessity: Since A is κ-closed, $gs_\kappa Cl(A) = \cap\{\text{all } \kappa - \text{closed sets containg A}\} = A$

Sufficiency: Since $A = gs_\kappa Cl(A)$and from (a) $gs_\kappa Cl(A)$is κ-closed. We get A is κ-closed.

(c) Let $A \subseteq B$.
$gs_\kappa Cl(A) = \cap\{\text{all } \kappa - \text{closed sets containg A}\} = \cap \mathcal{A}$ where $\mathcal{A}$ is the collection of all $\kappa -$ closed sets containg A
$gs_\kappa Cl(B) = \cap\{\text{all } \kappa - \text{closed sets containg B}\} = \cap \mathcal{B}$ where $\mathcal{B}$is the collection of all $\kappa -$ closed sets containg $\mathcal{B}$
Since $\mathcal{A} \subseteq \mathcal{B}$, $(\cap \mathcal{A}) \subseteq (\cap \mathcal{B})$. Therefore $gs_\kappa Cl(A) \subseteq gs_\kappa Cl(A)$.

(d) Let $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Therefore By (c)$gs_\kappa Cl(A) \subseteq gs_\kappa Cl(A \cup B)$ and $gs_\kappa Cl(B) \subseteq gs_\kappa Cl(A \cup B)$. Hence $gs_\kappa Cl(A) \cup gs_\kappa Cl(B) \subseteq gs_\kappa Cl(A \cup B)$.

(e) Suppose κ is $regular_\kappa$. Let $y \notin gs_\kappa Cl(A) \cup gs_\kappa Cl(B)$. Then $y \notin gs_\kappa Cl(A)$ and$y \notin gs_\kappa Cl(A)$. Then there exist two κ-open sets U and V such that $U \cap A = \phi$ and$V \cap B = \phi$. Since κ is a$regular_\kappa$ operation, by Theorem(3.11), $A \cap B$ is κ-open in (X,τ).Therefore $(U \cap A) \cap (V \cap B) = (U \cap V) \cap (A \cap B) = \phi$. Implies $x \notin gs_\kappa Cl(A \cup B)$. Hence,$gs_\kappa Cl(A \cup B) \subseteq gs_\kappa Cl(A) \cup gs_\kappa Cl(B)$.

(f) We know $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Hence by (c),$gs_\kappa Cl(A \cap B) \subseteq gs_\kappa Cl(A)$ and $gs_\kappa Cl(A \cap B) \subseteq gs_\kappa Cl(B)$. Therefore $gs_\kappa Cl(A \cap B) \subseteq gs_\kappa Cl(A) \cap gs_\kappa Cl(B)$.

(g) From (a), $gs_\kappa Cl(A) \subseteq gs_\kappa Cl(gs_\kappa Cl(A))$. Now by Theorem(3.19) for every κ-open set containing z in (X,τ), $V \cap gs_\kappa Cl(A) \neq \phi$. Therefore there exist a point $y \in V$ and $y \in gs_\kappa Cl(A)$. Again by Theorem (3.19), $V \cap A \neq \phi$. Which implies $z \in gs_\kappa Cl(A)$. Hence $gs_\kappa Cl(gs_\kappa Cl(A)) = gs_\kappa Cl(A)$.

Definition 3.23

An operation κ on GSO(X,τ) is said to be $\boldsymbol{open_\kappa}$operation if for every gs-open neighbourhood U of $x \in X$, there exists a κ-open set V such that $x \in V$ and $V \subset U^\kappa$

Theorem 3.24

Let $\kappa: GSO(X,\tau) \to P(X)$ be an operation on $GSO(X,\tau)$ and P and Q are subsets of X. Then the results below are true.

(a) $gsCl_\kappa(A)$ is gs-closed and $A \subseteq gsCl_\kappa(A)$.
(b) A is κ-closed if and only if $A = gsCl_\kappa(A)$.
(c) If (X,τ) is a κ-regular space then $gsCl_\kappa(A) = gscl(A)$.
(d) If $A \subseteq B$ then $gsCl_\kappa(A) \subseteq gsCl_\kappa(B)$.
(e) $gsCl_\kappa(A) \cup gsCl_\kappa(B) \subseteq gsCl_\kappa(A \cup B)$.
(f) If κ is is $regular_\kappa$, then $gsCl_\kappa(A) \cup gsCl_\kappa(B) = gsCl_\kappa(A \cup B)$.
(g) $gsCl_\kappa(A \cap B) \subseteq gsCl_\kappa(A) \cap gsCl_\kappa(B)$.
(h) If κ is $open\ _\kappa$operation then $gsCl_\kappa(A) = gs_\kappa Cl(A)$ and $gsCl_\kappa(gsCl_\kappa(A)) = gsCl_\kappa(A)$.

Proof

(a) Let $A \subseteq X$. Consider $x \in gscl(Cl_\kappa(A))$. Then for every gs-open neighbourhood V of x, $V \cap gsCl_\kappa(A) \neq \phi$. Let $y \in V \cap gsCl_\kappa(A)$. Since $V \subseteq V^\kappa$, $V^\kappa \cap gsCl_\kappa(A) \neq \phi$. Implies $x \in gsCl_\kappa(A)$. Therefore, $gsCl(gsCl_\kappa(A)) \subseteq gsCl_\kappa(A)$. Hence $gsCl_\kappa(A)$ is gs-closed.

Let $x \in A$ and U be any gs-open neighbourhood of x, then $x \in U \cap A$. Now $U \subseteq U^\kappa$ implies$x \in U^\kappa \cap A$ which $U^\kappa \cap A = \phi$. Therefore,$x \in gsCl_\kappa(A)$. Hence, $A \subseteq gsCl_\kappa(A)$.

(b) **Necessity:** Let A be κ-closed. Then$X\backslash A$ is κ-open. Claim:$gsCl_\kappa(A) \subseteq A$. Let $x \notin A$.Then$x \in X\backslash A$. Since $X\backslash A$ is a κ-open set containing x, by Definitin(3.3), there exists a gs-open set containing x such that $U^\kappa \subseteq X\backslash A$ which implies $U^\kappa \cap A = \phi$. Therefore $x \notin gsCl_\kappa(A)$. Hence $gsCl_\kappa(A) \subseteq A$.

Sufficiency: Let$A = gsCl_\kappa(A)$. Let $x \in X\backslash A$. Then$x \notin A = gsCl_\kappa(A)$, there exists a gs-open neighbourhood W of x such that $W^\kappa \cap A = \phi$ which implies$W^\kappa \subseteq X\backslash A$. By Definition(3.3), $X\backslash A$ is κ-open (i.e.)A is κ-closed.

(c) $gscl(A) \subseteq gsCl_\kappa(A)$ is proved in Remark(3.17). Let $x \notin gscl(A)$.Then there exists a gs-open neighbourhood U of x such that $U \cap A = \phi$. Since (X,τ) is a κ-regular space by Definition(3.11), for every $x \in X$, there exists a gs-neighbourhood $V$of x such that $V^\kappa \subseteq U$ and so $V^\kappa \cap A = \phi$ which implies $x \notin gsCl_\kappa(A)$. Therefore, $gsCl_\kappa(A) \subseteq gscl(A)$.

(d) Given $A \subseteq B$. Let $x \in gsCl_\kappa(A)$. By Definition(3.17), there exists a gs-open neighbourhood U of x such that $U^\kappa \cap A \neq \phi$which implies $V^\kappa \cap B \neq \phi$. Therefore $x \in gsCl_\kappa(A)$. Hence $gsCl_\kappa(A) \subseteq gsCl_\kappa(B)$.

(e) $A \subseteq (A \cup B)$ implies $gsCl_\kappa(A) \subseteq gsCl_\kappa(A \cup B)$ From(d). Similarly$B \subseteq (A \cup B)$ implies $gsCl_\kappa(B) \subseteq gsCl_\kappa(A \cup B)$.

Hence, $gsCl_\kappa(A) \cup gsCl_\kappa(B) \subseteq gsCl_\kappa(A \cup B)$.

(f) Let κ is a$regular_\kappa$ operation. Let $x \notin gsCl_\kappa(A) \cup gsCl_\kappa(B)$. $x \notin gsCl_\kappa(A)$ and $x \notin gsCl_\kappa(B)$. There exists gs-open neighbourhoods U and V of x such that $U^\kappa \cap A = \phi$ and$V^\kappa \cap B = \phi$. Since κ is a$regular_\kappa$ by Definition (3.10), there exists a gs-open neighbourhood C of x such that $C^\kappa \subseteq U^\kappa \cap V^\kappa$. Which implies $C^\kappa \cap (A \cup B) \subseteq (U^\kappa \cap V^\kappa) \cap (A \cup B) = \phi$ and $x \notin gsCl_\kappa(A \cup B)$. Therefore $gsCl_\kappa(A \cup B) \subseteq gsCl_\kappa(A) \cup gsCl_\kappa(B)$.

Hence gs$Cl_\kappa(A) \cup gsCl_\kappa(B) = gsCl_\kappa(A \cup B)$.

(g) $A \cap B \subseteq A$ implies gs$Cl_\kappa(A \cap B) \subseteq gsCl_\kappa(A)$ and $A \cap B \subseteq B$implies $gsCl_\kappa(A \cap B) \subseteq gsCl_\kappa(b)$ Therefore, $gsCl_\kappa(A \cap B) \subseteq gsCl_\kappa(A) \cap gsCl_\kappa(B)$.

(h) Let $y \notin gsCl_\kappa(A)$, there exists a gs-open neighbourhood U of y such that$(U^\kappa \cap A) = \phi$. Since the operationκ is $open_\kappa$, there exists a κ-open set V containing y such that $V \subseteq U^\kappa$implies $V \cap A$=ϕ and $x \notin gs_\kappa Cl(A)$ by Theorem(3.19). Hence $gs_\kappa Cl(A) \subseteq gsCl_\kappa(A)$.

Let $x \in gsCl_\kappa(A)$. Suppose $x \notin F$ where F is κ-closed and $A \subseteq F$. Then$x \in X\backslash F$, and $A \cap (X\backslash F) = \phi$. Since $X\backslash F$ is κ-open and $x \in X\backslash F$, there exists ags-open set W such that $x \in W$ and $W^\kappa \subseteq X\backslash F$. Which implies $A \cap W^\kappa = \phi$. Thus$x \notin gsCl_\kappa(A)$. Hence, if the operationκ is $open_\kappa$operation then$gsCl_\kappa(A) = gs_\kappa Cl(A)$.

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