CONTINUUM FORMULATION FOR LARGE AMPLITUDE FREE VIBRATION OF UNIFORM FIXED-HINGED BEAM

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ABSTRACT : A simplified continuum formulation using the classical Rayleigh-Ritz method is proposed to study the large amplitude vibrations of uniform fixed-hinged beam. The Rayleigh-Ritz method requires the assumption of admissible functions for the axial displacement and lateral deflection. Though simple accurate single term admissible functions for the lateral deflection and the axial displacement are used, the Rayleigh-Ritz method gives two nonlinear algebraic equations that have to be solved to obtain the nonlinear frequency. This complex task can be simplified if one of the displacement function can be expressed in terms of the other. This is achieved in this paper by using the property of constant tension that is developed in the beam due to large deflections, when both the ends of the beam are restrained to move axially. Using the property of constant tension, the axial displacement distribution along the beam can be obtained in terms of the lateral deflection function and the problem contains a single undetermined coefficient, which can be solved easily to obtain the nonlinear frequency. The application of this procedure in conjunction with the harmonic balance method is demonstrated to study the large amplitude vibration of beams. Using the proposed formulation, the ratio of the nonlinear to linear radian frequencies for various maximum amplitude ratios for the practically important fundamental mode, are obtained that compare very well with those available in the literature.

***Keywords* – continuum method, fixed-hinged beam, simple harmonic motion, duffing equation, uniform beam.**

INTRODUCTION

Large amplitude vibrations of beams, with von-Karman type nonlinearity, have been studied by many researchers after the famous work of Woinowsky-Krieger [1]. This formulation is based on the tension developed in the beam because of large deflections, with the ends restrained to move axially, and is considered as a measure of the geometric nonlinearity. The nonlinear partial differential equation, with the axial coordinate and time as independent variables, governing the large amplitude vibrations is obtained by incorporating this tension term in the formulation. The space variable is eliminated by assuming an admissible space mode and a nonlinear ordinary differential equation in time (temporal equation) is obtained in the form of a homogeneous Duffing equation. The solution of this equation is obtained in terms of the elliptic integrals. It is to be noted here that as the homogeneous Duffing equation is directly solved, no assumption need to be made on the nature of vibrations.

Subsequently many investigators formulated the large amplitude vibration problem of beams, using the approximate continuum or the versatile finite element[FE] formulations[2-8]. In these studies emphasis is given on the practically important fundamental mode that needs a lesser energy to get excited. Either an assumed time mode or a space mode approach is used with simplifying assumptions like the harmonic oscillations assumption and/or linearizing the strain-displacement relations. These assumptions inevitably introduce some error in the final solution. Detailed discussions on the effect of these assumptions are discussed in Ref. [10].

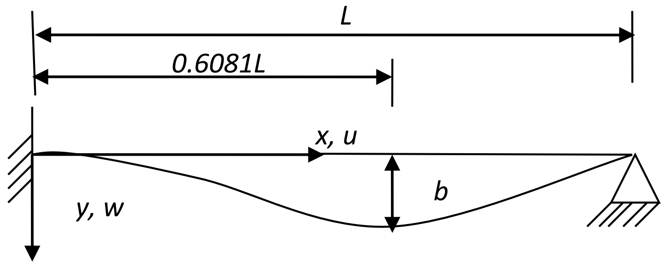

Fig.1. Large amplitude free vibration of fixed-Hinged beam.

I.EVALUATION OF AXIAL DISPLACEMENT *u*

In the general formulation proposed in this paper it is necessary to evaluate the axial displacement function *u*, once the admissible function for the lateral deflection *w*, that satisfies at least the geometric boundary conditions of the beam configuration, is known. This is achieved in the present formulation, where the von-Karman type nonlinearity exists as the ends of the beam are restrained to move axially. This strain – displacement relations, for the case of the beams with the ends restrained to move axially, from now onwards called simply as beams, is

$$\in_x = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \tag{1}$$

where $\in_x$ is the axial strain and x is the axial coordinate.

Multiplying Eq.(1) by EA, where E is the Young's modulus and A is the area of cross- section, considered as constant for simplicity in the present study, the axial load P in the beam is given by

$$P = EA\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] \tag{2}$$

For beams, the value of P is constant (even for tapered beams) to satisfy the axial force equilibrium. Differentiating Eq.(2) with respect to x and as $\frac{dP}{dx} = 0$, we get

$$\frac{d^2u}{dx^2} = -\frac{dw}{dx}\cdot\frac{d^2w}{dx^2} \tag{3}$$

For a given beam configuration, if the admissible function for the lateral deflection is known, then Eq.(3) can be integrated to evaluate the expression for u consistent with w. Note that the function u will have the same undetermined coefficient as in the assumed displacement w.

In the subsequent detailed presentation of this proposed simplified continuum formulation, a hinged-hinged beam is considered for which the admissible function for the lateral deflection w for the first mode of vibration is taken as

$$w = \frac{b}{\alpha}\left(\cos\frac{3\pi x}{2L} - \cos\frac{\pi x}{2L}\right) \tag{4}$$

where b is the undetermined coefficient and also represents the maximum deflection at the mid point of the beam and L is the length of the beam.

The function given in Eq.(4) is exact and satisfies the geometric boundary conditions that are essential in the Rayleigh-Ritz method as well as the natural boundary conditions and the symmetric conditions given by

$$w(0) = \frac{d^2w}{dx^2}(0) = w(L) = \frac{d^2w}{dx^2}(L) = \frac{dw}{dx}\left(\frac{L}{2}\right) = 0 \tag{5}$$

Integrating Eq.(3) twice and using the boundary conditions of u given by $u(0) = u(L) = 0$, the expression for u is obtained as

$$u = \left(\frac{b}{\alpha}\right)^2\left[\frac{7\pi}{16L}\sin\frac{\pi x}{L} - \frac{3\pi}{16L}\sin\frac{2\pi x}{L} + \frac{3\pi}{16L}\sin\frac{3\pi x}{L}\right] \tag{6}$$

Note that the expression for u given in Eq.(6) satisfies the antisymmetric condition

$$u\left(\frac{L}{2}\right) = 0 \tag{7}$$

II.EVALUATION OF THE RATIOS $(\omega_{NL_H}/\omega_L)^2$ WITH SHM ASSUMPTION FOR HINGED-HINGED BEAM

Using the nonlinear strain-displacement relation given in Eq. (1) and the curvature- displacement relation

$$\psi_x = -\frac{d^2w}{dx^2} \tag{8}$$

The strain energy U is given by

$$U = \frac{EA}{2}\int_0^L\left[\left(\frac{du}{dx}\right)^2 + \frac{du}{dx}\left(\frac{dw}{dx}\right)^2 + \frac{1}{4}\left(\frac{dw}{dx}\right)^4\right]dx + \frac{EI}{2}\int_0^L\left(\frac{d^2w}{dx^2}\right)^2 dx \tag{9}$$

where I is the area moment of inertia.

The kinetic energy T of the beam executing SHM, is

$$T = \frac{m\omega^2_{NL_H}}{2}\int_0^L w^2 dx \tag{10}$$

where ω_{NL_H} is the nonlinear radian frequency with the SHM assumption.

Using the assumed admissible function for w and the derived function u given in Eq.(4) and Eq.(6), and minimizing the Lagrangian ($U - T$) with respect to b as

$$\frac{d(U-T)}{db} = 0 \tag{11}$$

Eq.(11) gives the fundamental nonlinear frequency parameter λ_{NL_H} and using the linear fundamental frequency parameter λ_L (15), the ratios of the nonlinear to linear frequency parameters for the maximum amplitude ratio of b/r is

$$\frac{\lambda_{NL_H}}{\lambda_L} = \left[\frac{\omega_{NL_H}}{\omega_L}\right]^2 = 1 + \gamma\left(\frac{b}{r}\right)^2 \tag{12}$$

where λ is defined as $m\omega^2 L^4 / EI$, m is the mass density per unit length γ is a constant for a specific beam configuration and r is the radius of gyration. Equation (12), though is simpler to obtain with the assumption of SHM, does not represent the actual situation as the periodic nonlinear oscillations of the beam, do not exhibit the SHM. A correction factor is evaluated in the next section to correct the expression obtained in Eq. (12) using the HBM that eliminates the error due to the assumption of the SHM.

III.HARMONIC BALANCE METHOD

Solution of the Duffing equation with SHM assumption

To evaluate the correction factor mentioned in the previous section the following two steps are followed. The first step deals with the solution of the homogeneous Duffing equation using the assumption of the SHM and the second step deals with the correction of the solution obtained in the first step to eliminate the error involved in using the assumption. Based on the work of Woinowsky – Krieger [1], it is known that, the temporal equation is a homogeneous Duffing equation of hardening type with cubic nonlinearity.

The homogeneous Duffing equation, in its general form, is

$$\ddot{q} + \alpha_1 q + \alpha_2 q^3 = 0 \tag{13}$$

where q is the generalized amplitude of vibration, α_1 and α_2 are constants that depend on the boundary conditions of the beam and ($\dot{}$) denotes differentiation with respect to the time. Eq.(13) can be easily be solved by assuming the SHM assumption, given by

$$\ddot{q} = -\omega^2{}_{NL_H}\, q \tag{14}$$

Substituting Eq. (14) in Eq.(13) we get

$$-\omega^2{}_{NL_H} + \alpha_1 + \alpha_2 q^2 = 0 \tag{15}$$

Eq.(15) can be written, by neglecting the nonlinear term (third term), as

$$\omega^2{}_L = \alpha_1 \tag{16}$$

which is the linear radian frequency of the beam executing SHM. After dividing with $\omega_L{}^2$ or α_1 appropriately Eq.(15), after simplification, becomes

$$\frac{\omega^2{}_{NL_H}}{\omega^2{}_L} = 1 + \frac{\alpha_2}{\alpha_1} q^2 \tag{17}$$

The ratios of the nonlinear to linear radian frequencies can be obtained with reference to the maximum generalized amplitude q_m, which can also be denoted by the physical quantity b/r. It may be noted here that the value of $\gamma = \alpha_2 / \alpha_1$ depends on the boundary conditions of the beam and for the Fixed--hinged beam the value γ is 0.1267.

IV.CORRECTION FACTOR TO ELIMINATE THE SHM ASSUMPTION

The error induced in Eq.(17) due to the SHM assumption is eliminated by obtaining a more accurate solution of Eq. (13) by writing q as

$$q = q_m \sin \omega_{NL} t \tag{18}$$

where q_m is the maximum amplitude of vibration and ω_{NL} is the nonlinear radian frequency to be obtained without the assumption of the SHM.

Substituting Eq. (18) in Eq. (13), we get

$$\omega^2{}_{NL} q_m \sin \omega_{NL} t = \alpha_1 q_m \sin \omega_{NL} t + \alpha_2 q_m^3 \sin^3 \omega_{NL} t \tag{19}$$

Equation (19) does not exhibit the SHM and if the nonlinear term in Eq. (19) is neglected, the linear frequency will be obtained again as given in Eq. (16). By using the relations

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta \tag{20}$$

r

$\sin^3\theta$ =

b/r	Unsymmetric beam configuration	
	Fixed-Hinged beam	
	Present study	Gajbir *et al* [9]
0.0	1.0000	1.0000
0.2	1.0019	1.0019
0.4	1.0076	1.0077
0.6	1.0172	1.0172
0.8	1.0304	1.0304
1.0	1.0471	1.0471
2.0	1.1772	1.1758
3.0	1.3668	1.3615
4.0	1.5948	1.5838
5.0	1.8471	1.8293

$$\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta \quad (21)$$

and if the higher harmonic in Eq.(21) is neglected, as the interest is on the first harmonic, Eq.(21) can be approximated by

$$\sin^3\theta \approx \frac{3}{4}\sin\theta \quad (22)$$

From Eqs. (22) and (19) replacing θ by $\omega_{NL} t$, the ratio $\frac{\omega_{NL}}{\omega_L}$ can be obtained as

$$\frac{\omega^2{}_{NL}}{\omega_L^2} = 1 + \frac{3}{4}\gamma\, q_m^2 \quad (23)$$

Comparison of Eqs. (17) and (23) shows that the expressions for the ratios of the nonlinear to radian frequencies with or without the assumption of the SHM, differ by a factor 3/4 in the nonlinear part, and this factor is used to obtain a more accurate expression for $\omega^2{}_{NL} / \omega^2{}_L$.

Table.1.Non linear frequency parameter to the linear frequency parameter ratios for different *b/r* ratios

For the hinged-hinged beam considered in this section, Eq. (23) becomes

$$\frac{\omega^2{}_{NL}}{\omega_L^2} = 1 + 0.09509 q_m^2 \quad (24)$$

Equation(24) obtained using the HBM, though more accurate, still contains a small error because of neglecting the third harmonic in the final solution and this error is tolerable for all practical purposes.

V.NUMERICAL RESULTS AND DISCUSSION

The simplified continuum formulation presented in this paper has been applied to study the large amplitude vibration behavior (fundamental mode) of widely used slender, uniform fixed-hinged beam (Fig.1).. The quantitative effect of the geometric nonlinearity can be decided from the values of γ obtained for the different boundary conditions of the beams considered. Note that the value of γ is evaluated at the point where the lateral deflection *w* is the maximum. Obviously, this happens to be the central point of the beam for SS and CC boundary conditions. The assumed *w* functions in Table1 are normalized with respect to the value of the maximum deflection so that the function value becomes unity. With this normalization only, it can be inferred that the more the value of γ the more the geometric nonlinearity for a given beam boundary condition and *b/r* ratio.

VI.CONCLUSIONS

The effectiveness of the simplified continuum formulation proposed to study the large amplitude vibration (fundamental mode) of slender uniform beam, where the inplane displacement distribution is derived from the assumed admissible function for the lateral deflection is shown in this paper. As the inplane displacement distribution is obtained from a single term lateral deflection, only one of undetermined coefficient exists and thus the present formulation is much simpler. Application of the proposed formulation gives consistently accurate results when compared with those obtained by the FE analysis for the beam configurations. The method proposed here is general and can be applied to beam problems with other boundry conditions.

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