

On Fuzzy Ideal Based Fuzzy Zero Divisor Graphs On Γ -near ring

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Abstract

In a manner analogous to a Γ -near ring M , the fuzzy ideal based fuzzy zero divisor graph of a gamma near ring M can be defined as the undirected graph for some fuzzy ideal of M . The basic properties and possible structures of the graph are studied.

Keywords

Zero-divisor, Zero-divisor graph, gamma near ring, nilpotent.

1.Introduction

Research on the theory of fuzzy sets has been witnessing an exponential growth, both within Mathematics and in its application. Zadeh introduced the notation of fuzzy subset μ of a non-empty set X as a function from X to $[0, 1]$. Liu introduced and examined the notion of a fuzzy ideal of a Γ -near ring. Rosenfeld considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975. Among them most interesting graphs are the zero divisor graphs, because they involve both Γ -near ring theory and graph theory. Let M be Γ -near ring with identity and μ an fuzzy ideal of M . In the present paper, we introduce and investigate the fuzzy ideal based fuzzy zero divisor graph of M , denoted by $G(\mu)$. Throughout this paper we shall assume unless otherwise stated, that μ is not a non zero constant.

Thus there is a non-zero element y of M such that $\mu(y) \neq \mu(0)$.

2.Preliminaries

Definition 2.1: Let $(M, +)$ be a group and Γ be a non empty set. Then M is said to be a Γ -near-ring, if there exist a mapping $M \times \Gamma \times M \rightarrow M$ (The image of (x, α, y) is denoted by $(x \alpha y)$ satisfied the following conditions: (i) $(x + y) \alpha z = x \alpha z + y \alpha z$, (ii) $(x \alpha y) \beta z = x \alpha (y \beta z)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Definition 2.2 : A Fuzzy Γ -near ring is a function $\mu: M \rightarrow [0, 1]$, where $(M, +, \Gamma)$ is a Γ -near ring, that satisfies:

- i. $\mu \neq 0$
- ii. $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ for every x, y in M
- iii. $\mu(x \alpha y) \geq \min\{\mu(x), \mu(y)\}$ for every x, y in M

Definition 2.3: Let $\mu: M \rightarrow [0, 1]$. Then μ is said to be a fuzzy ideal of M if it satisfies the following condition :

- i. $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$
- ii. $\mu(-x) = \mu(x)$
- iii. $\mu(x) = \mu(y + x - y)$
- iv. $\mu(x\alpha y) \geq \mu(x)$
- v. $\mu(x\alpha(y + z) - x\alpha y) \geq \mu(z)$ for all $x, y, z \in M$ and $\alpha \in \Gamma$

The set of all fuzzy ideal of M is denoted by $FI(M)$.

Theorem 2.4: Let M be a Γ -near ring and $\mu \in FI(M)$. Then $\mu(x) \leq \mu(0)$ and $\mu(1) \leq \mu(x)$ for every x in M .

Proof:

The result is directly follows from definition.

Theorem 2.5: If M is commutative Γ -near ring & $\mu \in FI$, then $\mu(x\alpha y) \geq \max\{\mu(x), \mu(y)\}$, $\alpha \in \Gamma$.

Proof:

Since M is commutative, $x\alpha y = y\alpha x$
 $\Rightarrow \mu(x\alpha y) = \mu(y\alpha x)$. Since $\mu \in FI$, \Rightarrow
 $\mu(x\alpha y) \geq \mu(x)$ and
 $\mu(x\alpha y) = \mu(y\alpha x) \geq \mu(y)$
 $\Rightarrow \mu(x\alpha y) \geq \max\{\mu(x), \mu(y)\}$, $\alpha \in \Gamma$.

Definition 2.6: Let M be a commutative Γ -near-ring with I and let $Z(M)$ be its set of zero-divisors. The zero-divisor graph of M , denoted by $\Gamma(M)$ is the (undirected graph) with vertices $Z(M)^* = Z(M) \setminus \{0\}$, the set of non zero zero-divisors of M and for distinct $x, y \in Z(M)^*$, the vertices x and y are adjacent iff $x\gamma y = 0$ and $y\gamma x = 0$ for all $\gamma \in \Gamma$.

Definition 2.7: Let M be a Γ -near ring and $\mu \in FI(M)$. A μ zero divisor is an element $x \in M$ for which there exists $y \in M$ with $\mu(y) \neq \mu(0)$ such that $\mu(x\alpha y) = \mu(0)$. The set of μ -zero divisors in M will be denoted by $Z(\mu)$.

3.Fuzzy Zero Divisor Graph On Γ -Near Ring

Definition 3.1: Let M be a Γ -near ring and $\mu \in FI(M)$. We define a undirected Graph $G(\mu)$ with vertices $V(G(\mu)) = Z(\mu)^* = Z(\mu) - \mu^* = \{x \in Z(\mu) : \mu(x) \neq \mu(0)\}$, where distinct vertices x and y are adjacent if and only if $\mu(x\alpha y) = \mu(0)$, where $\mu^* = \{x \in M : \mu(x) = \mu(0)\}$

Remark 3.2: Let M be a Γ -near ring and $\mu \in FI(M)$. Clearly, if μ is a non zero constant, then $\Gamma(\mu) = \emptyset$.

Definition 3.3: Let M be a Γ -near ring and $\mu \in FI(M)$. We say μ is an F -integral domain if $Z(\mu) = \mu^*$.

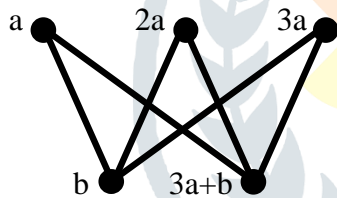
Definition 3.4: Let M be a Γ -near ring and in $\mu \in FI(M)$. An element $a \in M$ is said to be μ -nilpotent precisely when there exists a positive integer n such that $\mu(a^n) = \mu(0)$. The set of all μ -nilpotent of M is denoted by $\text{nil}(\mu)$ and we set $\text{nil}(\mu)^* = \text{nil}(\mu) - \mu^*$.

Example 3.5: In a Dihedral Group for the scheme $(10, 1, 1, 1, 10, 1, 1, 10)$ the α -table is given below. Take this Dihedral group as M (Γ -near ring).

α	0	a	2a	3a	b	a+b	2a+b	3a+b
0	0	0	0	0	0	0	0	0
a	0	a	a	a	0	a	a	0
2a	0	2a	2a	2a	0	2a	2a	0
3a	0	3a	3a	3a	0	3a	3a	0
b	b	b	b	b	b	b	b	b
a+b	b	a+b	a+b	a+b	b	a+b	a+b	b
2a+b	b	2a+b	2a+b	2a+b	b	2a+b	2a+b	b
3a+b	b	3a+b	3a+b	3a+b	b	3a+b	3a+b	b

Define $\mu: M \rightarrow [0,1]$ by $\mu(x) = \begin{cases} \frac{1}{2} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0 \end{cases}$

The corresponding fuzzy zero divisor graph is



Definition 3.6: Let M be a Γ -near ring and $\mu \in FI(M)$. We define a undirected graph $G_N(\mu)$ with vertices $V(G_N) = \text{nil}(\mu)^*$, where distinct vertices x and y are adjacent if and only if $\mu((x\alpha y)^n) = \mu(0)$, for some positive integer n , where $\mu^* = \{x \in M : \mu(x) = \mu(0)\}$

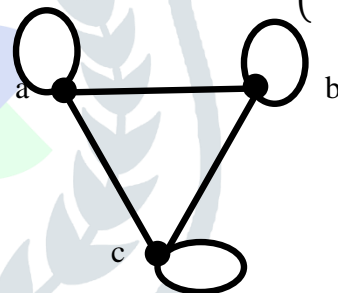
Remark 3.7: Assume that M is a Γ -near ring and $\mu \in FI(M)$. Let $\mu(x^n) = \mu(0)$ for some positive integer n . By Remark 3.2, there exists $0 \neq y \in M$ such that $\mu(y) \neq \mu(0)$. $\mu(x\alpha y) \geq \mu(x^n) = \mu(0)$ and by result. Hence $x^n \in Z(\mu)$. In particular, if $\mu(x) =$

$\mu(0)$, we conclude that $x \in Z(\mu)$. Moreover, $\mu(x^{n+k}) \geq \mu(x^n) = \mu(0)$. Hence $\mu(x^{n+k}) = \mu(0)$ for every positive integer k .

Example 3.8 : In a dihedral group for the scheme (10,10,10,10,1,1,10)

α	0	a	2a	3a	b	a+b	2a+b	3a+b
0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	a	a	0
2a	0	0	0	0	0	2a	2a	0
3a	0	0	0	0	0	3a	3a	0
b	b	b	b	b	b	b	b	b
a+b	b	b	b	b	b	a+b	a+b	b
2a+b	b	b	b	b	b	2a+b	2a+b	b
3a+b	b	b	b	b	b	3a+b	3a+b	b

Define $\mu: M \rightarrow [0,1]$ by $\mu(x) = \begin{cases} \frac{1}{2} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0 \end{cases}$



The graph of nilpotent elements.

Theorem 3.9: Let M be a commutative Γ -near ring and $\mu \in FI(M)$. Then the following hold:

- If $x \in \text{nil}(\mu)^*$ and $y \in Z(\mu)^*$, then $d_\mu(x,y) \leq 2$ in $G(\mu)$.
- Let $x \in Z(\mu) - \text{nil}(\mu)$, and $y \in \text{nil}(\mu)^*$ such that $x|z\alpha y^n$ for some positive integer n and $z \in M - Z(\mu)$. Then $d_\mu(x,y) \leq 2$ in $G(\mu)$.

Proof:

- Assume that $x \neq y$ and $\mu(x\alpha y) \neq \mu(0)$. Since $y \in Z(\mu)^*$ and $\mu(x\alpha y) \neq \mu(0)$,

there is a $z \in Z(\mu)^* - \{x\}$ such that

$\mu(zay) = \mu(0)$. Let n be the least positive integer such that $\mu(x^n az) = \mu(0)$ since x in $\text{nil}(\mu)^*$. Thus $d_\mu(x, y) \leq 2$. If $n=1$,

then $x - z - y$ is a path between x and y .

- ii. Assume that $x \neq y$ and $\mu(xay) \neq \mu(0)$. Since $x \in Z(\mu) - \text{nil}(\mu)$ and $\mu(xay) \neq \mu(0)$, there is a $w \in Z(\mu)^* - \{x, y\}$ such that $\mu(xaw) = \mu(0)$. Since $x|zay^n$ with $\mu(z) \neq \mu(0)$ (if not, $z \in \mu^* \subseteq Z(\mu)$ which is a contradiction and $\mu(xaw) = \mu(0)$, we get $\mu(zay^n aw) = \mu(0)$. If $\mu(y^n aw) \neq \mu(0)$, then $z \in Z(\mu)$ which is a contradiction. So we conclude $\mu(y^n aw) = \mu(0)$. Let m be the positive integer such that $\mu(way^n) = \mu(0)$. Let $m=1$, then $x - w - y$ is a path of length 2 from x to y . For $m \geq 2$, then $x - y^{m-1}aw - y$ is a path between x and y . Thus $d_\mu(x, y) \leq 2$ in $G(\mu)$.

Definition 3.10: Let M be a Γ -near ring and $\mu \in \text{FI}(M)$. A μ -unit of M is an element $a \in M$ if $\mu(a) \neq \mu(0)$ and for which there exist b in M such that $\mu(a\alpha b) = \mu(1)$. The set of μ -units of M is denoted by $U(\mu)$.

Example 3.11: Let M denote the Γ -near ring of integers modulo 5, for scheme $(0, 1, 4, 1, 4)$, the α -table is given by

α	0	1	2	3	4
0	0	0	0	0	0
1	0	1	4	1	4
2	0	2	3	2	3
3	0	3	2	3	2
4	0	4	1	4	2

Define $\mu: M \rightarrow [0, 1]$ by $\mu(x) = \begin{cases} \frac{1}{2} & \text{if } x \neq 0, 1 \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \end{cases}$

Therefore, $U(\mu) = \{1, 4\}$.

Theorem 3.12: Let M be a commutative Γ -near ring and $\mu \in \text{FI}(M)$. $V(G(\mu)) - \text{nil}(\mu)$ is totally disconnected if and only if $\text{nil}(\mu)$ is a prime ideal of M .

Proof:

Suppose that $V(G(\mu)) - \text{nil}(\mu)$ is totally disconnected. Let $x, y \notin \text{nil}(\mu)$ such that $xay \in \text{nil}(\mu)$. So there exists a positive integer n such that $\mu(x^n ay^n) = \mu(0)$. If $\mu(x^n) = \mu(0)$, then $x \in \text{nil}(\mu)$ which is a contradiction. So we may assume that $\mu(x^n) \neq \mu(0)$ and $\mu(y^n) \neq \mu(0)$. If $x^n = y^n$, then $\mu(x^{2n}) = \mu(0)$ thus $x \in \text{nil}(\mu)$, a contradiction. So we may assume that $x^n \neq y^n$. Thus $x^n, y^n \in V(G(\mu)) - \text{nil}(\mu)$ and $x^n - y^n$ is a path from x^n to y^n in $G(\mu)$ and this is a contradiction. Thus $xay \notin \text{nil}(\mu)$ and $\text{nil}(\mu)$ is a prime ideal of M . Conversely, assume that $\text{nil}(\mu)$ is a prime ideal of M . Let x and y be distinct elements of $V(G(\mu)) - \text{nil}(\mu)$. Suppose that $\mu(xay) = \mu(0)$. Then $xay \in \text{nil}(\mu)$. Hence either x or y belong to $\text{nil}(\mu)$, which is a contradiction. Hence the Proof.

Theorem 3.13: Let M be a commutative Γ -near ring and $\mu \in \text{FI}(M)$. Then $G(\mu)$ is connected with $\text{diam}(G(\mu)) \leq 3$.

Proof:

Let x and y be distinct vertices of $G(\mu)$. We split it into 5 cases.

Case 1: $\mu(x\alpha y) = \mu(0)$. Then x - y is a path in

$G(\mu)$.

Case 2: $\mu(x\alpha y) \neq \mu(0)$, $\mu(x^2) = \mu(0)$ and $\mu(y^2) = \mu(0)$. Then $\mu(x\alpha(x\alpha y)) = \mu(x^2\alpha y) \geq \max\{\mu(x^2), \mu(y)\} = \max\{\mu(0), \mu(y)\} = \mu(0)$. Thus $\mu(x\alpha(x\alpha y)) = \mu(0)$. Similarly $\mu(y\alpha(x\alpha y)) = \mu(0)$. Then x - $x\alpha y$ - y is a path in $G(\mu)$.

Case 3: $\mu(x\alpha y) \neq \mu(0)$, $\mu(x^2) = \mu(0)$ and $\mu(y^2) \neq \mu(0)$. Then there is an element $b \in Z(\mu)^* - \{x, y\}$ with $\mu(y\alpha b) = \mu(0)$. If $\mu(x\alpha b) = \mu(0)$, then x - b - y is a path between x and y . If $\mu(x\alpha b) \neq \mu(0)$, then x - $x\alpha b$ - y is a path.

Case 4: $\mu(x\alpha y) \neq \mu(0)$, $\mu(x^2) \neq \mu(0)$ and $\mu(x^2) = \mu(0)$. By case 3, the result follows.

Case 5: $\mu(x\alpha y) \neq \mu(0)$, $\mu(x^2) \neq \mu(0)$ and $\mu(y^2) \neq \mu(0)$. Then there are $a, b \in Z(\mu)^* - \{x, y\}$ with $\mu(x\alpha a) = \mu(0) = \mu(y\alpha b)$. If $a=b$, then x - a - y is a path. If $a \neq b$ and $\mu(a\alpha b) \neq \mu(0)$, then x - $a\alpha b$ - y is a path. If $a \neq b$ and $\mu(a\alpha b) = \mu(0)$, then x - a - b - y is a path.

Thus $G(\mu)$ is connected and $\text{diam}(G(\mu)) \leq 3$.

Theorem 3.14: Let M be a commutative Γ -near ring and $\mu \in \text{FI}(M)$. If $\Gamma(\mu)$ contains a cycle, then $\text{gr}(\Gamma(\mu)) \leq 4$.

Proof:

Suppose not, Assume that $\Gamma(\mu)$ contains a cycle x_0 - x_1 -...- x_n - x_0 such that $\text{gr}(\Gamma(\mu)) > 4$.

$\mu(x_i\alpha x_j) \neq \mu(0)$ for $i, j \in \{0, 1, \dots, n\}$ with $|i-j| \geq 2$ and $\mu(x_i\alpha x_{i+1}) = \mu(0)$.

Case 1: $x_1\alpha x_{n-1} \neq x_0$ and $x_1\alpha x_{n-1} \neq x_n$. Then $\mu(x_0\alpha x_n) = \mu(0)$ and $\mu(x_1\alpha x_{n-1}) \neq \mu(0)$. Since $|n-2| \geq 2$, and we have

$$\mu(x_0\alpha x_1\alpha x_{n-1}) \geq \max\{\mu(x_0\alpha x_1), \mu(x_{n-1})\}$$

$$= \max\{\mu(0), \mu(x_{n-1})\} = \mu(0)$$

Thus $\mu(x_0\alpha x_1\alpha x_{n-1}) = \mu(0)$.

Similarly $\mu(x_1\alpha x_{n-1}\alpha x_n) = \mu(0)$. We get cycle of length 3 (x_0 - $x_1\alpha x_{n-1}$ - x_n - x_0).

Case 2: $x_1\alpha x_{n-1} = x_0$. Since $\mu(x_0^2) = \mu(x_0\alpha x_1\alpha x_{n-1}) \geq \max\{\mu(0), \mu(x_{n-1})\} = \mu(0)$.

We claim that there is an element y of M such that $\mu(x_0\alpha y) \neq \mu(0)$ and $x_0\alpha y \neq x_0$.

Suppose not, then for every $y \in M$, either $\mu(x_0\alpha y) = \mu(0)$ or $x_0\alpha y = x_0$. Take $y = x_3$. By assumption, $\mu(x_0\alpha x_3) \neq \mu(0)$ and $x_0\alpha x_3 \neq x_0$, which is a contradiction. So there is an element y of M such that $\mu(x_0\alpha y) \neq \mu(0)$ and $x_0\alpha y \neq x_0$. If $x_0\alpha y \neq x_1$, then $\mu(x_0\alpha x_1\alpha y) \geq \max\{\mu(0), \mu(y)\} = \mu(0)$. Thus x_0 - x_n - $x_0\alpha y$ - x_0 is a 3-cycle in $\Gamma(\mu)$, $\alpha \in \Gamma$.

Case 3: $x_1\alpha x_{n-1} = x_n$. Then necessarily $\mu(x_n) = \mu(0)$ and there exists an element $y \in M$ such that $\mu(x_n\alpha y) \neq \mu(0)$ and $x_n\alpha y \neq x_n$. If $x_n\alpha y = x_n$ leads to a contradiction. Thus x_n - $x_n\alpha y$ - x_{n-1} - x_n is a cycle of length. This completes the proof.

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