On S - Near Rings and S' - Near Rings with Right Bipotency

[1] D.Radha, [2] C.Dhivya

[1] PG and Research Department of Mathematics

A.P.C Mahalaxmi College for Women, Thoothukudi,

Tamil Nadu, India.

[2] Research Scholar

A.P.C Mahalaxmi College for Women, Thoothukudi,

Tamil Nadu, India.

Abstract:-- A **right near ring** $(N,+,\cdot)$ is an algebraic system with two binary operations such that (i) $(N,+)$ is a group - (not necessarily abelian) with 0 as its identity element, (ii) $(N,\cdot)$ is a semigroup (we write xy for $x\cdot y$ for all x,y in N) and (iii) $(x+y)z = xz+yz$ for all x,y,z in N. We say that N is **zero symmetric** if $n0 = 0$ for all n in N. N is called an $\boldsymbol{S}$ **- near ring** or an $\boldsymbol{S'}$ **- near ring** according as $x \in Nx$ or $x \in xN$ for all $x \in N$. A subgroup M of N is called an **N-subgroup** if $NM \subset M$ and an **invariant N-subgroup** if, in addition, $MN \subset M$. An element a in N is said to be **distributive**, if $a(b+c) = ab+ac$ for all b and c in N; N is called **distributively generated (d.g.),** if the additive group of N is generated by the multiplicative semigroup of distributive elements of N.

A near ring N is defined to be **right bipotent** if $aN = a^2N$ for each a in N. In this paper, we have proved some more results on right bipotent near rings by using the concepts of S' - near ring ; subcommutativity ; regularity ; reduced property etc. It is proved that every right bipotent near ring is an S' - near ring and it is also S - near ring if it is also subcommutative. Every regular near ring is central and reduced if it is right bipotent. Some special characterizations are obtained in such a way that, a reduced right bipotent near ring is a near field if $N = N_d$ and it is a division ring if it is dgnr.

Keywords:-- S near ring, S'- near ring, near field, right bipotent near ring, subcommutative, nilpotent, right N - subgroup, zero divisors, regular near ring, division ring, distributively generated near ring.

1.Introduction

Near rings can be thought of as generalized rings: if in a ring we ignore the commutativity of addition and one distributive law, we get a near ring. Taussky [24] in 1936 and B.H.Neumann [13] in 1940 considered near rings in which addition need not be commutative. Since then the theory of near rings has been developed much. Later Frolich [6], Beidleman [2], Oswald [14] and many other researchers had done and have been doing extensive work on different aspects of near rings. Gunter Pilz [5] "Near rings" is an extensive collection of the work done in the area of near rings.

A near ring N is defined to be **left bipotent** if $Na = Na^2$ for each a in N. The definitions for S - Near ring and S' - Near ring are dealt in P(r,m) Near rings by R. Balakrishnan and S. Suryanarayanan in [1].

2.Preliminaries

Definition 2.1 [9]

N is said to be **subcommutative**, if $aN = Na$ for all $a \in N$.

Definition 2.2 [5]

An element $n \in N$ is called **nilpotent** if $n^k = 0$ for some positive integer k.

Definition 2.3 [8]

A near ring N is **regular** if for each a in N, there exists x in N such that $a = axa$.

Definition 2.4

An element e in N is called **idempotent** if $e^2 = e$.

Definition 2.5 [5]

An idempotent a in N is called a **central** if $ax = xa$ for all x in N.

Definition 2.6 [5]

Let $(P, +)$ be a group with 0 and let N be a near ring. Let $\mu: N \times P \to P$; (P, μ) is called an ***N*-group** if for all $p \in P$ and for all $n, n_1 \in N$ we have $(n + n_1)p = np + n_1p$ and $(nn_1)p = n(n_1p)$. N^P stands for N-groups.

Definition 2.7 [5]

A subgroup S of N^P with $NS \subset S$ is a ***N*-subgroup** of P.

Definition 2.8 [8]

An additive group A of N is called a **left *N*-subgroup** if $NA \subseteq A$ where $NA = \{ra / r \in N, a \in A\}$.

Definition 2.9 [8]

An additive group A of N is called a **right *N*-subgroup** if $AN \subseteq A$ where $AN = \{ar / r \in N, a \in A\}$.

Definition 2.10 [8]

For any subset A of a near ring N, Define $\sqrt{A} = \{x \in N / x^n \in A, for\ some\ n\}$.

Definition 2.11 [12]

An element $0 \neq x \in N$ is called a **right zero divisor** if $\exists\ 0 \neq a \in N$ such that $ax = 0$

Definition 2.12 [12]

An element $0 \neq x \in N$ is called a **left zero divisor** if $\exists\ 0 \neq a \in N$ such that $xa = 0$.

Definition 2.13 [5]

If all non zero elements of N are left (right) cancelable, we say that N fulfills the **left (right) cancellation law**.

Definition 2.14 [8]

N is called a **near-field** if it contains an identity and each non zero element has a multiplicative inverse.

Notation 2.15 [5]

Let $N_d = \{d \in N | d \text{ is distributive}\}$

Definition 2.16 [5]

If $N = N_d$, N is said to be distributive.

Definition 2.17 [1]

N is called an ***S* - near ring** according as $x \in Nx$ for all $x \in N$.

Definition 2.18 [1]

N is called an ***S'* - near ring** according as $x \in xN$ for all $x \in N$.

Definition 2.19 [25]

A near ring N is defined to be **right bipotent** if $aN = a^2N$ for each a in N.

3.Main Results

Theorem 3.1

Every Right Bipotent near ring is an S'- near ring.

Proof:

Let N be right bipotent. This implies $a^2N = aN$. Therefore $a \in a^2N = aN$. This implies $a \in aN$. Hence N is S'- near ring.

Corollary 3.2

Every S- near ring is S' - near ring if it is subcommutative with vice versa.

Proof:

Let N be S - near ring.

Then, $x \in Nx = xN$ for all x in N. This implies $x \in xN$. Hence N is S'- near ring.

Converse follows.

Result 3.3

Any right bipotent subcommutative near ring is an S - near ring.

Theorem 3.4

Homomorphic images of right bipotent S' - near rings are also such.

Proof:

Let $f: N \to N'$ be a homomorphism of near rings N onto N', and let N be a right bipotent S'- near ring. If $a \in N'$, there exists $b \in N$ such that $f(b) = a$. By assumption, we have $bN = b^2N$. Then $f(bN) = f(b)f(N) = aN'$ and $f(b^2N) = f(b^2)f(N) = [f(b)]^2 f(N) = a^2N'$. Thus $bN' = b^2N'$. Now since $b \in bN$, we have $a = f(b) \in f(bN) = aN'$.

Theorem 3.5

A regular near ring N is right bipotent if each idempotent in N is central.

Proof:

N is regular, so far a in N, there exists x in N such that $a = axa$. Let $ax = e$. Now, $(ax)^2 = (ax)(ax) = (axa)x = ax$. Therefore ax is an idempotent. Now $a = axa = aax$ (since idempotents are central) $= a^2x$. Hence $aN = a^2N$ and N is right bipotent.

Theorem 3.6

Let N be an S' - near ring, then N is regular iff for each $a\,(\neq 0)$ in N, there exists an idempotent e such that $aN = eN$.

Proof:

If N is a regular near ring, then for every a in N, there exists x in N such that $a = axa$

Let $ax = e$. Now, $(ax)^2 = (ax)(ax) = (axa)a = ax = e$. (i.e) $e^2 = e$. Therefore e is an idempotent and $aN = eN$. (for $aN = axaN \subseteq axN = eN \subseteq aN$). Conversely, Let N be an S' - near ring satisfying the given condition. For any $d \in N$, there exists an idempotent b such that $d \in dN = bN$. This implies $d = bu$ for some u in N. Also $b \in bN = dN$. This implies $b = dy$ for some y in N. Now $dyd = dybu = bbu = b^2u = bu = d$. Therefore $dyd = d$. Hence N is a regular near ring.

Theorem 3.7

A right bipotent near ring N is regular iff it is an S' - near ring.

Proof:

Let N be regular near ring. This implies for each a in N, there exists x in N such that $a = axa$. Let $ax = e$. Now $(ax)^2 = (ax)(ax) = (axa)x = ax$. Therefore ax is an idempotent. Now $a = axa = aax = a^2x \in a^2N = aN$. This implies $a \in aN$. Therefore every regular near ring is an S' - near ring. Conversely, Let N be a right bipotent S' - near ring. Then for each

a in N, $a \in aN = a^2N$ and so $a^2 = a^4z$ for some z in N. This implies $a^2a^2 = a^4za^2$. This gives $(a^2 - a^2za^2)a^2 = 0$ and $(a^2 - a^2za^2)a^2za^2 = 0$. $(a^2 - a^2za^2)^2$ $= (a^2 - a^2za^2)(a^2 - a^2za^2) =$ $(a^2 - a^2za^2)a^2 - (a^2 - a^2za^2)a^2za^2 = 0$. Therefore $(a^2 - a^2za^2)^2 = 0$. From this we get $a^2 - a^2za^2 = 0$. Hence $a^2 = a^2za^2$. Let $a^2z = e$. Now, $(a^2z)^2 = (a^2z)(a^2z) = (a^2za^2)z = a^2z$. Therefore a^2z is an idempotent and $aN = a^2N = a^2za^2N \subseteq a^2zN = eN \subseteq a^2N = aN$. Hence by Theorem 3.6, N is regular.

Theorem 3.8

A right bipotent near ring is an S' - near ring iff it has no non zero nilpotent elements.

Proof:

Let N be a right bipotent S' - near ring. Let $b \in N$ be nilpotent. For some positive n, $b^n = 0$. Then $b \in bN = b^2N = \cdots = b^nN$ and $b = 0$. Conversely, Let N be right bipotent with no non zero nilpotent elements. If $x \in N$, then $xN = x^2N$ so $x^2 = x^2y$ for some y in N. This implies $x^2 - x^2y = 0$. This gives $(x - xy)x = 0$. Also $(x - xy)xy = 0$. Now, $(x - xy)^2 =$ $(x - xy)(x - xy) = (x - xy)x - (x - x$ $y)xy = 0$. Hence $(x - xy)^2 = 0$. This implies $x - xy = 0$. This gives $x = xy$. Therefore $x \in xN$. Hence N is an S' - near ring.

Corollary 3.9

A right bipotent near ring is regular iff it is reduced.

Proof: Follows by Theorems 3.7 and 3.8

Theorem 3.10

An S' - near ring is right bipotent iff $A = \sqrt{A}$ for every right N-subgroup A of N.

Proof:

Clearly $A \subseteq \sqrt{A}$. Now let $a \in \sqrt{A}$, then $a^n \in A$ for some n. Also we have $aN = a^2N = \cdots = a^nN$ in a right bipotent near ring. Since N is an S' - near ring, $a \in aN = a^nN$. This gives $a = a^nb$ for some b in N. Thus $a \in A$, (since $a^n \in A$ and A is a right N-subgroup of N). Hence $\sqrt{A} \subseteq A$. Conversely, we have to prove that if N is an S' - near ring with the condition $A = \sqrt{A}$ for every right N-subgroup A of N then N is right bipotent. For $a \in N$, $a^3 \in a^2N$ and $a \in \sqrt{a^2N} = a^2N$. Then $aN \subseteq a^2N \subseteq aN$ and N is right bipotent.

Theorem 3.11

Let N be a right bipotent near ring with no zero divisors. If N has a non zero distributive element, then N is a near field.

Proof:

N is regular. Let d be a non zero distributive element in N, then there exists x in N such that $d = dxd$. Let $dx = e$. Now, $(dx)^2 = (dx)(dx) = (dxd)x = dx$. Therefore dx is an idempotent. If r is any element in N, then $r(d - dxd) = 0$. This implies $r(d - ed) = 0$. This gives $r - re = 0$ (since d is a distributive element). From this, we $r = re$. That is, e is a right identity in N. If $a \in N$ with $a \neq 0$ then $aN = a^2N$. Therefore, $ae = a^2y$ for some y in N. This implies $a(e - ay) = 0$. This gives $e - ay = 0$ (since $a \neq 0$). From this, we get $e = ay$. That is, y is a right inverse of a. Hence N is a near field.

Corollary 3.12

Let N be a right bipotent distributively generated (d.g.) near ring with no zero divisors then N is a division ring.

Proof:

By Theorem 3.11, N is a near field and so $(N,+)$ is abelian (see(6)). Moreover, a d.g. near ring with $(N,+)$ abelian is a ring (13). Therefore, N is a division ring.

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