M-open Sets in Bitopological Spaces

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Abstract
The aim of this paper is to introduce and investigate the concept of $\tau_i\tau_j$-Mclosed sets which are introduced in a bitopological space in analogy with Mclosed sets in topological spaces. Also Mclosure and Minterior operators in bitopological spaces are introduced. In addition, several properties of these notions and connections to several other known ones are provided.

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1 Introduction and Preliminaries

Levine in 1963 initiated a new types of open set called semiopen set [8]. A subset A of a space (X,τ) is called regular open (resp., regular closed) [10] if $A = int(cl(A))$ (resp., $A = cl(int(A))$. The delta interior [3] of a subset A of (X,τ) is the union of all regular open sets of X contained in A and is denoted by $\delta int(A)$. A subset A of a space (X,τ) is called δ-open [9] if $A = \delta int(A)$. The complement of δ-open set is called δ-closed. Alternatively, a set A of (X,τ) is called δ-closed [3] if $A = \delta cl(A)$, where $\delta cl(A) = \{x \in X: A \cap int(cl(U)) \neq \phi,\ U \in \tau$ and $x \in U\}$. A subset A of a space X is called θ-open [1] if $A = \theta int(A)$, where $\theta int(A) = \cup\ \{int(U): U \subseteq A,\ U \in \tau^c\}$, and a subset A is called θ-semiopen [2] (resp., δ- preopen [9] , e-open [4] and M-open [5]) if $A \subseteq cl(\theta int(A))$ (resp., $A \subseteq int(\delta cl(A))$, $A \subseteq cl(\delta int(A)) \cup int(\delta cl(A))$ and $A \subseteq cl(\theta int(A)) \cup int(\delta cl(A)))$, where $int()$, $cl()$, $\theta int()$, $\delta int()$ and $\delta cl()$ are the interior, closure, θ-interior, δ-interior and δ-closure operations, respectively. The notion of bitopological spaces (in short, Bts's) was first introduced by kelly [6].
Through out this paper, Let (X,τ_1,τ_2) or simply X be a Bts and $i,j \in \{1,2\}$. A subset S of a Bts X is said to be $\tau_{1,2}$-open [7] if $S = A \cup B$ where $A \in \tau_1$ and $B \in \tau_2$. A subset S of X is said to be $\tau_{1,2}$-closed if the complement of S is $\tau_{1,2}$-open. and $\tau_{1,2}$-clopen if S is both $\tau_{1,2}$-open and $\tau_{1,2}$-closed. For a subset A of X, the interior (resp., closure) of A with respect to τ_i will be denoted by $int_i(A)$ (resp., $cl_i(A)$) for $i = 1,2$. In this paper, we introduce and investigate the concept of $\tau_i\tau_j$-Mclosed sets which are introduced in a bitopological spaces in analogy with Mclosed sets in topological spaces. Also introduce Mclosure and Minterior operators in bitopological spaces. In addition, several properties of these notions and connections to several other known ones are provided.

2 M-open sets and their properties in bitopological spaces

Definition 2.1 *Let (X,τ_1,τ_2) be a Bts. A subset A of X is called $\tau_i\tau_j$-M-open (briefly, $\tau_i\tau_j$-M-o) if $A \subseteq cl_j(\theta int_i(A)) \cup int_i(\delta cl_j(A))$ and A is $\tau_i\tau_j$-Mclosed (in short, $\tau_i\tau_j$-M-c) if $X\backslash A$ is $\tau_i\tau_j$-M-o. A is pairwise M-open if it is both $\tau_i\tau_j$-M-o and $\tau_j\tau_i$-M-o.* Clearly A is $\tau_i\tau_j$-M-c if and only if $int_j(\theta cl_i(A)) \cap cl_i(\delta int_j(A)) \subseteq A$. We denote the family of all (i,j)-M-c (resp., (i,j)-M-o) sets in a Bts (X,τ_1,τ_2) by $D_{MC}(\tau_i,\tau_j)$ (resp., $D_{MO}(\tau_i,\tau_j)$).

Definition 2.2 *Let (X, τ_1, τ_2) be a Bts. A subset A of X is called $\tau_i\tau_j$-θ-semiopen (briefly, $\tau_i\tau_j$-θ-so) if $A \subseteq cl_j(\theta int_i(A))$, $\tau_i\tau_j$-δ-preopen (briefly, $\tau_i\tau_j$-δ-po) if $A \subseteq int_i(\delta cl_j(A))$, $\tau_i\tau_j$-e-open if $A \subseteq cl_j(\delta int_i(A)) \cup int_i(\delta cl_j(A))$.*

Proposition 2.1 *The following implications hold:*

1. τ_i-θ-o $\Rightarrow \tau_i\tau_j$-θ-so $\Rightarrow \tau_i\tau_j$-M-o $\Rightarrow \tau_i\tau_j$-e-o.
2. τ_i-θ-o $\Rightarrow \tau_i$-o $\Rightarrow \tau_i\tau_j$-δ-po $\Rightarrow \tau_i\tau_j$-M-o.

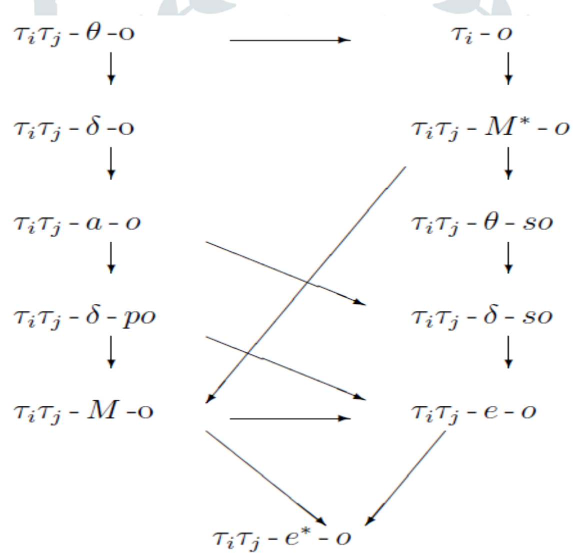
The converse of these implications need not be true as shown by the following examples,

Example 2.1 *In Bts's (X, τ_1, τ_2) and (X, τ_3, τ_4), $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}\}$, $\tau_2 = \{\phi, X, \{a\}, \{b, d\}, \{a, c\}, \{a, b, d\}\}$, $\tau_3 = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{d, c\}, \{a, d, c\}, \{b, c, d\}\}$ and $\tau_4 = \{\phi, X, \{a\}, \{b, d\}, \{a, c\}, \{a, b, d\}\}$. Then the set $\{b, c\}$ is a τ_1-o set that is not a τ_1-θ-o set; $\{b, c, d\}$ is a $\tau_1\tau_2$-θ-so set that is not a τ_1-θ-o set; $\{a, d\}$ is a $\tau_1\tau_2$-δ-po set that is not a τ_1-o set; $\{a, b\}$ is a $\tau_1\tau_2$-M-o set that is not a $\tau_1\tau_2$-θ-so set; $\{a, c\}$ is a $\tau_1\tau_2$-e-o set that is not a $\tau_1\tau_2$-M-o set; $\{b, d\}$ is a $\tau_3\tau_4$-M-o set that is not a $\tau_3\tau_4$-δ-po set.*

Remark 2.1 *$\tau_1\tau_2$-o sets and $\tau_1\tau_2$-M-o sets are independent of each other as seen from this following example.*

Example 2.2 *In Example 2.1, the subsets $\{b, d\}$ is $\tau_1\tau_2$-o but not $\tau_1\tau_2$-M-o set and the subsets $\{a, b\}$ is M-o set but not $\tau_1\tau_2$-o set.*

Remark 2.2 *According the Definitions 2, 2 and Proposition 2, the following diagram holds for a subset A of a space X:*



Note: $A \rightarrow B$ denotes A implies B, but not conversely.

Theorem 2.1 *In Bts (X, τ_1, τ_2), (1) Arbitrary union of $\tau_i\tau_j$-M-o sets are $\tau_i\tau_j$-M-o. (2) The intersection of an $\tau_i\tau_j$-M-o set with an τ_i-o set is an $\tau_i\tau_j$-M-o set. (3) The intersection of arbitary $\tau_i\tau_j$-M-c sets is $\tau_i\tau_j$-M-c.*

Proof. (1) Let $\{A_i, i \in I\}$ be a family of $\tau_i\tau_j$-M-o sets. Then $A_i \subseteq cl_j(\theta int_i(A_i)) \cup int_i(\delta cl_j(A_i))$, hence $\cup_i A_i \subseteq \cup_i (cl_j(\theta int_i(A_i)) \cup int_i(\delta cl_j(A_i))) \subseteq cl_j(\theta int_i(\cup_i A_i)) \cup int_i(\delta cl_j(\cup_i A_i))$, for all $i \in I$. Thus $\cup_i A_i$ is $\tau_i\tau_j$-M-o.

(2) and (3) are obvious.

Remark 2.3 *The intersection of any two $\tau_i\tau_j$-M-o sets is not $\tau_i\tau_j$-M-o set, in Example 2(`)@, the sets $A = \{a, b, c\}$ and $B = \{a, c, d\}$ are $\tau_1\tau_2$-M-o sets but $A \cap B = \{a, c\}$ is not $\tau_1\tau_2$-M-o set.*

Remark 2.4 *The family $D_{MC}(\tau_1, \tau_2)$ is generally not equal to the family $D_{MC}(\tau_2, \tau_1)$ as seen from the following example.*

Example 2.3 *In Example 2.1 the family* $D_{MC}(\tau_3,\tau_4)=\{\phi,X,\{a\},\{b\},\{c\},\{d\},\{c,d\},\{b,d\},\{b,c\},\{a,d\},\{a,c\},\{a,b\},\{b,c,d\},\{a,c,d\}\}$ *and* $D_{MC}(\tau_4,\tau_3)=\{\phi,X,\{a\},\{b\},\{c\},\{d\},\{a,c\},\{b,d\}\{b,c,d\}\}$. *Therefore* $D_{MC}(\tau_1,\tau_2)\neq D_{MC}(\tau_2,\tau_1)$.

Theorem 2.2 *In a Bts* (X,τ_1,τ_2), $\tau_1\subseteq\tau_2$ *and* M-*open* $(X,\tau_1)\subseteq$ M-*open* (X,τ_2) *then* $D_{MC}(\tau_2,\tau_1)\subseteq D_{MC}(\tau_1,\tau_2)$.

Proof. Let $A\in D_{MC}(\tau_2,\tau_1)$ that is A is an a (τ_2,τ_1)-M-c set. To Prove that $A\in D_{MC}(\tau_1,\tau_2)$. Let $G\in M$-open (X,τ_1) be such that $A\subseteq G$. Since M-open $(X,\tau_1)\subseteq M$-open (X,τ_2), we have $G\in M$-open (X,τ_2). As A is a (τ_2,τ_1)-M-c set, we have τ_1-$\delta pcl(A)\subseteq G$. Since $\tau_1\subseteq\tau_2$, we have τ_2-$\delta pcl(A)\subseteq\tau_1$-$\delta pcl(A)$ and it follows that τ_2-$\delta pcl(A)\subseteq G$. Hence A is a (τ_1,τ_2)-M-c. That is $A\in D_{MC}(\tau_1,\tau_2)$. Therefore $D_{MC}(\tau_2,\tau_1)\subseteq D_{MC}(\tau_1,\tau_2)$.

Theorem 2.3*Let A and B be subsets of* (X,τ_1,τ_2) *such that* $A\subseteq B$. *If A is an* $\tau_i\tau_j$-M-*o set in* (X,τ_1,τ_2), *then A is an* $\tau_i\tau_j$-M-*o set in* $(B,\tau_1\backslash B,\tau_2\backslash B)$.

Proof. If A is an $\tau_i\tau_j$-M-o set in X,

$$A\subseteq int_i(cl_j(A))\cup cl_j(int_i(A))$$

$$A\subseteq(int_i(cl_j(A))\cap B)\cup(cl_j(int_i(A))\cap B)$$

$A=int_{\tau_i\backslash B}(cl_{\tau_j\backslash B}(A))\cup cl_{\tau_j\backslash B}(int_{\tau_i\backslash B}(A))$.

Hence A is an $\tau_i\tau_j$-M-open set in $(B,\tau_1\backslash B,\tau_2\backslash B)$.

The converse of the Theorem 2(`)@ need not be true as shown by the following example, even when $A\in\tau_i$.

Example 2.4 *In Example 2.1,* $A=\{d\}\in\tau_1\backslash B$ *where* $B=\{a,b,d\}$. *Hence* A *is an* $\tau_1\tau_2$-M-*o set in* $(B,\tau_1\backslash B,\tau_2\backslash B)$, *but not an* $\tau_1\tau_2$-M-*o set in* (X,τ_1,τ_2).

Definition 2.3*Let* A *be a subset of* (X,τ_1,τ_2). *Then*

1. The intersection of all $\tau_i\tau_j$-M-c sets containing A is called the $\tau_i\tau_j$-M closure of A, denoted by $\tau_i\tau_j$-Mcl(A). i.e., τ_i,τ_j-$Mcl(A)=\bigcap\ \{U: A\subseteq U,\ U\in D_{MC}(\tau_i,\tau_j)\}$.
2. The union of all $\tau_i\tau_j$-M-o sets contained in A is called the $\tau_i\tau_j$-M interior of A, denoted by $\tau_i\tau_j$-$Mint(A)$. i.e., τ_i,τ_j-$Mint(A)=\bigcup\ \{U: U\subseteq A,\ U\in D_{MO}(\tau_i,\tau_j)\}$.

Theorem 2.4 *Let* A *and* B *be subsets of* (X,τ_1,τ_2) *and* $x\in X$. *Then,*

1. A is $\tau_i\tau_j$-M-c if and only if $\tau_i\tau_j$-$Mcl(A)=A$.
2. A is $\tau_i\tau_j$-M-o if and only if $\tau_i\tau_j$-$Mint(A)=A$.
3. $x\in\tau_i\tau_j$-$Mcl(A)$ if and only if for every $\tau_i\tau_j$-M-o set U containing x, $U\cap A\neq\phi$.
4. $x\in\tau_i\tau_j$-$Mint(A)$ if and only if there exists an $\tau_i\tau_j$-M-o set U such that $x\in U\subseteq A$.
5. If $A\subseteq B$, then $\tau_i\tau_j$-$Mint(A)\subseteq\tau_i\tau_j$-$Mint(B)$ and $\tau_i\tau_j$-$Mcl(A)\subseteq\tau_i\tau_j$-$Mcl(B)$.

Theorem 2.5 *Let* $\{A_\alpha:\alpha\in\Delta\}$ *be a family of subsets of* X. *Then*

1. $\tau_i\tau_j$-$Mint(\bigcap\ \{A_\alpha:\alpha\in\Delta\})\subseteq\bigcap\ \{\tau_i\tau_j$-$Mint(A_\alpha):\alpha\in\Delta\})$.
2. $\bigcup\ \{\tau_i\tau_j$-$Mint(A_\alpha):\alpha\in\Delta\}\subseteq\tau_i\tau_j$-$Mint(\bigcup\ \{A_\alpha:\alpha\in\Delta\})$.

Theorem 2.6 *The following are equivalent for a subset* A *of* X:

1. A is $\tau_i\tau_j$-M-o.
2. $A=\tau_i\tau_j$-$\delta pint(A)\cup\tau_i\tau_j$-$\theta sint(A)$.
3. $A\subseteq\tau_i\tau_j$-$\delta pcl(\tau_i\tau_j$-$\delta pint(A))$.

Proof. (1) ⇒ (2): Let A be an $\tau_i\tau_j$-M-o set. Then $A\subseteq cl_j(\theta int_i(A))\cup int_i(\delta cl_j(A))$ and $\tau_i\tau_j$ - $\delta pint(A)\cup\tau_i\tau_j$ - $\theta sint(A)=(A\cap cl_j(\theta int_i(A)))\cup(A\cap int_i(\delta cl_j(A)))$ $=A\cap(cl_j(\theta int_i(A))\cup int_i(\delta cl_j(A)))=A$. Hence (2) holds.

(2) ⇒ (3) : $A=\tau_i\tau_j$ - $\delta pint(A)\cup\tau_i\tau_j$ - $\theta sint(A)=\tau_i\tau_j$ - $\delta pint(A)\cup(A\cap cl_j(\theta int_i(A)))\subseteq$

$\tau_i\tau_j$ - $\delta pint(A) \cup cl_j(\theta int_i(A))$. Now since $\tau_i\tau_j$ - $\delta pint(A) \subseteq \tau_i\tau_j$ - $\delta pcl(\tau_i\tau_j$ - $\delta pint(A))$, $cl_j(B) \subseteq \tau_i\tau_j$ - $\delta pcl(B)$ and $int_i(B) \subseteq \tau_i\tau_j$ - $\delta pcl(B)$ for every subset $B \subseteq X$, then $A \subseteq \tau_i\tau_j$-$\delta pcl(\tau_i\tau_j$-$\delta pint(A))$. Thus (3) holds.

(3) ⇒ (1) : We have $A \subseteq \tau_i\tau_j$ - $\delta pcl(\tau_i\tau_j$ - $\delta pint(A)) = \tau_i\tau_j$ - $\delta pint(A) \cup cl_j(\theta int_i(A)) \subseteq cl_j(\theta int_i(A)) \cup int_i(\delta cl_j(A))$. Thus (1) holds.

Corollary 2.1*The following are equivalent for a subset A of X:*

1. A is $\tau_i\tau_j$-M-c.
2. $A = \tau_i\tau_j$-$\delta pcl(A) \cup \tau_i\tau_j$-$\theta scl(A)$.
3. $A \subseteq \tau_i\tau_j$-$\delta pint(\tau_i\tau_j$-$\delta pcl(A))$.

Corollary 2.2 *The following hold:*

1. Every $\tau_i\tau_j$-M-o set is a disjoint union of an $\tau_i\tau_j$-δ-po set and an $\tau_i\tau_j$-θ-so set.
2. If A is an $\tau_i\tau_j$-M-o set and $\theta int_i(A) = \phi$, then A is an $\tau_i\tau_j$-δ-po set.

Proof. 1. Follows from part (2) of Corollary 3(`)@ and the fact that,

$\tau_i\tau_j$-$\delta pint(A) \backslash \tau_i\tau_j$-$\theta sint(A) = \tau_i\tau_j$-$\delta pint(A) \backslash (A \cap cl_j(\theta int_i(A)))$

$= \tau_i\tau_j$-$\delta pint(A) \backslash (cl_j(\theta int_i(A))$,

which is $\tau_i\tau_j$-δ-po.

2. Obvious.

Theorem 2.7 *For a subset A of X*

1. $\tau_i\tau_j$-$Mcl(A) = \tau_i\tau_j$-$\theta scl(A) \cap \tau_i\tau_j$-$\delta pcl(A)$.
2. $\tau_i\tau_j$-$Mint(A) = \tau_i\tau_j$-$\theta sint(A) \cup \tau_i\tau_j$-$\delta pint(A)$.

Proof. We only prove part (1), as the proof of (2) is similar. Clearly, $\tau_i\tau_j$-$Mcl(A) \subseteq \tau_i\tau_j$-$\theta scl(A) \cap \tau_i\tau_j$-$\delta pcl(A)$. Moreover, as $\tau_i\tau_j$-$Mcl(A)$ is $\tau_i\tau_j$-M-c,

$\tau_i\tau_j$-$Mcl(A) \supseteq int_i(\theta cl_j(\tau_i\tau_j$-$Mcl(A))) \cap clj(\delta int_i(\tau_i\tau_j$-$Mcl(A)))$

$\supseteq int_i(\theta cl_j(A)) \cap cl_j(\delta int_i(A))$.

Thus $\tau_i\tau_j$-$Mcl(A) \supseteq A \cup int_i(\theta cl_j(A)) = \tau_i\tau_j$-$\theta scl(A) \cap \tau_i\tau_j$-$\delta cl(A)$.

Corollary 2.3*For a subset A of X*

1. $\tau_i\tau_j$-$Mcl(\theta int_i(A)) = \theta int_i(\tau_i\tau_j$-$Mcl(A)) = \theta int_i(cl_j(\theta int_i(A)))$.
2. $\tau_i\tau_j$-$Mint(\delta cl_j(A)) = \delta cl_j(\tau_i\tau_j$-$Mint(A)) = \delta cl_j(int_i(\delta cl_j(A)))$.
3. $\tau_i\tau_j$-$Mcl(\tau_i\tau_j$-$\theta sint(A)) = \tau_i\tau_j$-$\theta scl(\tau_i\tau_j$-$\theta sint(A))$.
4. $\tau_i\tau_j$-$Mint(\tau_i\tau_j$-$\theta scl(A)) = \tau_i\tau_j$-$\theta sint(\tau_i\tau_j$-$\theta scl(A))$.
5. $\tau_i\tau_j$-$\theta sint(\tau_i\tau_j$-$Mcl(A)) = \tau_i\tau_j$-$\theta scl(A) \cap cl_j(\theta int_i(A))$.
6. $\tau_i\tau_j$-$\theta scl(\tau_i\tau_j$-$Mint(A)) = \tau_i\tau_j$-$\theta sint(A) \cup int_i(\delta cl_j(A))$.
7. $\tau_i\tau_j$-$\delta pint(\tau_i\tau_j$-$Mcl(A)) = \tau_i\tau_j$-$Mcl(\tau_i\tau_j$-$\delta pint(A)) = \tau_i\tau_j$-$\delta pint(\tau_i\tau_j$-$\delta pcl(A))$.
8. $\tau_i\tau_j$-$\delta pcl(\tau_i\tau_j$-$Mint(A)) = \tau_i\tau_j$-$Mint(\tau_i\tau_j$-$\delta pcl(A)) = \tau_i\tau_j$-$\delta pcl(\tau_i\tau_j$-$\delta pint(A))$.
9. $\tau_i\tau_j$-$\delta pint(\tau_i\tau_j$-$Mcl(A)) = \tau_i\tau_j$-$Mcl(\tau_i\tau_j$-$\delta pint(A)) = \tau_i\tau_j$-$\theta int(\tau_i\tau_j$-$\theta scl(A)) \cap \tau_i\tau_j$-$\delta pcl(A)$.
10. $\tau_i\tau_j$-$\delta pcl(\tau_i\tau_j$-$Mint(A)) = \tau_i\tau_j$-$Mint(\tau_i\tau_j$-$\delta pcl(A)) = \tau_i\tau_j$-$\theta scl(\tau_i\tau_j$-$\theta sint(A)) \cup \tau_i\tau_j$-$\delta pint(A)$.
11. $\tau_i\tau_j$-$Mint(\tau_i\tau_j$-$Mcl(A)) = \tau_i\tau_j$-$Mcl(\tau_i\tau_j$-$Mint(A))$.

Theorem 2.8. *If A and B be subsets of X. Then*

1. $\tau_i\tau_j$-$Mcl(X) = X$ and $\tau_i\tau_j$-$Mcl(\phi) = \phi$.
2. $A \subseteq \tau_i\tau_j$-$Mcl(A)$.
3. If B is any $\tau_i\tau_j$-M-c set containing A, then $\tau_i\tau_j$-$Mcl(A) \subseteq B$.

Proof. Follows from Definition 2(`)@

Theorem 2.9. *Let A and B be subsets of X and $i,j \in \{1,2\}$ be fixed integers. If $A \subseteq B$, then $\tau_i\tau_j$-$Mcl(A) \subseteq \tau_i\tau_j$-$Mcl(B)$.*

Proof. Let $A \subseteq B$. By Definition 2(`)@ $\tau_i\tau_j$-$Mcl(B) = \bigcap \{F: B \subseteq F \in D_{MC}(\tau_i,\tau_j)\}$. If $B \subseteq F \in D_{MC}(\tau_i,\tau_j)$, since $A \subseteq B$, $A \subseteq B \subseteq F \in D_{MC}(\tau_i,\tau_j)\}$, we have $\tau_i\tau_j$ - $Mcl(A) \subseteq F$. Therefore $\tau_i\tau_j$ - $Mcl(A) \subseteq \bigcap \{F: B \subseteq F \in D_{MC}(\tau_i,\tau_j)\} = \tau_i\tau_j$ - $Mcl(B)$. That is $\tau_i\tau_j$-$Mcl(A) \subseteq \tau_i\tau_j$-$Mcl(B)$.

Theorem 2.10*Let A be a subset of X. If $\tau_1 \subseteq \tau_2$ and M-o $(X,\tau_1) \subseteq$ M-o (X,τ_2), then (τ_1,τ_2)-$Mcl(A) \subseteq (\tau_2,\tau_1)$-$Mcl(A)$.*

Proof. By Definition 2.3, (τ_1,τ_2)-$Mcl(A) = \bigcap \{F: A \subseteq F \in D_{MC}(\tau_1,\tau_2)\}$. Since $\tau_1 \subseteq \tau_2$ and M -open $(X,\tau_1) \subseteq M$ -open (X,τ_2) in (X,τ_1,τ_2) then $D_{MC}(\tau_2,\tau_1) \subseteq D_{MC}(\tau_1,\tau_2)$ this implies $D_{MC}(\tau_2,\tau_1) \subseteq D_{MC}(\tau_1,\tau_2)$. Therefore $\bigcap \{F: A \subseteq F \in D_{MC}(\tau_1,\tau_2)\} \subseteq \bigcap \{F: A \subseteq F \in D_{MC}(\tau_2,\tau_1)\}$. That is (τ_1,τ_2)-$Mcl(A) = \bigcap \{F: A \subseteq F \in D_{MC}(\tau_1,\tau_2)\} \subseteq \bigcap \{F: A \subseteq F \in D_{MC}(\tau_2,\tau_1)\} = (\tau_2,\tau_1)$-$Mcl(A)$. Hence,$(\tau_1,\tau_2)$-$Mcl(A) \subseteq (\tau_2,\tau_1)$-$Mcl(A)$.

Theorem 2.11.*Let A be a subset of X and $i,j \in \{1,2\}$ be fixed integers, then $A \subseteq \tau_i\tau_j$-$Mcl(A) \subseteq \tau_i$-$cl(A)$.*

Proof. By Definition 2.3, it follows that $A \subseteq \tau_i\tau_j$-$Mcl(A)$. Now to prove that $\tau_i\tau_j$-$Mcl(A) \subseteq \tau_i$-$cl(A)$. By Definition of closure, τ_j-$cl(A) = \{F \subseteq X: A \subseteq F$ and F is τ_i-c}. If $A \subseteq F$ and F is τ_i-c set, then F is $\tau_i\tau_j$-M-c, as every τ_i-c set is $\tau_i\tau_j$-M-c. Therefore $\tau_i\tau_j$-$Mcl(A) = \bigcap \{F \subseteq X: A \subseteq F$ and F is τ_i-M-c} $\subseteq \bigcap \{F \subseteq X: A \subseteq F$ and F is τ_i-c}=τ_i-$cl(A)$. Hence $\tau_i\tau_j$-$Mcl(A) \subseteq \tau_i$-$cl(A)$.

Example 2.5*Let $X = \{a,b,c,d\}$, $\tau_1 = \{\phi, X, \{a\}, \{b,c\}, \{a,b,c\}\}$ and $\tau_2 = \{\phi, X, \{b\}, \{b,d\}\}$. Then τ_2 -c sets are $\{X, \phi, \{a,c\}, \{a,c,d\}\}$ and $(1,2)$ - M -c sets are $\{\phi, X, \{d\}, \{a\}, \{c\}, \{c,d\}, \{a,d\}, \{a,c\}, \{b,c,d\}, \{a,c,d\}\}$. Take $A = \{b\}$. Then τ_2-$cl(A) = X$ and $(1,2)$-$Mcl(A) = \{b,c,d\}$. Now $A \subseteq (1,2)$-$Mcl(A)$, but $A \neq (1,2)$-$Mcl(A)$. Also $(1,2)$-$Mcl(A) \subseteq \tau_2$-$cl(A)$, but $\tau_i\tau j$-$Mcl(A) \neq \tau_j$-$cl(A)$.*

Theorem 2.12*Let A be a subset of X and $i,j \in \{1,2\}$ be fixed integers. If A is $\tau_i\tau_j$-M-c, then $\tau_i\tau_j$-$Mcl(A) = A$.*

Proof. Let A be a $\tau_i\tau_j$-M-c subset of X. We know that $A \subseteq \tau_i\tau_j$-$Mcl(A)$. Also $A \subseteq A$ and A is $\tau_i\tau_j$-M-c. By Theorem 11(`)@ (iii), $\tau_i\tau_j$-$Mcl(A) \subseteq A$. Hence $\tau_i\tau_j$-$Mcl(A) = A$.

Theorem 2.13*The operator $\tau_i\tau_j$-Mcl in Definition 2, (i) is the Kurutowski closure operator on X.*

Proof. (i) $\tau_i\tau_j$-$Mcl(\phi) = \phi$, by Theorem 11(`)@ (i).

(ii) $E \subseteq \tau_i\tau_j$-$Mcl(E)$ for any subset E in X by Theorem 11(`)@ (ii).

(iii) Suppose E and F are two subsets of X. It follows from Theorem 3(`)@, that $\tau_i\tau_j$-$Mcl(E) \subseteq \tau_i\tau_j$-$Mcl(E \cup F)$ and that $\tau_i\tau_j$-$Mcl(F) \subseteq \tau_i\tau_j$-$Mcl(E \cup F)$. Hence we have $\tau_i\tau_j$-$Mcl(E) \cup \tau_i\tau_j$-$Mcl(F) \subseteq \tau_i\tau_j$-$Mcl(E \cup F)$. Now if $x \notin \tau_i\tau_j$-$Mcl(E) \cup \tau_i\tau_j$-$Mcl(F)$, then $x \notin (i,j)$-$Mcl(E)$ and $x \notin (i,j)$-$Mcl(F)$, it follows that there exist $A, B \in D_{MC}(\tau_i\tau_j)$ such that $E \subseteq A$, $x \notin A$ and $F \subseteq B$, $x \notin B$. Hence $E \cup F \subseteq A \cup B$, $x \notin A \cup B$. Since $A \cup B$ is $\tau_i\tau_j$-M-c and $A, B \in D_{MC}(\tau_i,\tau_j)$, then $A \cup B \in D_{MC}(\tau_i,\tau_j)$ so $x \notin \tau_i\tau_j$-$Mcl(E \cup F)$. Then we have $\tau_i\tau_j$-$Mcl(E \cup F) \subseteq \tau_i\tau_j$-$Mcl(E) \cup \tau_i\tau_j$-$Mcl(F)$. From the above dicussions we have $\tau_i\tau_j$-$Mcl(E \cup F) = \tau_i\tau_j$-$Mcl(E) \cup \tau_i\tau_j$-$M\mathrm{cl}(F)$.

(iv) Let E be any subset of X. By the definition of $\tau_i\tau_j$-Mcl, $\tau_i\tau_i$-$Mcl(E) = \bigcap \{A \subseteq X: E \subseteq A \in D_{MC}(\tau_i,\tau_j)\}$. If $\{E \subseteq A \in D_{MC}(\tau_i,\tau_j)\}$, then $\tau_i\tau_i$-$Mcl(E) \subseteq A$. Since A is a $\tau_i\tau_j$-M-c set containig $\tau_i\tau_j$-$Mcl(E)$, by Theorem 11(`)@ (iii), $\tau_i\tau_j$-$Mcl\{\tau_i\tau_j$-$Mcl(E)\} \subseteq A$. Hence $\tau_i\tau_j$ - $Mcl\{\tau_i\tau_j$ - $Mcl(E)\} \subseteq \bigcap \{A \subseteq X: E \subseteq A \in D_{MC}(\tau_i,\tau_j)\} = \tau_i\tau_j$ - $Mcl(E)$. Conversely $\tau_i\tau_j$-$Mcl(E) \subseteq \tau_i\tau_j$-$Mcl\{\tau_i\tau_j$-$Mcl(E)\}$ is true by Theorem 11(`)@ (iii). Then we have $\tau_i\tau_j$-$Mcl(E) = \tau_i\tau_j$-$Mcl\{\tau_i\tau_j$-$Mcl(E)\}$. Hence $\tau_i\tau_j$-Mcl is a Kuraowski closure operator on X.

From this Theorem $\tau_i\tau_j$-Mcl defines the new topology on X.

Definition 2.4.*Let $i,j \in \{1,2\}$ be two fixed integers. Let τ_M-(τ_i,τ_j) be topology on X*

generated by (τ_i, τ_j) *- M cl in the usual manner. That is* τ_M *-* $(\tau_i, \tau_j) = \{E \subseteq X: (\tau_i, \tau_j)\text{-}Mcl(E^c) = E^c\}$*.*

Theorem 2.14. *Let* X *be a Bts and* $i, j \in \{1,2\}$ *be two fixed integers, then* $\tau_i \subseteq \tau_M(\tau_i, \tau_j)$*.*

Proof. Let $G \in \tau_i$, it follows that G^c is τ_i-c. By Proposition 2 (ii), G^c is (τ_i, τ_j)-M-c. Therefore (τ_i, τ_j)-$Mcl(G^c) = G^c$, by Theorem 3(`)@. That is $G \in \tau_M(\tau_i, \tau_j)$ and hence $\tau_i \subseteq \tau_M(\tau_i, \tau_j)$.

Remark 2.5. *Containment relation in the above Theorem 3(`)@ may be proper as seen from the following Example.*

Example 2.6 *Let* $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ *and* $\tau_2 = \{\phi, X, \{b\}, \{b, d\}\}$*. Then* (τ_1, τ_2)*-*M*-c sets are* $\{\phi, X, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{c, d\}, \{b, c, d\}, \{a, c, d\}\}$ *and* $\tau_M(\tau_1, \tau_2) = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{b, c, d\}, \{a, b, d\}\}$*. Clearly* $\tau_2 \subseteq \tau_M(\tau_1, \tau_2)$ *but* $\tau_2 \neq \tau_M(\tau_1, \tau_2)$*.*

Theorem 2.15 *Let* (X, τ_1, τ_2) *be a Bts and* $i, j \in \{1,2\}$ *be two fixed integers. If a subset* E *of* X *is* $\tau_i\tau_j$*-*M*-c, then* E *is* $\tau_M(\tau_i, \tau_j)$*.*

Proof. Let a subset E of X be $\tau_i\tau_j$-M-c. By Theorem 3(`)@ $\tau_i\tau_j$-$Mcl(E) = E$. That is $\tau_i\tau_j$-$Mcl\{(E^c)^c\} = (E^c)^c$, it follows that $E^c \in \tau_M(\tau_1, \tau_2)$. Therefore E is $\tau_M(\tau_i, \tau_j)$.

Example 2.7 *For* (X, τ_1, τ_2) *of Example 2.6, the subset* $A = \{b, c\}$ *is* $\tau_M(\tau_1, \tau_2)$*, but not* $\tau_1\tau_2$*-*M*-c.*

Theorem 2.16. *If* $\tau_1 \subseteq \tau_2$ *and* M*-open* $(X, \tau_1) \subseteq M$*-open* (X, τ_2) *in* X*, then* $\tau_M(\tau_2, \tau_1) \subseteq \tau_M(\tau_1, \tau_2)$*.*

Proof. Let $G \in \tau_M(\tau_2, \tau_1)$, then (τ_2, τ_1)-$Mcl(G^c) = G^c$. To prove that $G \in \tau_M(\tau_1, \tau_2)$. That is to prove (τ_1, τ_2)-$Mcl(G^c) = G^c$. Now (τ_1, τ_2)-$Mcl(G^c) = \bigcap \{F \subseteq X: G^c \subseteq F \in D_M(\tau_1, \tau_2)\}$. Since $\tau_1 \subseteq \tau_2$ and M-open $(X, \tau_1) \subseteq M$-open (X, τ_2), by Therom 2(`)@ $D_M(\tau_2, \tau_1) \subseteq D_M(\tau_1, \tau_2)$. Thus $\bigcap \{F \subseteq X: G^c \subseteq F \in D_M(1,2)\} \subseteq \bigcap \{F \subseteq X: G^c \subseteq F \in D_M(2,1)\}$. That is (τ_1, τ_2)-$Mcl(G^c) \subseteq (\tau_2, \tau_1)$-$Mcl(G^c)$, and so (τ_1, τ_2)-$Mcl(G^c) \subset G^c$.

Conversely $G^c \subseteq (\tau_1, \tau_2)$-$Mcl(G^c)$ is true by the Theorem 11(`)@ (ii). Then we have (τ_1, τ_2)-$Mcl(G^c) = G^c$. That is $G \in \tau_M(\tau_1, \tau_2)$ and hence $\tau_M(\tau_1, \tau_2) \subseteq \tau_M(\tau_2, \tau_1)$.

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