FIXED POINTS IN WEAK NON-ARCHIMEDEAN INTUITIONISTIC GENERALIZED FUZZY METRIC SPACES

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Abstract:

In this paper, we prove a weak non-Archimedean intuitionistic generalized fuzzy metric space by changing triangular inequality with a similar approach [10,11] and study some properties of the topology induced by a weak non- Archimedean intuitionistic fuzzy metric. Also, we prove a common fixed point theorem in weak non-Archimedean intuitionistic generalized fuzzy metric space for generalized ψ-ϕ-contractive mappings.

Keywords: Weak Non-Archimedean, Fuzzy Metric Space, Generalized Fuzzy Metric Space, ψ- Φ-Contractive Mappings.

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1. Introduction And Preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [13] in 1965. Since that time, to use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and applications. Especially, Kaleva and Seikkala [6], Kramosil and Michalek [7], Georege and Veeramani [3] have introduced the concept of fuzzy metric space in different ways. Grabiec [4] initiated the study of fixed point theory in fuzzy metric spaces, which is parallel to fixed point theory in probabilistic metric space. Many authors followed this concept by introducing and investigating the different types of fuzzy contractive mappings Mihet [8], who realized this strong condition, defined a new fuzzy contraction called ψ-contraction which enlarges the class of fuzzy contractive mappings of Gregori and Sapena and proved fixed point theorems under different hypotheses in fuzzy metric space in the meaning of Kramosil and Michalek. For instance, he assumed that the space under consideration is a non-Archimedean generalized fuzzy metric spaces and he proved a fixed point theorem for fuzzy ψ-contractive mapping in this space [9]. Recently, Vetro [11] introduced the concept of weak non-

Archimedean fuzzy metric space and proved common fixed point results for a pair of generalized contractive type mappings. Also, he presents that every non-Archimedean fuzzy metric space is itself a weak non-Archimedean fuzzy metric space. In 2018, Jeyaraman et.al., proved common fixed point theorems in weak non-Archimedean intuitionistic generalized fuzzy metric spaces.

On the other hand, Atanassov [1] introduced and studied the notion of intuitionistic fuzzy set by generalizing the notion of fuzzy set. An intuitionistic fuzzy set gives both a membership degree and a non membership degree. Using the idea of intuitionistic fuzzy set, Park [10] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to George and Veeramani [3] and proved some known results of metric spaces for intuitionistic fuzzy metric space.

The aim of this paper, we prove a weak non-Archimedean intuitionistic generalized fuzzy metric space by changing triangular inequality with a similar approach [10,11] and study some properties of the topology induced by a weak non- Archimedean intuitionistic fuzzy metric. Also, we prove a common fixed point theorem in weak non-Archimedean intuitionistic generalized fuzzy metric space for generalized ψ-ϕ-contractive mappings.

Definition: 1.1

A 5-tuple (X, $\mathcal{M}$, $\mathcal{N}$, *, $\Diamond$) is said to be an Intuitionistic Generalized Fuzzy Metric Space (shortly IGFM-Space), if X is an arbitrary set, * is a continuous t-norm, $\Diamond$ is a continuous t- conorm and $\mathcal{M}$ and $\mathcal{N}$ are fuzzy sets on $X^3 \times (0, \infty)$ satisfying the following conditions:

for all x, y , z , $a \in X$ and s, t > 0

(IGFM 1) $\mathcal{M}(x, y, z, t) + \mathcal{N}(x, y, z, t) \leq 1$,

(IGFM 2) $\mathcal{M}(x, y, z, t) > 0$,

(IGFM 3) $\mathcal{M}(x, y, z, t) = 1$ if and only if $x = y = z$,

(IGFM 4) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, where p is a permutation function,

(IGFM 5) $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, t + s)$,

(IGFM 6) $\mathcal{M}(x, y, z, .) : (0, \infty) \to [0, 1]$ is continuous,

(IGFM 7) $\mathcal{N}(x, y, z, t) > 0$,

(IGFM 8) $\mathcal{N}(x, y, z, t) = 0$ if and only if $x = y = z$,

(IGFM 9) $\mathcal{N}(x, y, z, t) = \mathcal{N}(p\{x, y, z\}, t)$, where p is a permutation function,

(IGFM 10) $\mathcal{N}(x, y, a, t) \Diamond \mathcal{N}(a,z, z, s) \leq \mathcal{M}(x, y, z, t + s)$,

(IGFM 11) $\mathcal{N}(x, y, z, .): (0, \infty) \to [0, 1]$ is continuous.

Then, ($\mathcal{M}$, $\mathcal{N}$) is called an intuitionistic generalized fuzzy metric space on X.

The function $\mathcal{M}(x, y, z, t)$ and $\mathcal{N}(x, y, z, t)$ denote the degree of nearness and degree of non-nearness between x, y and z with respect to t, respectively.

Remark: 1.2

Every fuzzy metric space (X, $\mathcal{M}$, $\mathcal{N}$, *, $\Diamond$) is an intuitionistic generalized fuzzy metric space of the form (X, $\mathcal{M}$, 1- $\mathcal{M}$, *, $\Diamond$) such that t-norm * and t-conorm$\Diamond$ are associated,

ie., $x \Diamond y = 1- ((1-x)*(1-y))$ for any x, y $\in$X

Remark:1.3

In intuitionistic generalized fuzzy metric space X. $\mathcal{M}$ (x, y, z,.) is non-decreasing and $\mathcal{N}$(x, y, z, .) is non-increasing for all x, y, z $\in$X .

In the above definition, if the triangular inequality (IGFM 5) and (IGFM 10) are replaced by the following:

$\mathcal{M}(x, y, z, \max\{t,s\}) \geq \mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s)$

$\mathcal{N}(x, y, z, \min\{t,s\}) \leq \mathcal{N}(x, y, a, t) \Diamond \mathcal{N}(a, z, z, s)$

Or equivalently $\mathcal{M}(x, y, z, t) \geq \mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, t)$

$\mathcal{N}(x, y, z, t) \leq \mathcal{N}(x, y, a,t) \Diamond \mathcal{N}(a, z, z, t)$

then (X,$\mathcal{M}$, $\mathcal{N}$, *, $\Diamond$) is called non-Archimedean intuitionistic generalized fuzzy metric space. It is easy to check that the triangle inequality (NA) implies (IGFM 5) and (IGFM 10), that is, every non-Archimedean intuitionistic generalized fuzzy metric space is itself an intuitionistic generalized fuzzy metric spaces.

Example: 1.4

Let X be a non-empty set with at least two elements. Define $\mathcal{M}$ (x, y, z, t) by: If we define the intuitionistic generalized fuzzy set (X, $\mathcal{M}$, $\mathcal{N}$) by $\mathcal{M}(x, x, x, t) = 1$, $\mathcal{N}(x, x, x, t) = 0$ for all $x \in X$ and $t > 0$, and $\mathcal{M}(x, y, z, t) = 0$, $\mathcal{N}(x, y, z, t) = 1$, for $x \neq y \neq z$ and $0 < t \leq 1$ and $\mathcal{M}(x, y, z, t) = 1$, $\mathcal{N}(x, y, z, t) = 0$, for $x \neq y \neq z$ and $t > 1$.

Then (X, $\mathcal{M}$, $\mathcal{N}$, *,$\Diamond$) is a non-Archimedean intuitionistic generalized fuzzy metric space with arbitrary continuous t-norm * and t- conorm $\Diamond$.

Clearly (X, $\mathcal{M}$, $\mathcal{N}$, *,$\Diamond$) is also an intuitionistic generalized fuzzy metric spaces.

2. Weak Non- Archimedean Intuitionistic Generalized Fuzzy Metric Spaces

Definition:2.1

In Definition 1.1, if the triangular inequality (IGFM 5) and (IGFM 10) are replaced by the following:

$\mathcal{M}(x, y, z, t) \geq \max \{\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, t/2), \mathcal{M}(x, y, a, t/2) * \mathcal{M}(a, z, z, t) \}$

$\mathcal{N}(x,y, z, t) \leq \min \{\mathcal{N}(x, y, a, t) \Diamond \mathcal{N}(a, z, z, t/2), \mathcal{N}(x, y, a, t/2) \Diamond \mathcal{N}(a, z, z, t) \}$,

for all $x, y, z \in X$ and $t > 0$, then (X, $\mathcal{M}$, $\mathcal{N}$, *, $\Diamond$) is said to be a weak non- Archimedean (WNA) intuitionistic generalized fuzzy metric spaces.

Obviously, every non-Archimedean intuitionistic generalized fuzzy metric space is itself a weak non-Archimedean intuitionistic generalized fuzzy metric spaces.

The inequality (WNA) does not imply that $\mathcal{M}$ (x, y, z, .) is non decreasing and $\mathcal{N}$(x, y, z, .) is non increasing. Thus a weak non-Archimedean intuitionistic generalized fuzzy metric space is not necessarily an intuitionistic generalized fuzzy metric spaces.

Example: 2.2

Let X = [0, ∞) and define $\mathcal{M}$ (x, y, z, t), $\mathcal{N}$(x, y, z, t) by

$\mathcal{M}$ (x, y, z, t) = $\begin{cases} 1, & x = y = z \\ \frac{t}{t+1}, & x \neq y \neq z' \end{cases}$ $\mathcal{N}$(x, y, z, t) = $\begin{cases} 0, & x = y = z \\ \frac{1}{t+1}, & x \neq y \neq z \end{cases}$ for all t > 0.

(X, $\mathcal{M}$, $\mathcal{N}$, *, ◊) is a weak non-Archimedean intuitionistic generalized fuzzy metric space with a * b = ab and a ◊ b = a + b – ab, for every a, b ∈ [0, 1].

Remark : 2.3

(i) For any $r_1, r_2 \in (0, 1)$ with $r_1 > r_2$, there exist $r_3, r_4 \in (0, 1)$ such that $r_1 * r_3 \geq r_2$ and $r_1 \geq r_4 \lozenge r_2$.

(ii) For any $r_5 \in (0, 1)$, there exist $r_6, r_7 \in (0, 1)$ such that $r_6 * r_6 \geq r_5$ and $r_7 \lozenge r_7 \leq r_5$.

Definition: 2.4

Let (X, $\mathcal{M}$, $\mathcal{N}$, *,◊) be a weak non-Archimedean intuitionistic generalized fuzzy metric space and let r ∈ (0,1), t > 0 and x ∈X.

The set B(x, r, t) = {y ∈X: $\mathcal{M}$ (x, y, z, t) >1 –r, $\mathcal{N}$(x, y, z, t) < r}is called the open ball with centre *x* and radius *r* with respect to t.

Proposition: 2.5

Let (X, $\mathcal{M}$, $\mathcal{N}$,*, ◊) be a weak non-Archimedean intuitionistic generalized fuzzy metric space, then every open ball is an open set.

Proof

Let B(x, r, t) be an open ball with centre *x* and radius *r* with respect to t.Now,

y ∈ B(x, r, t) implies $r_0 = \mathcal{M}(x, y, z, t) > 1 - r$ and $\mathcal{N}(x, y, z, t) < r$.

Then, there exist s ∈(0,1) such that $r_0 > 1 - s > 1 - r$.

Hence, from remark (2.3).There exist $r_1, r_2 \in (0, 1)$ such that $r_0 * r_1 > 1 - s$ and $(1 - r_0) \lozenge (1 - r_2) \leq s$.

Put $r_3 = \max\{r_1, r_2\}$and consider the open ball $B(y, 1 - r_3, t/2)$.

We claim thatB $(y, 1 - r_3, t/2) \subset B(x, r, t)$.Now, let $z \in B(y, 1 - r_3, t/2)$.

Then $\mathcal{M}(a, z, z, t/2) > r_3$ and $\mathcal{N}(a, z, z, t/2) < 1 - r_3$.

Therefore,$\mathcal{M}(x, y, z, t) \geq \mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, t/2) \geq r_0 * r_3 \geq r_0 * r_1 > 1 - s > 1 - r$ and

$\mathcal{N}(x, y, z, t) \leq \mathcal{N}(x, y, a, t) \lozenge \mathcal{N}(a, z, z, t/2) \leq (1 - r_0) \lozenge (1 - r_3)$

$\leq (1 - r_0) \lozenge (1 - r_2) \leq s < r$.

Thus $z \in B(x, r, t)$ and hence $B(y, 1 - r_3, t/2) \subset B(x, r, t)$.

Remark: 2.6

Let $(X, \mathcal{M}, \mathcal{N}, *, \Diamond)$ be a weak non-Archimedean intuitionistic generalized fuzzy metric space, the family $\tau = \{A \subset X: \forall x \in A, \exists t > 0 \text{ and } r \in (0,1) \text{ such that } B(x, r, t) \subset A\}$ is a topology on X.

Proposition: 2.7

Every weak non-Archimedean intuitionistic generalized fuzzy metric space is Hausdorff.

Proof:

Let $(X, \mathcal{M}, \mathcal{N}, *, \Diamond)$ be a weak non-Archimedean intuitionistic generalized fuzzy metric space and $x, y, z \in X$, with $x \neq y \neq z$. Then $0 < \mathcal{M}(x, y, z, t) < 1$ and $0 < \mathcal{N}(x, y, z, t) < 1$.

Put $r_1 = \mathcal{M}(x, y, z, t)$, $r_2 = \mathcal{N}(x, y, z, t)$ and $r = \max\{r_1, 1, 1 - r_2\}$.

For each $r_0 \in (r, 1)$, from Remark (2.3) there exist r_3 and r_4 such that $r_3 * r_3 \geq r_0$ and

$(1 - r_4) \Diamond (1 - r_4) \leq 1 - r_0$.

Put $r_5 = \max\{r_3, r_4\}$and considerthe open balls $B(x, 1 - r_5, t)$ and

$B(y, 1 - r_5, t/2)$. Clearly $B(x, 1 - r_5, t) \cap B(y, 1 - r_5, t/2) = \emptyset$.

For if there exists $z \in B(x, 1 - r_5, t) \cap B(y, 1 - r_5, t/2)$, then

$r_1 = \mathcal{M}(x, y, z, t) \geq \mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, t/2) \geq r_5 * r_5 \geq r_3 * r_3 \geq r_0 > r_1$ and

$r_2 = \mathcal{N}(x, y, z, t) \leq \mathcal{N}(x, y, a, t) \Diamond \mathcal{N}(a, z, z, t/2) \leq (1 - r_5) \Diamond (1 - r_5) \leq (1 - r_4) \Diamond (1 - r_4) \leq 1 - r_0 < r_2$.

which is a contradiction.

Proposition: 2.8

Let $(X, \mathcal{M}, \mathcal{N}, *, \Diamond)$ be a weak non-Archimedean intuitionistic generalized fuzzy metric space and τ be the topology on X induced by the weak non-Archimedean intuitionistic generalized fuzzy metric. Then for a sequence $\{x_n\}$ in X, $x_n \to x$ if and only if

$\mathcal{M}(x_n, x, x, t) \to 1$ and $\mathcal{N}(x_n, x, x, t) \to 0$ as $n \to 1$ for all $t > 0$.

Proof:

Fix $t > 0$. Suppose $x_n \to x$. Then for $r \in (0,1)$, there exist $n_0 \in N$ such that $x_n \in B(x, r, t)$ for all $n \geq n_0$. Then $1 - \mathcal{M}(x_n, x, x, t) < r$ and $\mathcal{N}(x_n, x, x, t) < r$ and hence $\mathcal{M}(x_n, x, x, t) \to 1$ and $\mathcal{N}(x_n, x, x, t) \to 0$ as $n \to \infty$.

Conversely, if for each $t > 0$, $\mathcal{M}(x_n, x, x, t) \to 1$ and $\mathcal{N}(x_n, x, x, t) \to 0$ as $n \to \infty$, then for

$r \in (0,1)$, there exist $n_0 \in N$ such that $1 - \mathcal{M}(x_n, x, x, t) < r$ and $\mathcal{N}(x_n, x, x, t) < r$ for all $n \geq n_0$.

It follows that $\mathcal{M}(x_n, x, x, t) > 1 - r$ and $\mathcal{N}(x_n, x, x, t) < r$ for all $n \geq n_0$.

Thus $x_n \in B(x, r, t)$ for all $n \geq n_0$ and hence $x_n \to x$.

Definition: 2.9

Let (X, $\mathcal{M}$, $\mathcal{N}$, *,◊) be a weak non-Archimedean intuitionistic generalized fuzzy metric space. A sequence $\{x_n\}$ in X called a Cauchy sequence, iffor each r ∈ (0, 1) and t > 0 there exist $n_o \in$ N such that $\mathcal{M}(x_n, x_m, x_m, t) > 1-r$ and $\mathcal{N}(x_n, x_m, x_m, t) < r$ for all m, n ≥ n_0.

Definition: 2.10

The weak non-Archimedean intuitionistic generalized fuzzy metricspace (X, $\mathcal{M}$, $\mathcal{N}$, *,◊) is said to be complete if every Cauchy sequence is convergent.

Lemma: 2.11

Let (X, $\mathcal{M}$, $\mathcal{N}$, *,◊) be a weak non-Archimedean intuitionistic generalized fuzzy metric space and $\{x_n\} \subset X$ be a sequence convergent to $x \in X$ then $\lim_{n\to\infty} \mathcal{M}(y, x_n, x_n, t) = \mathcal{M}(y, x, x, t)$ and $\lim_{n\to\infty} \mathcal{N}(y, x_n, x_n, t) = \mathcal{N}(y, x, x, t)$.

Proof:

Since $\mathcal{M}(y,x_n, x_n, t) \geq \mathcal{M}(y, x, x, t) * \mathcal{M}(x, x_n, x_n, t/2)$,

$\mathcal{M}(y, x, x, t) \geq \mathcal{M}(y, x, x_n, t) * \mathcal{M}(x, x, x_n, t/2)$ and

$\mathcal{N}(y, x_n, x_n, t) \leq \mathcal{N}(y, x, x, t) \lozenge \mathcal{N}(x, x_n, x_n, t/2)$

$\mathcal{N}(y, x, x, t) \leq \mathcal{N}(y, x, x_n, t) \lozenge \mathcal{N}(x, x, x_n, t/2)$ as taking $n \to \infty$, we have

$\mathcal{M}(y, x, x, t) \geq \lim_{n\to\infty} \mathcal{M}(y, x_n, x_n, t) \geq \mathcal{M}(y, x, x, t)$ and

$\mathcal{N}(y, x, x, t) \leq \lim_{n\to\infty} \mathcal{N}(y, x_n, x_n, t) \leq \mathcal{N}(y, x, x, t)$.

3. Main Results

In this section, we define generalized ψ - ϕ-contractive mappings and prove a common fixed point theorem.

Let Ψ be the class of all mappings ψ: [0,1] → [0,1] and Φ be the class of all mappings ϕ: [0,1] → [0,1] such that

(i) ψ is non increasing and continuous,

(ii) $\psi(t) > t$ for all $t \in (0,1)$,

(iii) ϕ is nondecreasing and continuous,

(iv) $\phi(t) < t$ for all $t \in (0,1)$.

Lemma: 3.1

If $\psi \in \Psi$ and $\phi \in \Phi$, then $\psi(1) = 1$ and $\phi(0) = 0$.

Lemma: 3.2

If $\psi \in \Psi$ and $\phi \in \Phi$, then $\lim_{n\to\infty} \psi^n(t) = 1$ and $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \in (0, 1)$.

Proof:

Suppose that $\lim_{n\to\infty} \psi^n(t_0) = l < 1$ for some $t_0 \in (0,1)$. By the monotonicity and continuity of , we have $l = \lim_{n\to\infty} \psi^{n+1}(t_0) = \psi(\lim_{n\to\infty} \psi^n(t_0)) = \psi(l) > l$.

By the same way, assume that $\lim_{n\to\infty} \phi^n(t_0) = m > 0$ for some $t_0 \in (0, 1)$. By the monotonicity and continuity of ϕ, we have $m = \lim_{n\to\infty} \phi^{n+1}(t_0) = \phi(\lim_{n\to\infty} \phi^n(t_0)) = \phi(m) < m$,

which is a contradiction.

Definition: 3.3

Let $(X, \mathcal{M}, \mathcal{N}, *, \lozenge)$ be a weak non-Archimedean intuitionistic generalized fuzzy metric space, $\psi \in \Psi$ and $\phi \in \Phi$. Let f, g: X →X,(f, g) is a pair of generalized ψ–ϕ-contractive mappings if the following implications hold: for every x, y, z ∈X and t ∈(0,1)

$\mathcal{M}(x, y, z, t) > 0 \Longrightarrow \mathcal{M}(f(x), g(y), g(z), t) \geq \psi(m(x, y, z, t))$

$\mathcal{N}(x, y, z, t) < 1 \Longrightarrow \mathcal{N}(f(x), g(y), g(z), t) \leq \phi(n(x, y, z, t))$, where

$m(x, y, z, t) = \min\{\mathcal{M}(x, y, z, t), \mathcal{M}(x, f(x), f(x), t), \mathcal{M}(y, g(y), g(y), t), \mathcal{M}(z, g(z), g(z), t)\}$

$n(x, y, z, t) = \max\{\mathcal{N}(x, y, z, t), \mathcal{N}(x, f(x), f(x), t), \mathcal{N}(y, g(y), g(y), t), \mathcal{N}(z, g(z), g(z), t)\}$

Theorem: 3.4

Let $(X, \mathcal{M}, \mathcal{N}, *, \lozenge)$ be an complete weak non-Archimedean intuitionistic generalized fuzzy metric space and f, g: X →X, (f, g) is a pair of generalized ψ-ϕ-contractive mappings. If there exist $x_0 \in X$ such that $\mathcal{M}(x_0, f(x_0), f(x_0), t) > 0$ and $\mathcal{N}(x_0, f(x_0), f(x_0), t) < 1$ for all $t > 0$, then f and g have a unique common fixed point.

Proof:

Let $x_0 \in X$ be such that $\mathcal{M}(x_0, f(x_0), f(x_0), t) > 0$ and $\mathcal{N}(x_0, f(x_0), f(x_0), t) < 1$for all $t > 0$.

Fix $x_0 \in X$ and define the sequence (x_n) by

$x_1 = f(x_0), x_2 = g(x_1), \ldots, x_{2n+1} = f(x_{2n}), x_{2n+2} = g(x_{2n+1}), \ldots$

we have for all $t > 0$

$$\begin{aligned}
\mathcal{M}(x_1, x_2, x_3, t) &= \mathcal{M}(f(x_0), g(x_1), f(x_2), t) \\
&\geq \psi(m(x_0, x_1, x_2, t)) \\
&= \psi(\mathcal{M}(x_0, x_1, x_2, t)) > 0,
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}(x_2, x_3, x_4, t) &= \mathcal{M}(g(x_1), f(x_2), g(x_3), t) \\
&\geq \psi(m(x_1, x_2, x_3, t)) \\
&\geq \psi(\mathcal{M}(x_1, x_2, x_3, t)) \\
&= \psi^2(\mathcal{M}(x_0, x_1, x_2, t)) > 0 \text{ and}
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}(x_1, x_2, x_3, t) &= \mathcal{N}(f(x_0), g(x_1), f(x_2), t) \\
&\leq \phi(n(x_0, x_1, x_2, t))
\end{aligned}$$

$= \phi(\mathcal{N}(x_0, x_1, x_2, t)) < 1,$

$\mathcal{N}(x_2, x_3, x_4, t) = \mathcal{N}(g(x_1), f(x_2), g(x_3), t)$

$\leq \phi(n(x_1, x_2, x_3, t))$

$\leq \phi(\mathcal{N}(x_1, x_2, x_3, t))$

$= \phi^2(\mathcal{N}(x_0, x_1, x_2, t)) < 1.$

Generally, for each $n \in \mathbb{N}$, we get

$\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t) \geq \psi^n(\mathcal{M}(x_0, x_1, x_2, t))$, $\mathcal{N}(x_n, x_{n+1}, x_{n+2}, t) \leq \phi^n(\mathcal{N}(x_0, x_1, x_2, t))$.

By Lemma (3.2), as $n \to \infty$, we deduce that

$\lim_{n\to\infty} \mathcal{M}(x_{n+2}, x_{n+1}, x_n, t) = 1, \quad \lim_{n\to\infty} \mathcal{N}(x_{n+2}, x_{n+1}, x_n, t) = 0.$

Now, we show that $\{x_n\}$ is a Cauchy sequence. If $\{x_n\}$ is not a Cauchy, then there are $r \in (0,1)$ and $t > 0$ such that for each $k \in \mathbb{N}$ there exist $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ and

$\mathcal{M}(x_{m(k)}, x_{n(k)}, x_{n(k)}, t) \leq 1 - r$ and $\mathcal{N}(x_{m(k)}, x_{n(k)}, x_{n(k)}, t) \geq r$.

Then we can assume that $m(k)$ are odd numbers, $n(k)$ are even numbers and set

$p(k) = \min\{m(k) : \mathcal{M}(x_{m(k)}, x_{n(k)}, x_{n(k)}, t) \leq 1-r, m(k) \text{ is odd number}\}$

$q(k) = \min\{m(k) : \mathcal{N}(x_{m(k)}, x_{n(k)}, x_{n(k)}, t) \geq r, m(k) \text{ is odd number}\}.$

We have

$1-r \geq \mathcal{M}(x_{p(k)}, x_{n(k)}, x_{n(k)}, t)$

$\geq \mathcal{M}(x_{p(k)-2}, x_{n(k)}, x_{n(k)}, t) * \mathcal{M}(x_{p(k)-2}, x_{p(k)}, x_{p(k)}, t/2)$

$\geq \mathcal{M}(x_{p(k)-2}, x_{n(k)}, x_{n(k)}, t) * \mathcal{M}(x_{p(k)-2}, x_{p(k)-1}, x_{p(k)}, t/2) * \mathcal{M}(x_{p(k)-1}, x_{p(k)}, x_{p(k)}, t/4)$

$\geq (1-r) * \mathcal{M}(x_{p(k)-2}, x_{p(k)-1}, x_{p(k)-1}, t/2) * \mathcal{M}(x_{p(k)-1}, x_{p(k)}, x_{p(k)}, t/4)$

$r \geq \mathcal{N}(x_{q(k)}, x_{n(k)}, x_{n(k)}, t)$

$\geq \mathcal{N}(x_{q(k)-2}, x_{n(k)}, x_{n(k)}, t) \lozenge \mathcal{N}(x_{q(k)-2}, x_{q(k)}, x_{q(k)}, t/2)$

$\geq \mathcal{N}(x_{q(k)-2}, x_{n(k)}, x_{n(k)}, t) \lozenge \mathcal{N}(x_{q(k)-2}, x_{q(k)-1}, x_{q(k)}, t/2) \lozenge \mathcal{N}(x_{q(k)-2}, x_{q(k)}, x_{q(k)}, t/4)$

$\geq r \lozenge \mathcal{N}(x_{q(k)-2}, x_{q(k)-1}, x_{q(k)}, t/2) \lozenge \mathcal{N}(x_{q(k)-2}, x_{q(k)}, x_{q(k)}, t/4)$ as $k \to \infty$, we obtain

$\lim_{n\to\infty} \mathcal{M}(x_{p(k)}, x_{n(k)}, x_{n(k)}, t) = 1-r$ and $\lim_{n\to\infty} \mathcal{N}(x_{q(k)}, x_{n(k)}, x_{n(k)}, t) = r$.

Now, from

$\mathcal{M}(x_{p(k)}, x_{n(k)}, x_{n(k)}, t) \geq \mathcal{M}(x_{p(k}x_{n(k)+1}, x_{n(k)+1}, t) * \mathcal{M}(x_{n(k)+1}, x_{n(k)}, x_{n(k)}, t/2)$

$\geq \mathcal{M}(x_{p(k)+1}, x_{n(k)+1}, x_{n(k)+1}, t) * \mathcal{M}(x_{p(k)}, x_{p(k)+1}, x_{p(k)+1}, t/2) *$

$\mathcal{M}(x_{n(k)+1}, x_{n(k)}, x_{n(k)}, t/2)$

$\geq \psi(m(x_{p(k)}, x_{n(k)}, x_{n(k)}, t) * \mathcal{M}(x_{p(k)}, x_{p(k)+1}, x_{p(k)+1}, t/2) *$

$\mathcal{M}(x_{n(k)+1}, x_{n(k)}, x_{n(k)}, t/2))$ and

$\mathcal{N}(x_{q(k)}, x_{n(k)}, x_{n(k)}, t) \leq \mathcal{N}(x_{q(k)}, x_{n(k)+1}, x_{n(k)+1}, t) \lozenge \mathcal{N}(x_{n(k)+1}, x_{n(k)}, x_{n(k)}, t/2)$

$\leq \mathcal{N}(x_{q(k)+1}, x_{n(k)+1}, x_{n(k)+1}, t) \lozenge \mathcal{N}(x_{q(k)}, x_{q(k)+1}, x_{q(k)+1}, t/2) \lozenge$

$$\mathcal{N}(x_{n(k)+1}, x_{n(k)}, x_{n(k)}, t/2)$$
$$\leq \phi(n(x_{q(k)}, x_{n(k)}, x_{n(k)}, t) \Diamond \mathcal{N}(x_{q(k)}, x_{q(k)+1}, x_{q(k)+1}, t/2) \Diamond$$
$$\mathcal{N}(x_{n(k)+1}, x_{n(k)}, x_{n(k)}, t/2)).$$

$m(x_{p(k)}, x_{n(k)}, x_{n(k)}, t) = \min\{\mathcal{M}(x_{p(k)}, x_{n(k)}, x_{n(k)}, t), m(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}, t),$
$\mathcal{M}(x_{p(k)}, x_{p(k)+1}, x_{p(k)+1}, t)\}$ and
$n(x_{q(k)}, x_{n(k)}, x_{n(k)}, t) = \max\{\mathcal{N}(x_{q(k)}, x_{n(k)}, x_{n(k)}, t) \Diamond \mathcal{N}(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}, t),$
$\mathcal{N}(x_{q(k)}, x_{q(k)+1}, x_{p(k)+1}, t)\}.$

Since ψ and ϕ continuous taking limit as $k \to \infty$, we get

$$1-r \geq \psi(1-r)*1*1 = \psi(1-r) > 1-r \text{ and } r \leq \phi(r) \Diamond 0 \Diamond 0 = \phi(r) < r,$$

which are contradictions. Therefore $\{x_n\}$is a Cauchy sequence. Since X iscomplete, there exist $x \in X$ such that $\lim_{n\to\infty} x_n = x$. If $f(x) \neq x$, then there exist $t > 0$ such that $\mathcal{M}(x, f(x), f(x), t) < 1$ and $\mathcal{N}(x, f(x), f(x), t) > 0$. From

$\mathcal{M}(f(x), x_{2n}, x_{2n+1}, t) = \mathcal{M}(f(x), g(x_{2n-1}), f(x_{2n}), t) \geq \psi(m(x, x_{2n-1}, x_{2n}, t))$ and

$\mathcal{N}(f(x), x_{2n}, x_{2n+1}, t) = \mathcal{N}(f(x), g(x_{2n-1}), f(x_{2n}), t) \leq \phi(n(x, x_{2n-1}, x_{2n}, t))$

by Lemma (3.2), as $n \to \infty$, we obtain

$$\mathcal{M}(f(x), x, x, t) \geq \psi(\mathcal{M}(f(x), x, x, t)) > \mathcal{M}(f(x), x, x, t) \text{and}$$
$$\mathcal{N}(f(x), x, x, t) \leq \phi(\mathcal{N}(f(x), x, x, t)) < \mathcal{N}(f(x), x, x, t),$$

which are contradictions. Therefore $f(x) = x$.

Analogously, we obtain that $g(x) = x$ and thus x is a common fixed point of f and g.

Now, we prove the uniqueness of the common fixed points of f, g.

Assume that $x, y, z \in X$ are two common fixed points of f and g. If $x\neq y\neq z$, then there exist $t > 0$ such that $\mathcal{M}(x, y, z, t) < 1$ and $\mathcal{N}(x, y, z, t) > 0$ and hence

$\mathcal{M}(x, y, z, t) = \mathcal{M}(f(x), g(y), g(z), t) \geq \psi(\mathcal{M}(x, y, z, t)) > \mathcal{M}(x, y, z, t)$and

$\mathcal{N}(x, y, z, t) = \mathcal{N}(f(x), g(y), g(z), t) \leq \phi(\mathcal{N}(x, y, z, t)) < \mathcal{N}(x, y, z, t)$

which are contradictions. Therefore $x = y = z$.

Corollary: 3.5

Let$(X, \mathcal{M}, \mathcal{N}, *, \Diamond)$ be an complete weak non-Archimedean intuitionistic generalized fuzzy metric space and $f: X \to X$, (f, f) is a pair of generalized ψ-ϕ-contractive mappings. If there exist $x_0 \in X$ such that $\mathcal{M}(x_0, f(x_0), f(x_0), t) > 0$ and $\mathcal{N}(x_0, f(x_0), f(x_0), t) < 1$ for all $t > 0$, then f have a unique fixed point.

Proof:

In Theorem (3.4), if we take f = g the proof is obvious.

Corollary: 3.6

Let (X, $\mathcal{M}$, *) be an complete weak non-Archimedean generalized fuzzy metric space and f, g : X →X, (f, g) is a pair of generalized ψ-ϕ-contractivemappings. If there exist $x_0 \in X$ such that $\mathcal{M}(x_0, f(x_0), f(x_0), t) > 0$, for all t > 0, then f and g have a unique common fixed point.

Proof:

Since every weak non-Archimedean generalized fuzzy metric space is weak non-Archimedean intuitionistic generalized fuzzy metric space, in Theorem (3.2),

if we take a ◊ b =1 −((1 −a) *(1 −b)) and $\mathcal{N}(x, y, z, t) = 1 - \mathcal{M}(x, y, z, t)$ the proof is finished.

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