A Note on weakly ω -I -Continuous Functions

P. Revathi[1] and A. Vadivel[2]

[1]Govt. Polytechnic for Women, TPK Road, Madurai-625 011; Department of Mathematics (FEAT), Annamalai University, Annamalai Nagar-608 002, Tamil Nadu,
Post Graduate and Research Department of Mathematics, Government Arts College (Autonomous). Karur, Tamil Nadu-639 005; Department of Mathematics, Annamalai University, Annamalai Nagar-608 002, India.

Abstract

In [9], Vadivel et al. introduced and investigated the notions of weakly ω-I-continuous and weak*-ω-I-continuous functions in ideal topological spaces. In this paper, we investigate their relationships with continuous and θ-continuous functions.

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1. Introduction

In a topological space (X,τ), the closure and the interior of any subset A of X will be denoted by ClA and $IntA$, respectively. An ideal is defined as a collection I of subsets of X satisfing the following two conditions: (1) If $A \in I$ and $B \subset A$, then $B \in I$; (2) If $A \in I$ and $B \in I$, then $A \cup B \in I$. Let (X,τ) be a topological space and I an ideal of subsets of X. An ideal topological space is a topological space (X,τ) with an ideal I on X and is denoted by (X,τ,I). For a subset $A \subset X$, $A^*(I) = \{x \in X : U \cap A \notin I$ for each open neighbourhood U of $x\}$ is called the local function of A with respect to I and τ [5]. We simply write A^* instead of $A^*(I)$ in case there is no chance for confusion. For every ideal topological space (X,τ,I), there exixts a topology $\tau^*(I)$ (briefly, τ^*), finer than τ, generated by $\beta(I,\tau) = \{U - A : U \in \tau$ and $A \in I\}$, but in general $\beta(I,\tau)$ is not always a topology [3] Additionally, $Cl^*A = A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(I)$.

The following Lemma is useful in the sequel.

Lemma 1.1 [3] *Let (X,τ,I) be an ideal topological space and A, B subsets of X. Then the follwing properites hold.*

(a) If $A \subset B$, then $A^* \subset B^*$.

(b) $A^* = Cl(A^*) \subseteq Cl(A)$.

(c) $(A^*)^* \subseteq A^*$.

(d) If $U \in \tau$, then $U \cap A^* \subseteq (U \cap A)^*$.

Vadivel et al. [9] introduced the notions of weakly ω-I-continuous and weakly*-I-continuous functions in ideal topological spaces. Also ω-I-continuous functions were studied by Jeyanthi et al. [4]. In Theorem 2.8 (below), we obtain an improvement of Theorems 2.5 and 2.8 of [4].

Definition 1.1 *A function $f:(X,\tau) \to (Y,\varphi,I)$ is said to be weakly-ω-I-continuous (briefly weakly ω-I-c) if for each $x \in X$ and each open neighbourhood V of $f(x)$, there exists a ω-open neighbourhood U of x such that $f(U) \subset Cl^*V$.*

Let A be a subset of an ideal topological space. The $*$-frontier of A is defined by $A^* - intA$ and is denoted by $fr^* A$.

Definition 1.2 *A function $f:(X,\tau)\to(Y,\varphi,I)$ is said to be weak*-I-continuous (breifly weak*-ω-I-c) [9] if for each open set V in Y, $f^{-1}(fr^* V)$ is closed in X.*

We will use the following Theorem.

Theorem 1.1 *[9] A function $f:(X,\tau)\to(Y,\varphi,I)$ is continuous if and only if it is both weakly ω-I-continuous and weak*-ω-I-continuous.*

2. Weakly ω-I-Continuouity and θ-continuity

In some cases weakly ω-I-continuity or weak*-ω-I-continuity implies continuity or θ-continuity. Now we can give related results.

Theorem 2.1 *Let (Y,φ,I_f) be an ideal topological space such that U is infinite for every $U\in\varphi$ and if $U,V\in\varphi$ such that $U\cap V=\phi$ then $U=\phi$ or $V=\phi$ and I_f denotes the ideal of finite subsets of Y. Then a function $f:(X,\tau)\to(Y,\varphi,I_f)$ is continuous if and only if it is weak*-ω-I_f-c.*

Proof The necessity is clear by Theorem 1.1.

Sufficiency. We first show that $V^*=Y$ for each non empty open set V in Y. Let $y\in Y$ and $y\in U\in\varphi$. Then $U,V\in\varphi$ and $U\cap V\neq\phi$ by hypothesis. Thus $U\cap V$ is infinite by hypothesis. Therefore $U\cap V\notin I_f$. Then $y\in V^*$. Hence $V^*=Y$. Therefore, f is weakly ω-I_f-c and by Theorem 1.1, f is continuous.

Example 2.1 *Let $(Y,\varphi,I_f)=(R,\varphi,I_f)$, where φ is the left ray or right ray or cofinite topology. Then, for (R,φ), the conditions in Theorem 2.1 are satisfied. Hence a function $f:(X,\tau)\to(R,\varphi,I_f)$ is continuous if and only if it is weak*-ω-I_f-c.*

Definition 2.1 *A function $f:(X,\tau)\to(Y,\varphi)$ is said to be θ-continuous [1] (resp. weakly continuous [6]) at x_0 if for each open neighbourhood V of $f(x_0)$, there is an open neighbourhood U of x_0 such that $f(ClU)\subseteq ClV$ (resp. $f(U)\subseteq Cl(V)$). f is said to be θ-continuous (resp. weakly continuous) if it is θ-continuous (resp. weakly continuous) at each point of X.*

It is well known that continuity implies θ-continuity and θ-continuity implies weak continuity. Vadivel et al. [9]showed that weak-ω-I-continuity strictly lies between continuity and weak continuity. Therefore, we have the following diagram:

$$\begin{array}{ccc} \text{continuity} & \Rightarrow & \theta\text{-continuity} \\ \Downarrow & & \Downarrow \\ \text{weak-}\omega\text{-}I\text{-continuity} & \Rightarrow & \text{weak continuity} \end{array}$$

Remark 2.1 *By the two examples stated below, we show that θ-continuity and weak-ω-I-continuity are independent of each other.*

Example 2.2 *Let $X=\{a,b,c,d,e\}$, $\tau=\{X,\phi,\{a\},\{c\},\{a,c\},\{a,b,c\}\}$ and $I=P(X)$. Define the function $f:X\to X$ as $f=\{(a,e),(b,d),(c,c),(d,b),(e,d)\}$. Then f is θ-continuous but not weak-ω-I-continuous.*

(i) Let $x=a,b$ or $e\in X$ and $V=X\in\tau$ such that $f(x)\in V$. Then, there exists $U=X\in\omega(\tau)$ such that $x\in X=U$ and $f(ClU)\subseteq X\subseteq ClX=X$.

(ii) Let $c \in X$ and $V = \{c\}$, $V = \{a,c\}$, or $V = \{a,b,c\} \in \tau$ such that $f(c) = c \in V$. Then there exists a ω-open set $U = \{c\} \subseteq X$ such that $c \in U$ and $f(ClU) = f(\{b,c,d,e\}) = \{d,c,b\} \subseteq Cl(\{c\}) = \{b,c,d,e\} \subseteq Cl(\{a,c\}) = X = Cl(\{a,b,c\})$.

(iii) Let $d \in X$ and $V = X$ or $V = \{a,b,c\} \in \tau$ such that $f(d) = b \in V$. Then there exists a ω-open set $U = X \subseteq X$ such that $d \in U$ and $f(ClU) = f(X) \subseteq X \subseteq Cl(\{a,b,c\}) = X = ClX$.

By (i), (ii), (iii), f is θ-continuous. On the other hand, for $V = \{a,b,c\} \in \tau$, $f^{-1}(V) = f^{-1}(\{a,b,c\}) = \{c,d\} \notin \tau$. Therefore f is not continuous. Then f is not weak-ω-I-continuous.

Example 2.3 *Let* $X = \{1,2,3,4\}$, $\tau = \{X,\phi,\{1,2,3\},\{3\},\{3,4\}\}$ *and* $Y = \{a,b,c,d\}$, $\sigma = \{Y,\phi,\{a,b\},\{b\},\{d\},\{b,d\},\{a,b,d\},\{b,c,d\}\}$ *and* $I = \{\phi\}$ *, Then* $\beta(I,\sigma) = \sigma = \sigma^*$. *Therefore* $ClV = Cl^*V$ *for every* $V \in \sigma$. *Define the function* $f : X \to Y$ *as* $f = \{(1,a),(2,b),(3,c),(4,d)\}$. *Then* f *is weak-*ω*-*I*-continuous but not* θ*-continuous.*

First we show that $f:(X,\tau) \to (Y,\sigma,I)$ is weak-ω-I-continuous

(i) Let $\{1\} \in X$ and $V = \{a,b\}, V = \{a,b,d\}$ or $V = Y \in \sigma$ such that $f(1) = a \in V$. Then, there exists a ω-open set $U = \{1,2,3\} \subseteq X$ such that $\{1\} \in U$ and $f(U) = \{a,b,c\} \subseteq Cl^*(\{a,b\}) = \{a,b,c\} \subset Cl^*(\{a,b,d\}) = Cl^*(Y) = Y$.

(ii) Let $\{2\} \in X$ and $V = \{b\}$, $V = \{a,b\}, V = \{b,d\}, V = \{a,b,d\}, V = \{b,c,d\}$ or $V = Y \in \sigma$ such that $f(2) = b \in V$. Then there exists a ω-open set $U = \{1,2,3\} \subseteq X$ such that $\{2\} \in U$ and $f(U) = \{a,b,c\} \subseteq Cl^*(\{b\}) = \{a,b,c\} \subseteq Cl^*(\{b,d\}) \subseteq Cl^*(\{a,b,d\}) = Cl^*(\{b,c,d\}) = Cl^*(Y) = Y$.

(iii) Let $\{3\} \in X$ and $V = \{b,c,d\}$ or $V = Y \in \sigma$ such that $f(3) = \{c\} \in V$. Then there exists a ω-open set $U = \{3\} \subseteq X$ such that $\{3\} \in U$ and $f(U) = \{c\} \subseteq Cl^*(\{b,c,d\}) = Cl^*(Y) = Y$.

(iv) Let $\{4\} \in X$ and $V = \{d\}, V = \{b,d\}, V = \{a,b,d\}, V = \{b,c,d\}$ or $V = Y \in \sigma$ such that $f(4) = d \in V$. Then there exists a ω-open set $U = \{3,4\} \subseteq X$ such that $\{4\} \in U$ and $f(U) = \{c,d\} \subseteq Cl^*(\{d\}) = \{c,d\} \subseteq Cl^*(\{b,d\}) = Cl^*(\{a,b,d\}) = Cl^*(\{b,c,d\}) = Cl^*(Y) = Y$. By (i), (ii), (iii) and (iv), f is weakly ω-I-continuous.

Now we show that $f:(X,\tau) \to (Y,\sigma)$ is not θ-continuous.

Let $\{1\} \in X$ and $V = \{a,b\} \in \sigma$ such that $f(1) = a \in V \in \sigma$. But, for every open set $U \subseteq X$ such that $\{1\} \in U$, where $U = \{1,2,3\}$ or $U = X$, $ClU = X$. Then $f(ClU) = Y \neq ClV = \{a,b,c\}$. Therefore $f:(X,\tau) \to (Y,\sigma)$ is not θ-continuous.

Theorem 2.2 *Let* (Y,σ) *be a regular space. Then for a function* $f:(X,\tau) \to (Y,\sigma,I)$*, the following properties are equivalent:*

(a) f is continuous;

(b) f is θ-continuous;

(c) f is weakly ω-I-c;

(d) f is weakly continuous.

Proof It is shown in Theorem 2 of [6] that if (Y,σ) is regular space, then a function $f:(X,\tau) \to (Y,\sigma)$ is weakly continuous if and only if f is continuous.

Theorem 2.3 *Let* (Y,σ,I) *be any ideal topological space such that* $Y - V \subseteq V^*$ *for every* $V \in \sigma$. *Then*

(a) Every function $f:(X,\tau) \to (Y,\sigma,I)$ is θ-continuous and weakly ω-I-c.

(b) A function $f:(X,\tau) \to (Y,\sigma,I)$ is continuous if and only if it is weak*-ω-I-c.

Proof (a) By hypothesis $Cl^*(V)=Y$ for every $V\in\sigma$ and every function f is weakly ω-I-c. Furthermore $ClV=Y$ for every $V\in\sigma$ since $Cl^*V\subseteq ClV$. Thus every function f is θ-continuous.

(b) By (a), every function $f:(X,\tau)\to(Y,\sigma,I)$ is weakly ω-I-c and hence by Theorem 1.1 f is continuous.

Let (X,τ) be a space with an ideal I on X and $D\subseteq X$. Then $I_D=\{D\cap A: A\in I\}$ is obviously an ideal on D.

Theorem 2.4 *If $f:(X,\tau)\to(Y,\sigma,I)$ is weakly ω-I-c, D is a dense subset in the topological space (Y,σ^*) and $f(X)\subseteq D$, then $f:(X,\tau)\to(D,\sigma_D,I_D)$ is weakly ω-I_D-c, where (D,σ_D) is a subspace of (Y,σ).*

Proof Let $x\in X$ and W be any open set of D containing $f(x)$. That is $f(x)\in W\in\sigma_D$. Then there exists a $V\in\sigma$ such that $W=D\cap V$. Since $f:(X,\tau)\to(Y,\sigma,I)$ is weakly ω-I-c and $f(x)\in V\in\sigma$, there exists $U\in\tau$ such that $x\in U$ and $f(U)\subseteq Cl^*(V)$. If D is a dense subset in the topological space (Y,σ^*), then D is a dense subset in the topological space (Y,σ) since $Cl^*D\subseteq ClD$. Since $\sigma\subseteq\sigma^*$, $V\in\sigma^*$. So $Cl^*(D\cap V)=Cl^*V$ since D is dense. Thus $f(U)\subseteq Cl^*V\cap f(X)\subseteq Cl^*(D\cap V)\cap D=Cl^*V\cap D$. Since $W=D\cap V$, $Cl^*_DW=Cl^*V\cap D$ by **Error! Reference source not found.**, Lemma [4]. So $f(U)\subseteq Cl^*_DW$. Hence we obtain that $f:(X,\tau)\to(D,\sigma_D,I_D)$ is weakly ω-I_D-c.

Vadivel et al. [9]also introduced the notions of weakly FI^*-spaces.

An ideal topological space (X,τ,I) is said to be a weakly FI^*-space if $ClA\subseteq A^*$ for every ω-open set $A\subseteq X$.

First we investigate some properties of weakly FI^*-spaces.

Theorem 2.5 *An ideal topological space (X,τ,I) is a weakly FI^*-space if and only if $U\notin I$ for every $U\in\tau-\{\phi\}$.*

Proof Necessity. Let (X,τ,I) be an weakly FI^*-space. Suppose that there exists $U\in\tau-\{\phi\}$ such that $U\in I$. Let $x\in U$. Then $x\in ClU$. Also since $x\in U\in\tau$ and $U\cap U=U\in I$, $x\notin U^*$. Thus $ClU\not\subseteq U^*$ and this is contradiction.

Sufficiency. Let $U\notin I$ for every $U\in\tau-\{\phi\}$. Let A be an open subset of X. If $x\in ClA$ then $U\cap A\neq\phi$ for each open neighbourhood U of x. Also since $U\cap A\in\tau-\{\phi\}, U\cap A\notin I$. Hence we obtain that $x\in A^*$. Thus $ClA\subset A^*$. Therefore (X,τ,I) is a weakly FI^*-space.

Recall that an ideal I of subsets of X in an ideal space (X,τ,I) is said to be codense if $\tau\cap I=\{\phi\}$.

Corollary 2.1 *An ideal topological space (X,τ,I) is a weakly FI^*-space if and only if I is codense.*

The following Theorem will be useful in the sequel.

Theorem 2.6 *Let (X,τ,I) be a weakly FI^*-space and $A\in\tau$. Then the following properties hold:*

(a) $A^*=Cl^*A=(ClA)^*=Cl(A^*)=ClA$,

(b) $Cl^*(ClA)=Cl(Cl^*A)=Cl^*(A^*)$.

Proof (a) If $A\in\tau$ then $ClA\subset A^*$ since (X,τ,I) is a weakly FI^*-space. Thus $(ClA)^*\subseteq(A^*)^*\subseteq A^*$ by Lemma 1.1. Also since $A\subseteq ClA, A^*\subseteq(ClA)^*$ by Lemma 1.1.

Therefore $A^* = (ClA)^*$. Since $Cl^*A = A \cup A^*$ and (X,τ,I) is a weakly FI^*-space, $Cl^*A = A^*$. Thus $Cl(A^*) = A^*$ by Lemma 1.1 (b). Also since (X,τ,I) is a weakly FI^*-space, $ClA = A^*$ by Lemma 1.1 (b). Hence we obtain $A^* = Cl^*A = (ClA)^* = Cl(A^*) = ClA$.

(b) For $A \in \tau$, $Cl^*(ClA) = (ClA)^* \cup ClA = Cl(A^*) \cup ClA = Cl(A^* \cup A) = Cl(Cl^*A) = Cl^*(A^*)$.

Thorem 2.7 *Let (X,τ) be a regular space and (Y,σ,I) be a weakly FI^*-space. A function $f:(X,\tau)\to(Y,\sigma,I)$ is θ-continuous if and only if it is weakly ω-I-c.*

Proof Necessity. Let f be θ-continuous, $x \in X$ and V be any open set of Y containing $f(x)$. Since f is θ-continuous, there exists a ω-open neighbourhood U of x such that $f(ClU) \subseteq ClV$. Then since (Y,σ,I) is a weakly FI^*-space, $f(U) \subseteq f(ClU) \subseteq ClV \subseteq V^* \subseteq V^* \cup V = Cl^*V$. Thus f is weakly ω-I-c.

Sufficiency. Let f be weakly ω-I-c, $x \in X$ and V be an open set of Y containing $f(x)$. Since f is weakly ω-I-c there exists a ω-open neighbourhood U of x such that $f(U) \subseteq Cl^*(V)$. Since $\sigma \subseteq \sigma^*$, $f(U) \subseteq Cl^*(V) \subseteq Cl(V)$. Since (X,τ) is a regular space, there exists an open neighbourhood H of x such that $x \in H \subseteq ClH \subseteq U$. Then $f(ClH) \subseteq ClV$. Thus f is θ-continuous.

Corollary 2.2 *Let (X,τ) be a regular space and (Y,σ,I) be a weakly FI^*-space. For a function $f:(X,\tau)\to(Y,\sigma,I)$, the following properties are equivalent:*

(a) f is θ-continuous;

(b) f is weakly ω-I-c;

(c) f is weakly continuous.

Proof (a) $\Rightarrow$ (b). This is shown in Theorem 2.7

(b) $\Rightarrow$ (c). It is shown in Remark 3.1 of [9] that every weakly ω-I-c function is weakly continuous.

(c) $\Rightarrow$ (b). Let $x \in X$ and V be any open neighbourhood of $f(x)$. Since f is weakly continuous, there exists a ω-open neighbourhood U of x such that $f(U) \subseteq ClV$. Since (Y,σ,I) is a weakly FI^*-space, by Theorem 2.6 we have $f(U) \subseteq Cl^*V$. This shows that f is weakly ω-I-c.

Theorem 2.8 *Let (Y,σ,I) be a weakly FI^*-space. For a function $f:(X,\tau)\to(Y,\sigma,I)$, the following properties are equivalent:*

(a) f is weakly ω-I-c;

(b) f is weakly continuous;

(c) $Cl(f^{-1}(V)) \subseteq f^{-1}(Cl^*(V))$ for every $V \in \sigma$.

Proof (a) $\Leftrightarrow$ (b). This follows from Remark 3.1 of 2.9 and the proof of (c) $\Rightarrow$ (b) in Corollary 2.2.

(b) $\Rightarrow$ (c). Let f be weakly continuous. By Theorem 4 of [7], $Cl(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$ for every open set V of Y. Since (Y,σ,I) is a weakly FI^*-space, by Theorem 2.6 we obtain $Cl(f^{-1}(V)) \subseteq f^{-1}(Cl^*(V))$.

(c) $\Rightarrow$ (b). Let V be any open set of Y. Since (Y,σ,I) is a weakly FI^*-space, $Cl^*(V) = Cl(V)$ and hence we have $Cl(f^{-1}(V)) \subseteq f^{-1}(ClV)$. It follows from Theorem 7 of [8] that f is weakly continuous.

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