RELATION BETWEEN CHROMATIC, DOMINATOR, DOMINATOR CHROMATIC NUMBER OF MIDDLE GRAPH OF SOME SPECIAL GRAPH

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ABSTRACT

Graph theory is one of the popular research areas in mathematics. It is flooded with many parameters. Knowing the importance of graph theoretic parameters we have taken the parameters domination number, chromatic number and domination coloring number, for our study. In this paper, we proposed to study the interrelation between domination number, chromatic number and domination coloring number of middle graph of some special graphs such as path, cycle and star graph.

KEYWORDS: *Chromatic, dominator, dominator chromatic number, coloring, middle graph of some special graph.*

1.1 INTRODUCTION

In graph theory domination and coloring are two popular research areas which have been extensively studied. [3] The concept of graph domination was introduced by Berge C in 1958 in the name of coefficient of external stability. Later on it was renamed as domination in 1962 by Ore O [2]. A decade later, Cockayne EJ and stephen TH in 1977 introduced the notation ϒ (G) to denote dominator number of a graph G [4]. Chromatic number χ (G) is defined as the minimum number of color required to color the vertices of G in such a way that no two adjacent vertices receive the same color. If χ (G) =K, We say that G is k-chromatic [1]. In this paper, interrelation between coloring, domination and domination coloring number of path, cycle and star graphs and their middle graphs are studied and presented.

1.2 BASIC DEFINITIONS

To study the interrelation between this parameter, we need to understand some of the basic graph theoretic definitions and preliminary results. Such definitions and results are presented in the following section.

Definition 1: [5]

A subset S of a vertex set V of a graph G= (V,E) is said to be a ***dominating set*** if for every vertex v ∈ V – S there exist at least one vertex in S is adjacent to v.

A dominating set S is said to be ***Minimal Dominating set*** if no subset of S is a Dominating set. Minimum cardinality of minimal dominating set is said to be the ***Domination Number*** and is denoted by **ϒ (G)**.

Definition 2: [5]

The **chromatic Number χ (G)** of a graph G which is defined to be the minimum number of colors required to color the vertices of G in such a way that no two adjacent vertices receive the same color.

Definition 3:

A dominator coloring of the graph G is a proper coloring in which each vertex of the graph G dominates at-least one entire color class. The minimum number of color classes in a dominator coloring of a graph G is called **dominator *chromatic number*** and is denoted by $\boldsymbol{\chi_d(G)}$.

Definition 4:

A path $P_n$is a graph with a set of vertices V ={v_1, v_2, … . . v_n} such that v_iv_{i+1}belongs to E. A walk is a sequence ($v_0e_1v_1e_2v_2e_3$ … . . e_nv_n) where e_i is the edge $v_{i-1}v_i$ for i =1, 2 ...n. A trail is a walk where all the edges are distinct.

Definition 5:

A cycle is defined as walk with at least three vertices, where all the vertices are distinct, and where the end vertices coincide. Cycle of length n is denoted by C_n. Cycles are called odd if they have odd length and even if they have even length.

A graph is called connected if for every pair $(x,\ y)$ of distinct vertices there is a path between x and y. A forest is a graph with no cycles and a tree is a connected graph with no cycle.

Definition 6:

A **star** S_k is the **complete bipartite** graph $K_{1,k}$. A tree with one internal node and k leaves. Alternatively some authors define S_k to be the tree of **order** k with maximum **diameter** 2.

Definition 7: [6]

The **middle graph** of a graph G, denoted by M (G) and is defined as follows. The vertex set of M (G) is V (G) ∪E (G). Two vertices x, y in the vertex set of M (G) are adjacent in M (G) in case one of the following holds.

1. x, y are in E(G) and x, y are adjacent in G.
2. x is in V (G), y is in E (G) and x, y is incident in G.

1.3 PRELIMINARY RESULTS [7]

1. The star $K_{1,n}$ of order $n \geq 2$ has $\Upsilon(k_{1,n}) = 1$, $\chi(k_{1,n}) = 2$, and $\chi_d(k_{1,n}) = 2$.

2. The path P_n of order $n \geq 2$ has $\Upsilon(P_n) = \lceil \frac{n+1}{3} \rceil$, $\chi(P_n) = 2$, and $\chi_d(P_n) = \begin{cases} 1 + \lceil \frac{n}{3} \rceil & \text{if } n = 2,3,4,5,7 \\ 2 + \lceil \frac{n}{3} \rceil & \text{otherwise} \end{cases}$

3. The cycle C_n of order $n \geq 3$ has $\Upsilon(C_n) = \lceil \frac{n}{3} \rceil$, $\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$

$$\chi_d(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil & \text{if } n = 4 \\ \lceil \frac{n}{3} \rceil + 1 & \text{if } n = 5 \\ \lceil \frac{n}{3} \rceil + 2 & \text{otherwise} \end{cases}$$

4. The middle graph of star $[M(k_{1,n})]$ of order $n \geq 2$ has $\Upsilon[M(k_{1,n})] = n$ and $\chi[M(k_{1,n})] = n+1$.

5. The middle graph of path $[M(P_n)]$ of order $n \geq 2$ has $\Upsilon[M(P_n)] = \begin{cases} \frac{n}{2} + 1 & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$

   $\chi[M(P_n)] = 3$ if n is even and odd.

6. The middle graph cycle $[M(C_n)]$ of order $n \geq 3$ has $\Upsilon[M(C_n)] = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$

   $\chi[M(C_n)] = \begin{cases} 3 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd} \end{cases}$

1.4 MAIN RESULTS

Lemma: 1 [7]

Let G be a connected graph. Then $\max\{\chi(G), \Upsilon(G)\} \leq \chi(G) + \Upsilon(G) \leq \chi_d(G)$. Then bounds are sharp.

Lemma: 2 [7]

For any graph G, $\chi(G) \leq \chi_d(G)$. Lemma: 3

For all graph G, $\Upsilon(G) + \chi(G) \leq n + 1$.

Theorem: 1 [7]

Let G be a graph such that G= M (C_n) with n ≥ 3, then $\chi_d(G) = \frac{n+1}{2} + 2$ for odd n and $\chi_d(G) = \frac{n}{2} + 2$ for even n.

Proof:

Let $C_n : v_1, v_2, \ldots v_n, v_{n+1}(=v_1)$ be a path of length n and let $v_i v_{i+1}=e_i$ for i = 1, 2….n-1 and $v_1 v_n=e_n$. By the definition of middle graph, M(C_n) has the vertex set V(C_n) ∪E(C_n)={ v_i|1≤i≤n} ∪{e_i | 1 ≤i≤ n} in which each v_i is adjacent to e_i and e_{i-1} and each e_i is adjacent to v_{i+1} and v_i for i = 2,3…… n-1 and v_1 is adjacent to e_1 and e_n, also e_n is adjacent to v_1 and v_n. And for odd n, the minimal dominator color class partition is given by, $\{\{e_1\}, \{e_3\} \ldots \{e_n\}, \{e_2, e_4, \ldots e_{n-1}\}, \{v_1, \ldots v_n\}\}$ for odd n .And each vertex in color class partition dominates at least one color class. Hence $\chi_d(G) = \frac{n+1}{2} + 1 + 1 = \frac{n+1}{2} + 2$. And for even n, the minimal dominator color class partition is given by, $\{\{e_1\}, \{e_3\} \ldots \{e_{n-1}\}, \{e_2, e_4, \ldots e_n\}, \{v_1, \ldots v_n\}\}$ for even n .And each vertex in color class partition dominates atleast one color class. Hence $\chi_d(G) = \frac{n}{2} + 1 + 1 = \frac{n}{2} + 2$.

Theorem: 2 [7]

Let G be a graph such that, G= M (C_n) with n ≥3, then $\chi_d(G) = \chi(G) + \Upsilon(G) - 2$, for odd n and $\chi_d(G) = \chi(G) + \Upsilon(G) - 1$, for even n.

Proof:

Let G be a graph such that, G= M (C_n)) with n ≥ 3

For odd n, clearly $\Upsilon(G) = \frac{n+1}{2}$ and $\chi(G) = 4$.

Now, $\chi(G) + \Upsilon(G) - 2 = 4 + \frac{n+1}{2} - 2$

$= \frac{n+1}{2} + 2$.

And also by theorem 1 $\chi_d(G) = \frac{n+1}{2} + 2$

For even n, clearly $\Upsilon(G) = \frac{n}{2}$ and $\chi(G) = 3$.

Now, $\chi(G) + \Upsilon(G) - 1 = 3 + \frac{n}{2} - 1 = \frac{n}{2} + 2$

And also by theorem 1 $\chi_d(G) = \frac{n}{2} + 2$.

Theorem: 3 [7]

Let G be a graph such that, G= M (P_n) with n ≥ 2, then $\chi_d(G) = \frac{n+1}{2} + 2$ for odd n and $\chi_d(G) = \frac{n}{2} + 3$, for even n.

Proof:

Let $P_n: v_{1},\ldots, v_{n+1}$ be a path of length n and let $v_i v_{i+1} = e_i$. By the definition of middle graph, M (P_n) has the vertex set $V(P_n) \cup E(P_n) = \{v_i \mid 1 \le i \le n+1\} \cup \{e_i / 1 \le i \le n\}$ in which each v_i adjacent to e_i and e_i is adjacent to v_{i+1}. Also e_i is adjacent to e_{i+1}.

Case 1: If n is even.

The minimal dominator color class partition is given by, $\{\{e_1\}, \{e_3\} \ldots \{e_{n-1}\}, \{e_2, e_4, \ldots, e_n\}, \{v_1, v_2, \ldots, v_n\}, \{v_{n+1}\}\}$. And each vertex in color class partition dominates at least one color class. Hence $\chi_d(G) = \frac{n}{2} + 1+1+1 = \frac{n}{2} + 3$.

Case 2: If n is odd.

The minimal dominator color class partition is given by, $\{\{e_1\}, \{e_3\} \ldots \{e_n\}, \{e_2, e_4, \ldots, e_{n-1}\}, \{v_1, \ldots, v_n, v_{n+1}\}\}$. And each vertex in color class partition dominates atleast one color class. Hence $\chi_d(G) = \frac{n+1}{2} + 1+1 = \frac{n+1}{2} + 2$.

Theorem: 4 [7]

Let G be a graph such that, G= M (P_n) with n ≥2, then $\chi_d(G) = \chi(G) + \Upsilon(G) - 1$

Proof:

Let G be a graph such that, G= M (P_n) with n ≥2. For odd n, clearly $\Upsilon(G) = \frac{n+1}{2}$ and $\chi(G) = 3$.

Now, $\chi(G) + \Upsilon(G) - 1 = 3 + \frac{n+1}{2} - 1$

$= \frac{n+1}{2} + 2$

And also by theorem 3 $\chi_d(G) = \frac{n+1}{2} + 2$

For even n, clearly $\Upsilon(G) = \frac{n}{2} + 1$ and $\chi(G) = 3$.

Now, $\chi(G) + \Upsilon(G) - 1 = 3 + \frac{n}{2} + 1 - 1 = \frac{n}{2} + 3$

And also by theorem 3 $\chi_d(G) = \frac{n}{2} + 3$.

Theorem: 5

Let S be a graph such that S= [M ($k_{1,n}$)] with n≥2. Then χ_d[M ($k_{1,n}$)] = n+1.

Proof:

Let V= $\{v, v_1, v_2, \ldots \ldots v_n\}$ be a vertex set and E = $\{e_1 \ldots\ldots e_n\}$ be the edge set of $k_{1,n}$. In which $vv_i = e_i$, for i=1, 2, 3......n. By the definition of middle graph [M ($k_{1,n}$)] , its vertex set is V∪E= $\{v, v_1, \ldots . v_n\} \cup \{e_1 \ldots\ldots e_n\}$. The vertex v is exactly adjacent to $v_1, v_2, v_3, \ldots \ldots .$ *and* v_n in ($k_{1,n}$). In M ($k_{1,n}$) the vertex v is exactly adjacent to $e_1, e_2, \ldots \ldots$ *and* e_n also v_1 is adjacent to e_1, v_2 is adjacent to e_2 ,and so on $v_n$is adjacent to e_n. Therefore, the minimal dominator color class partition is given by $\{\{e_1\}, \{e_2\} \ldots\ldots\ldots \{e_n\}, \{v, v_1, v_2, \ldots \ldots v_n\}\}$. Hence χ_d [M ($k_{1,n}$)] = n+1.

***T*heorem: 6**

Let S be a graph such that S= [M ($k_{1,n}$)] with n≥2. Then χ_d[M ($k_{1,n}$)] = n+1 and ϒ [M ($k_{1,n}$)] = n.

Proof:

Let S be a graph such that S= [M ($k_{1,n}$)] with n ≥ 2 And also by Theorem 5 χ_d [M ($k_{1,n}$)] = n+1.

Hence χ_d [M ($k_{1,n}$)] = n+1. Since ϒ [M ($k_{1,n}$)] = n.

Hence ϒ [M ($k_{1,n}$)] = n. Lemma: 4

Let G be a star graph. Then max {χ [M($k_{1,n}$)], ϒ[M($k_{1,n}$)]} ≤ χ_d[M($k_{1,n}$)] ≤ χ[M($k_{1,n}$)]+ ϒ[M($k_{1,n}$)] The bounds are sharp.

Lemma: 5

For any middle graph of star following cases: 1. χ [M(($k_{1,n}$)]= χ_d [M ($k_{1,n}$)]
2. ϒ [M ($k_{1,n}$)]: χ [M ($k_{1,n}$)]-1 or ϒ [M ($k_{1,n}$)]: χ_d [M ($k_{1,n}$)]-1.

1.5 RELATION BETWEEN THE PARAMETERS

Graphs	χ	Υ	χ_d	Relation
Path (P_n)	$\frac{n}{2}$, if n is even $\frac{n-1}{2}$, if n is odd	$\frac{n}{2}$, if n is even $\frac{n-1}{2}$, if n is odd	$\frac{n}{2}$, if n is even $\frac{n-1}{2}$, if n is odd	$\chi = \Upsilon = \chi_d$ $\chi = \Upsilon = \chi_d$
Cycle (C_n)	$\frac{n}{2}$, if n is even $\frac{n}{2}$, if n is odd	$\frac{n}{2}$, if n is even $\frac{n-1}{2}$, if n is odd	$\frac{n}{2}$, if n is even $\frac{n}{2}$, if n is odd	$\chi = \Upsilon = \chi_d$ $\chi = \chi_d$
Star ($K_{1,n}$)	$\chi(k_{1,n}) = 2$	$\Upsilon(k_{1,n}) = 1$	$\chi_d(k_{1,n}) = 2$	$\chi(k_{1,n}) = \chi_d(k_{1,n})$ $\Upsilon(k_{1,n})$: $\chi(k_{1,n})$-1 OR $\Upsilon(k_{1,n})$: $\chi_d(k_{1,n})$-1

Graphs	χ	Υ	χ_d	Relation
Path[M (P_n)]	$\frac{n}{2}$+1, if n is even $\frac{n+1}{2} + 1$, if n is odd	$\frac{n}{2}$+1, if n is even n , if n is odd	$\frac{n}{2}$+3, if n is even $\frac{n+1}{2} + 2$, if n is odd	**$\chi = \Upsilon$, χ_d:Υ+2** χ_d : χ +1
Cycle[M (C_n)]	$\frac{n}{2}$+1, if n is even $\frac{n+1}{2} + 1$, if n is odd	$\frac{n}{2}$, if n is even $\frac{n+1}{2} + 1$, if n is Odd	$\frac{n}{2}$+2 , if n is even $\frac{n+1}{2} + 2$, if n is odd	**χ_d:Υ +2,** **χ_d:χ +1** **χ_d: χ +1**
Star [M ($K_{1,n}$)]	$\chi[M(k_{1,n})]$=n+1	Υ [M($k_{1,n}$)]=n	χ_d[M($k_{1,n}$)]=n+1	χ[M($k_{1,n}$)]= χ_d [M($k_{1,n}$)] Υ [M($k_{1,n}$)]: χ[M($k_{1,n}$)]-1 OR Υ [M($k_{1,n}$)]: χ_d [M($k_{1,n}$)]-1

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