FUZZY ALTERNATING DIRECTION IMPLICIT METHOD FOR SOLVING PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS IN THREE DIMENSIONS

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ABSTRACT

In this paper, we solved parabolic partial differential equations in three-dimensions using the finite difference method such as Alternating Direction Implicit method. Here we solved heat equation using Alternating Direction Implicit method and also we applied fuzzy concepts to solve heat equation using this Alternating Direction Implicit method.

Keywords: parabolic equation, alternating direction implicit method, heat transform, finite difference

I.INTRODUCTION

Partial differential equations form the basis of very many mathematical models of physical, chemical and biological phenomena, and more recently they spread into economics, financial forecasting, image processing and other fields [9]. Parabolic partial differential equations in three space dimensions with over-specified boundary data feature in the mathematical modeling of many important phenomena [3].

Calculation of the solution of fuzzy partial differential equations is in general very difficult. We can find the exact solution only in some special cases. The theory of fuzzy differential equations has attracted much attention in recent times because this theory represents a natural way to model dynamical systems uncertainty. The concept of the fuzzy derivative was first introduced by Chang and Zadeh [1]; it was followed up by Dubois and prade [4], who used the extension principle in their approach. The study of fuzzy differential equations has been initiated as an independent subject in conjunction with fuzzy valued analysis[2] and [10] and set-valued differential equations.

In this paper, we apply fuzzy to solve Alternating direction Implicit method with the parabolic partial differential equations in three dimensions.

PRELIMINARIES

The basic definition of fuzzy numbers is given in [5]

We denote the set of all real numbers by $\mathbb{R}$ and the set of all fuzzy numbers on $\mathbb{R}$ is indicated by $\mathbb{R}_F$.

A fuzzy number is mapping u : $\mathbb{R} \to [0,1]$ with the following properties:

(a) u is upper-semicontinuous,
(b) u is fuzzy convex, i.e., $u(\lambda x+(1-\lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$, $\lambda \in [0,1]$,
(c) u is normal, i.e., $\exists x_0 \in \mathbb{R}$ for which $u(x_0) = 1$,
(d) Supp u = $\{ x \in \mathbb{R} \mid u(x) > 0\}$ is the support of the u, and its closure cl(supp u) is compact.

Definition 2.1[5]

An arbitrary fuzzy numbers represented by an ordered pair of functions $(\underline{u}(\alpha), \overline{u}(\alpha))$, $0 \leq \alpha \leq 1$ that, satisfies the following requirements:

- $\underline{u}(\alpha)$ is a bounded left continuous non decreasing function over [0,1], with respect to any α.
- $\overline{u}(\alpha)$ is a bounded left continuous non increasing function over [0,1], with respect to any α.
- $\underline{u}(\alpha) \leq \overline{u}(\alpha)$, $0 \leq \alpha \leq 1$

Then the α-level set $[u]^\alpha$ of a fuzzy set u on $\mathbb{R}$ is defined as:

$[u]^\alpha = \{x \in \mathbb{R}; u(x) \geq \alpha\}$, for each $\alpha \in (0,1]$,

And for $\alpha = 0$

$[u]^0 = \overline{\bigcup_{\alpha \in (0,1]} [u]^\alpha}$

Where $\bar{A}$ denotes the closure of A.

For u, v $\in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$ the sum u+v and the product λu are defined by $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda u]^\alpha = \lambda[u]^\alpha, \forall \alpha \in [0,1]$ where $[u]^\alpha + [v]^\alpha$ means the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda[u]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$.

The metric structure is given by the Hausdroff distance D: $\mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\}$,

$D(u,v) = sup_{\alpha \in (0,1]} \max\{|\underline{u}(\alpha) - \underline{v}(\alpha)|, |\overline{u}(\alpha) - \overline{v}(\alpha)|\}$.

Where $[u]^\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)]$ and $[v]^\alpha = [\underline{v}(\alpha), \overline{v}(\alpha)]$.

$(\mathbb{R}_F, D)$ is a complete metric space and the following properties are well known: D(u+w, v+w) = D(u, v) $\quad \forall u, v, w \in \mathbb{R}_F$,

D(ku,kv) = |k |D(u, v) $\quad \forall k \in \mathbb{R}, u, v \in \mathbb{R}_F$,

D(u+v, w+e)≤ D(u, w) + D(v,e) $\quad \forall u, v, w, e \in \mathbb{R}_F$.

Model of equation [8]:

In the case of three dimensions, the mathematical model is such an initial and boundary value problem is given by [8] as follows:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, (0 \leq x, y, z \leq 1, t \geq 0) \qquad \text{.............(1)}$$

$$\tilde{u}(x, y, z, 0) = g(x, y, z), \qquad (0 \leq x, y, z \leq 1) \qquad \text{............... (2)}$$

$$\tilde{u}(0, y, z, t) = f_1(y, z, t), \tilde{u}(1, y, z, t) = f_2(y, z, t), \qquad (0 \leq y, z \leq 1, t \geq 0) \quad \text{.............. (3)}$$

$$\tilde{u}(x, 0, z, t) = f_3(x, z, t), \tilde{u}(x, 1, z, t) = f_4(x, z, t), \qquad (0 \leq x, z \leq 1, t \geq 0). \quad \text{.............. (4)}$$

$$\tilde{u}(x, y, 0, t) = f_5(x, y, t), \tilde{u}(x, y, 1, t) = f_6(x, y, t), \qquad (0 \leq x, y \leq 1, t \geq 0 \quad \text{.............. (5)}$$

Where $\tilde{u}(x, y, z, t)$ denoting temperature or concentration of chemical, while g, f_1, f_2, f_3, f_4, f_5 and f_6 are known functions and heat transferred in three dimension system of Length L, Width W and Depth D with grid points in cubic.

III. FUZZY ALTERNATING DIRECTION IMPLICIT METHOD

The alternating direction implicit method was first suggested by Douglas, Peaceman and Richard for solving the heat equation in two spatial variables and alternating direction implicit methods have proved valuable in the approximation of the solutions of parabolic and elliptic differential equations in two and three variables [6,7]

In the ADI approach, the finite difference equations are written in terms of quantities at three x levels. However three different finite difference approximations are used alternately, one to advance the calculations from (n+1) plane to the (n+2) plane and the third to advance the calculations from (n+2) plane to the (n+3) plane [8]

Assume Ũ is a fuzzy function of the independent crisp variables x and t. subdivided the x-t plane into sets of equal rectangles of sides Δx, Δy, Δz, Δt by equally spaced grid lines parallel to Ox [12]

Denote the parametric for of fuzzy number, $\tilde{U}_{i,j}$ as follows $\tilde{U}_{i,j} = (\underline{U}_{i,j}, \overline{U}_{i,j})$

Then by Taylor's theorem and definition of standard difference

$(D_x D_x)\tilde{U}_{i,j} = ((D_x D_x)\tilde{U}_{i,j}\ (D_y D_y)\tilde{U}_{i,j}\ (D_z D_z)\tilde{U}_{i,j})$

Then, we advance th3e solution of the parabolic partial differential equation in three dimensions from the nth plane to (n+1)th plane by replacing $\frac{\partial^2 \underline{u}}{\partial x^2}$ by implicit finite difference approximation at the (n+1)th plane.

Similarly, $\frac{\partial^2 \underline{u}}{\partial y^2}$ and $\frac{\partial^2 \underline{u}}{\partial z^2}$ are replaced by an explicit finite difference approximation at the nth plane.

With these approximation eq(1) in pa rabolic model can be written as

$$\frac{\underline{u}^{n+1}_{i,j,k} - \underline{u}^{n}_{i,j,k}}{\Delta t} = \frac{\underline{u}^{n+1}_{i-1,j,k} - 2\underline{u}^{n+1}_{i,j,k} + \underline{u}^{n+1}_{i+1,j,k}}{(\Delta x)^2} + \frac{\underline{u}^{n}_{i,j-1,k} - 2\underline{u}^{n}_{i,j,k} + \underline{u}^{n}_{i,j+1,k}}{(\Delta y)^2} + \frac{\underline{u}^{n}_{i,j,k-1} - 2\underline{u}^{n}_{i,j,k} + \underline{u}^{n}_{i,j,k+1}}{(\Delta z)^2} \quad \ldots\ldots\ldots\ldots.(6)$$

We set (Δx=Δy=Δz)= h and Δt=k then we have a square and multiply equation (6) by Δt then we get

$$\underline{u}^{n+1}_{i,j,k} - \underline{u}^{n}_{i,j,k} = \frac{k}{h^2}(\underline{u}^{n+1}_{i-1,j,k} - 2\underline{u}^{n+1}_{i,j,k} + \underline{u}^{n+1}_{i+1,j,k} + \underline{u}^{n}_{i,j-1,k} - 2\underline{u}^{n}_{i,j,k} + \underline{u}^{n}_{i,j+1,k} + \underline{u}^{n}_{i,j,k-1} - 2\underline{u}^{n}_{i,j,k} + \underline{u}^{n}_{i,j,k+1})$$

Let r = $\frac{k}{h^2}$, we get

$$\underline{u}^{n+1}_{i,j,k} - \underline{u}^{n}_{i,j,k} = r(\underline{u}^{n+1}_{i-1,j,k} - 2\underline{u}^{n+1}_{i,j,k} + \underline{u}^{n+1}_{i+1,j,k}) + r(\underline{u}^{n}_{i,j-1,k} - 2\underline{u}^{n}_{i,j,k} + \underline{u}^{n}_{i,j+1,k}) + r(\underline{u}^{n}_{i,j,k-1} - 2\underline{u}^{n}_{i,j,k} + \underline{u}^{n}_{i,j,k+1})$$

And

$$\underline{u}^{n+1}_{i,j,k} - r(\underline{u}^{n+1}_{i-1,j,k} - 2\underline{u}^{n+1}_{i,j,k} + \underline{u}^{n+1}_{i+1,j,k}) = \underline{u}^{n}_{i,j,k} + r(\underline{u}^{n}_{i,j-1,k} - 2\underline{u}^{n}_{i,j,k} + \underline{u}^{n}_{i,j+1,k}) + r(\underline{u}^{n}_{i,j,k-1} - 2\underline{u}^{n}_{i,j,k} + \underline{u}^{n}_{i,j,k+1})$$

We simply and rearrange the above equation and we get

$$-ru_{i-1,j,k}^{n+1} + (1+2r)u_{i,j,k}^{n+1} - ru_{i+1,j,k}^{n+1} = (1-4r)u_{i,j,k}^{n} + r(u_{i,j-1,k}^{n} + u_{i,j+1,k}^{n}) + r(u_{i,j,k-1}^{n} - 2u_{i,j,k}^{n} + u_{i,j,k+1}^{n}) \quad \ldots\ldots\ldots\ldots(7)$$

Also, we advance the solution from the(n+1)th plane to (n+2)th plane by replacing $\frac{\partial^2 u}{\partial y^2}$ by implicit finite difference approximation at (n+2)th plane.

Similarly, $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial z^2}$ are replaced by an explicit difference approximation at the (n+1)th plane.
With these approximation equation (1) in parabolic model becomes

$$\frac{u_{i,j,k}^{n+2}-u_{i,j,k}^{n+1}}{\Delta t} = \frac{u_{i-1,j,k}^{n+1}-2u_{i,j,k}^{n+1}+u_{i+1,j,k}^{n+1}}{(\Delta x)^2} + \frac{u_{i,j-1,k}^{n+2}-2u_{i,j,k}^{n+2}+u_{i,j+1,k}^{n+2}}{(\Delta y)^2} + \frac{u_{i,j,k-1}^{n+1}-2u_{i,j,k}^{n+1}+u_{i,j,k+1}^{n+1}}{(\Delta z)^2} \ldots\ldots\ldots\ldots(8)$$

We set (Δx=Δy=Δz) = h and Δt=k then we have a square region and multiply equation (8) by Δt then we get,

$$u_{i,j,k}^{n+2} - u_{i,j,k}^{n+1} = \frac{k}{h^2}\left(u_{i-1,j,k}^{n+1} - 2u_{i,j,k}^{n+1} + u_{i+1,j,k}^{n+1} + u_{i,j-1,k}^{n+2} - 2u_{i,j,k}^{n+2} + u_{i,j+1,k}^{n+2} + u_{i,j,k-1}^{n+1} - 2u_{i,j,k}^{n+1} + u_{i,j,k+1}^{n+1}\right)$$

Let r = $\frac{k}{h^2}$ we get,

$$u_{i,j,k}^{n+2} - u_{i,j,k}^{n+1} = r(u_{i-1,j,k}^{n+1} - 2u_{i,j,k}^{n+1} + u_{i+1,j,k}^{n+1}) + r(u_{i,j-1,k}^{n+2} - 2u_{i,j,k}^{n+2} + u_{i,j+1,k}^{n+2})$$

And

$$u_{i,j,k}^{n+2} - r(u_{i,j-1,k}^{n+2} - 2u_{i,j,k}^{n+2} + u_{i,j+1,k}^{n+2}) = u_{i,j,k}^{n+1} + r(u_{i-1,j,k}^{n+1} - 2u_{i,j,k}^{n+1} + u_{i+1,j,k}^{n+1}) + r(u_{i,j,k-1}^{n+1} - 2u_{i,j,k}^{n+1} + u_{i,j,k+1}^{n+1})$$

We simplify and rearrange the above equation and we get,

$$-ru_{i,j-1,k}^{n+2} + (1+2r)u_{i,j,k}^{n+2} - ru_{i,j+1,k}^{n+2} = (1-4r)u_{i,j,k}^{n+1} + r(u_{i-1,j,k}^{n+1} + u_{i+1,j,k}^{n+1}) + r(u_{i,j,k-1}^{n+1} + u_{i,j,k+1}^{n+1}) \quad \ldots\ldots\ldots\ldots(9)$$

Now, we advance the solution from the(n+2)th plane to (n+3)th plane by replacing $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ by explicit finite difference approximation at (n+2)th plane.

Then $\frac{\partial^2 u}{\partial z^2}$ by an implicit finite difference approximation at the (n+3)th plane. With these approximation equation (1) in parabolic model becomes,

$$\frac{u^{n+3}_{i,j,k}-u^{n+2}_{i,j,k}}{\Delta t} = \frac{u^{n+2}_{i-1,j,k}-2u^{n+2}_{i,j,k}+u^{n+2}_{i+1,j,k}}{(\Delta x)^2}+\frac{u^{n+2}_{i,j-1,k}-2u^{n+2}_{i,j,k}+u^{n+2}_{i,j+1,k}}{(\Delta y)^2}+\frac{u^{n+3}_{i,j,k-1}-2u^{n+3}_{i,j,k}+u^{n+3}_{i,j,k+1}}{(\Delta z)^2} \quad \ldots\ldots\ldots\ldots(10)$$

Put $\Delta t = k$ and (Δx=Δy=Δz) = h

$$\frac{u^{n+3}_{i,j,k}-u^{n+2}_{i,j,k}}{k} = \frac{u^{n+2}_{i-1,j,k}-2u^{n+2}_{i,j,k}+u^{n+2}_{i+1,j,k}}{(h)^2}+\frac{u^{n+2}_{i,j-1,k}-2u^{n+2}_{i,j,k}+u^{n+2}_{i,j+1,k}}{(h)^2}+\frac{u^{n+3}_{i,j,k-1}-2u^{n+3}_{i,j,k}+u^{n+3}_{i,j,k+1}}{(h)^2}$$

Multiply equation (10) by k when (Δx=Δy=Δz) = h then, we have a square region and we get

$$u^{n+3}_{i,j,k}-u^{n+2}_{i,j,k} = \frac{k}{h^2}\left(u^{n+2}_{i-1,j,k}-2u^{n+2}_{i,j,k}+u^{n+2}_{i+1,j,k}+u^{n+2}_{i,j-1,k}-2u^{n+2}_{i,j,k}+u^{n+2}_{i,j+1,k}+u^{n+3}_{i,j,k-1}-2u^{n+3}_{i,j,k}+u^{n+3}_{i,j,k+1}\right)$$

Let r = $\frac{k}{h^2}$ we get,

$$u^{n+3}_{i,j,k}-u^{n+2}_{i,j,k} = r\left(u^{n+2}_{i-1,j,k}-2u^{n+2}_{i,j,k}+u^{n+2}_{i+1,j,k}\right)+r\left(u^{n+2}_{i,j-1,k}-2u^{n+2}_{i,j,k}+u^{n+2}_{i,j+1,k}\right)+r\left(u^{n+3}_{i,j,k-1}-2u^{n+3}_{i,j,k}+u^{n+3}_{i,j,k+1}\right)$$

And

$$u^{n+3}_{i,j,k}-r\left(u^{n+3}_{i,j,k-1}-2u^{n+3}_{i,j,k}+u^{n+3}_{i,j,k+1}\right) = u^{n+2}_{i,j,k}+r\left(u^{n+2}_{i-1,j,k}-2u^{n+2}_{i,j,k}+u^{n+2}_{i+1,j,k}\right)+r\left(u^{n+2}_{i,j-1,k}-2u^{n+2}_{i,j,k}+u^{n+2}_{i,j+1,k}\right)$$

We simplify and rearrange the above equation we get,

$$-ru^{n+3}_{i,j,k-1}+(1+2r)u^{n+3}_{i,j,k}-ru^{n+3}_{i,j,k+1} = (1-4r)u^{n+2}_{i,j,k}+r\left(u^{n+2}_{i-1,j,k}+u^{n+2}_{i+1,j,k}\right)+r\left(u^{n+2}_{i,j-1,k}+u^{n+2}_{i,j+1,k}\right) \quad \ldots\ldots\ldots\ldots(11)$$

Expressed from the above equations (7), (9) and (11) by the system AX=B

$$
\begin{bmatrix}
1+2r & -r & \cdots & \cdots & & & & 0 & 0 \\
-r & 1+2r & -r & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
0 & -r & 1+2r & -r & & & & & \vdots \\
0 & 0 & -r & 1+2r & -r & & & & \vdots \\
\vdots & & & \ddots & \ddots & \ddots & & & \vdots \\
\vdots & & & & \ddots & \ddots & \ddots & \cdots \quad \cdots & \vdots \\
\vdots & & & & & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & & & & \ddots & \ddots & \vdots \\
\vdots & & & & & - & r & 1+2r \quad -r & 0 \\
0 & 0 & & & \cdots & \cdots & \cdots & -r \quad 1+2r & -r \\
0 & 0 & 0 & & & \cdots & \cdots & \cdots \quad -r & 1+2r
\end{bmatrix}
\begin{bmatrix}
\underline{u}^{n+1}_{2,j,k} \\
\underline{u}^{n+1}_{3,j,k} \\
\underline{u}^{n+1}_{4,j,k} \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\underline{u}^{n+1}_{m1-2,j,k} \\
\underline{u}^{n+1}_{m1-1,j,k}
\end{bmatrix}
=
\begin{bmatrix}
(1-4r)\underline{u}^{n}_{2,j,k} + r[\underline{u}^{n}_{2,j-1,k} + \underline{u}^{n}_{2,j+1,k} + \underline{u}^{n}_{2,j,k-1} + \underline{u}^{n}_{2,j,k+1}] \\
(1-4r)\underline{u}^{n}_{3,j,k} + r[\underline{u}^{n}_{3,j-1,k} + \underline{u}^{n}_{3,j+1,k} + \underline{u}^{n}_{3,j,k-1} + \underline{u}^{n}_{3,j,k+1}] \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
(1-4r)\underline{u}^{n}_{m1-1,j,k} + r[\underline{u}^{n}_{m1-1,j-1,k} + \underline{u}^{n}_{m1-1,j+1,k} + \underline{u}^{n}_{m1-1,j,k-1} + \underline{u}^{n}_{m1-1,j,k+1}]
\end{bmatrix}
$$

Similarly, applying the above procedure with remainder of equations

These systems are of a tridiagonal linear system of equations and can be solved by the Gauss elimination.

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