VON NEUMANN REGULAR SEMIRING

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Abstract

We consider the congruence and ideals of a semiring. Then we construct a Regular semiring and discuss the formulation of ideals on Regular semiring and commutative on Regular semiring. We have proved some important theorem.

Keywords: Regular semiring, Ideal, Commutative, congruence.

INTRODUCTION

Γ- semiring was first studied by M. K. Rao[8] as a generalization of Γ- ring and semiring. It is noted that Γ- ring were considered by N. Nobusawa [7] in 1964. The concepts of Γ- semiring and its sub Γ-semiring with a left (right) unity was studied by
J. Luch [6] and M. K. Rao in [8]. The ideals, prime ideals, semiprime ideals, k-ideals and h-ideals of a Γ- semiring respectively by S. Kyuno[5] and M. K. Rao[8].

In Γ- semiring, the properties of their ideals, prime ideals, semiprime ideals and their generalizations play an important role in structure theory. The properties of an ideal in Γ- semiring and Regular semiring are different from the properties of the usual ring ideals. It is noted that the theory of Regular semiring has been enriched with the help of the operator of a Γ- semiring by Dutta and Sardar[3].

PRELIMINARIES

2.1 Ring[7]:

A ring is a set R, together with two operations ⊕ and ∗, which has the following properties:

- R is a commutative group under ⊕
- R is associative under ∗
- Multiplicative identity: There is an element 1 such that $r * 1 = 1 * r = r$ for all $r \in R$
- The operation ∗ distributes over ⊕: $a * (b \oplus c) = (a * b) \oplus (a * c)$
  $(a \oplus b) * c = (a * c) \oplus (b * c)$

2.2 Semiring[1]:

A semiring is a set R with two binary operations + and . , called addition and multiplication, such that:

- (R, +) is a commutative monoid with identity element 0: 1. (a + b) + c = a + (b + c)
2. 0 + a = a + 0 = a
3. a + b = b + a
(R, .) is a monoid with identity element 1: 1. (a . b) . c = a . (b . c)
2. 1 . a = a .1 = a
Multiplication left and right distributes over addition: 1. a . (b + c) = (a . b) + (a . c)
2. (a + b) . c = (a . c) + (b . c)
Multiplication by 0 annihilates R: 1. 0 . a = a . 0 = 0

2.3 Γ-Ring[2]:

Let R and Γ be two addition abelian groups. If for all x, y, z ∈ R and for all α, β ∈ Γ the conditions

(1) $X \alpha y \in R$
(2) $(x + y) \alpha z = x \alpha z + y \alpha z$; $x (\alpha + \beta) z = x \alpha z + x \beta z$
(3) $(x \alpha y) \beta z = x \alpha (y \beta z)$

are satisfied, then R is called a Γ- ring.

2.4 Γ- Semiring[3]:

Let (R, +) and (Γ, +) be a commutative semigroup. Then we call R is a Γ- semiring if there exists a map $R \times \Gamma \times R \rightarrow R$, written (x, γ, y) by $x\gamma y$, such that it satisfies the following axioms for all x, y, z ∈ R and $\gamma, \beta \in \Gamma$:

(1) $x \gamma (y+z) = x \gamma y + x \gamma z$ and $(x+y) \gamma z = x \gamma z + y \gamma z$;
(2) $x (\gamma + \beta) y = x \gamma y + x \beta y$;
(3) $(x \gamma y) \beta z = x \gamma (y \beta z)$.
In the above case, we call R = (R, Γ) a Γ-semiring.

2.5 Regular[2]:

An element 'a' of a ring R is said to be regular if and only if there exists an element x of R such that a x a = a. The ring R is regular if and only if each element of R is regular.

2.6 Regular Semiring:

Let (R, +) and (Γ, +) be a commutative semigroup. Then we call R be a Regular semiring if there exists a map $R \times \Gamma \times R \rightarrow R$, written (x, α, y) by xαy, such that it satisfies the following axioms for all x, y, z ∈ R and α, β ∈ Γ:

(1) $x\,\alpha\,(y + z) = x\,\alpha\,y + x\,\alpha\,z$ and $(x + y)\,\alpha\,z = x\,\alpha\,z + y\,\alpha\,z$;
(2) $x\,(\alpha + \beta)\,y = x\,\alpha\,y + x\,\beta\,y$;
(3) $(x\,\alpha\,y)\,\beta\,z = x\,\alpha\,(y\,\beta\,z)$.

In the above case, we call R = (R, Γ) a Regular semiring.

2.7 Zero Element of Γ- Semiring [4]:

A *Γ*-semiring R is said to have a zero element if there exists an element 0 ∈ R such that 0 + x = x = x + 0 and 0 γ x = 0 = x γ 0 for all x ∈ R and γ ∈ Γ. Also, a *Γ*-semiring R is said to be commutative if x γ y = y γ x for all x, y ∈ R and γ ∈ Γ.

2.8 Zero Element of Regular Semiring:

A Regular semiring R is said to have a zero element if there exists an element 0 ∈ R such that 0 + x = x = x + 0 and 0 α x = 0 = x α 0 for all x ∈ R and α ∈ Γ. Also, a Regular semiring R is said to be commutative if x α y = y α x for all x, y ∈ R and α ∈ Γ.

Example: 2.9

Let R= (Z^+, +) be the semigroup of positive integers and let Γ = ($2Z^+$, +) be the semigroup of even positive integers. Then, R is a Regular semiring.

III. IDEALS OF A REGULAR SEMIRING

R will be a Regular semiring unless otherwise specified. The following lemmas are easy to prove.

3.1 Ideal of a Γ-Semiring[4]:

A subsemigroup I of R is called an ideal of (R, Γ), if IΓR ⊆ I and R Γ I ⊆ I, where by I Γ R we mean the set {x γ r | x ∈ I, r ∈ R, γ ∈ Γ}. If R is a Γ- semiring with zero element, then it is easy to verify that every ideal I of(R, Γ) has the zero element.

3.2 Ideal of a Regular Semiring:

A subsemigroup S of R is called an ideal of (R, Γ), if SΓR ⊆ S and R ΓS ⊆ S, where by SΓR we mean the set {x α r | x ∈ S, r ∈ R, α ∈ Γ}. If R is a Regular semiring with zero element, every ideal S of(R, Γ) has the zero element.

Lemma: 3.3

Let Λ be a non-empty index set and $\{S_\lambda\}_{\lambda\in\Lambda}$ be a family of ideals of (R, Γ).
Then, $\bigcap_{\lambda\in\Lambda} S_\lambda$ is an **i**deal of (R, Γ).

Lemma: 3.4

Let Ł(R, Γ) be the set of all ideals of (R, Γ). Then (Ł (R, Γ), ⊆, ∧, ∨) is a complete lattice, where S ∧ T = S ∩ T and S ∨ T = < S ∪ T > is the unique smallest ideals containing S∪T.

Theorem: 3.5

Let R be a Regular semiring with zero and S be an ideal of (R, Γ). Then $\frac{R}{S}$ = {x + S\ x ∈ R} is a Regular semiring with the mapping.

$$*: \frac{R}{S} \times \Gamma \times \frac{R}{S} \to \frac{R}{S}$$

Defined by

(x + S) * α * (y + S) = x α y + S. For all x, y ∈ R and α ∈ Γ.

Proof:

We first define an operation ⊕ on $\frac{R}{S}$.

(x + S) ⊕ (y + S) = x + y + S,

for all x + S, y + S ∈ $\frac{R}{S}$. Consequently, we can verify that ($\frac{R}{S}$, ⊕) is a commutative semigroup and we have the following equalities:

(x + S) ∗ α ∗ ((y+ S) ⊕ (z + S) = (x + S) ∗ α ∗ (y + z + S)
= x α (y + z) + S
= x α y + x α z + S
= (x α y + S) ⊕ (x α z + S)
= (x + S) ∗ α ∗ (y + S) ⊕ (x + S) ∗ α ∗ (z + S).

Similarly, we have

((x + S) ⊕ (y + S)) ∗ α ∗ (z + S) = (x + S) ∗ α ∗ (z + S) ⊕ (y + S) ∗ α ∗ (z + S),
(x + S) ∗ (α + β) ∗ (y + S) = (x + S) ∗ α ∗ (y + S) ⊕ (x + S) ∗ β ∗ (y + S)
((x + S) ∗ α ∗ (y + S)) ∗ β ∗ (z + S) = (x + S) ∗ α ∗ ((y + S) ∗ β ∗ (z + S)).

These shows that $\frac{R}{S}$ is a Regular semiring.

IV. COMMUTATIVE AND CONGRUENCES ON A REGULAR SEMIRING

In this section, we consider a commutative Regular semiring R. We have the following theorems.

4.1 Commutative Semiring[3]:

A commutative semiring is one whose multiplication is commutative. An idempotent semiring is one whose addition is idempotent: a + a = a, that is, (R, +, 0) is a join- semilattice with zero.

4.2 Commutative Regular Semiring[3]:

A Regular semiring R is said to be commutative Regular semiring if x y = y x, x + y = y + x for all x, y ∈ R and α ∈ Γ.

Theorem: 4.3

The following conditions on an ideal S of a commutative Regular semiring R with zero are equivalent:

(1) H + (0 : S)= (H Γ S : S) for all ideals H of (R, Γ).

(2) H Γ S = K Γ S implies that (0 : S)+ H = (0 : S)+ K for all ideals Hand K of (R, Γ),

where (S : A) = {x ∈ R | xαa ∈ S, for all a ∈ A and α ∈ Γ} for any ∅ ≠ A ⊆ R.

Theorem: 4.4

Let R be a commutative Regular semiring. If S is an ideal of (R, Γ), ∅ ≠ A ⊆ R and α∈Γ then the following statements hold:

(1) S ⊆ (S : A) ⊆ (S : A Γ A) ⊆ (S : A α A) for all α ∈ Γ.

(2) If A ⊆ S, then (S : A)= R.

Theorem:4.5

Let R be a commutative Regular semiring. If S is an ideal of (R, Γ), ∅ ≠ A ⊆ R. Then

$(S : A)= \bigcap_{a\in A}(S : a) = (S : A \setminus S)$.

An equivalence relation θ on (R, Γ) is said to be a congruence if for all x, y, z ∈ R and α ∈ Γ, we have

X θ y ⇒ (x + z) θ (y + z),
X θ y ⇒ (x α z) θ (y α z) and (z α x) θ (z α y).
By R:θ, we mean that the set of all equivalence classes of the elements of R with respect to the mapping θ, that is, R : θ = {θ (x) | x ∈ R}.

Lemma: 4.6

Let θ be a congruence relation on (R, Γ). Then

$$\theta(x+y) = \theta(\theta(x)+\theta(y)) \text{ and } \theta(x\alpha y) = \theta(\theta(x)\alpha\theta(y)) \text{ for all } x, y \in R \text{ and } \alpha \in \Gamma.$$

Proof:

We observe that $\theta(x + y) \subseteq \theta(\theta(x) + \theta(y))$ and $\theta(x\alpha y) \subseteq \theta(\theta(x)\alpha\theta(y))$. By routine checking,wecaneasilyverifythattheaboveequalities hold.

Theorem: 4.7

Let θ be congruence on (R, Γ). Define $\oplus$ on $R : \theta$ by

$\theta(x) \oplus \theta(y) = \theta(x + y)$ for all $x, y \in R$. Then, $(R: \theta, \oplus)$ is a Regular semiring with the following map

$\odot: (R: \theta) \times \Gamma \times (R: \theta) \to (R : \theta)$,
defined by $\theta(x) \odot \alpha \odot \theta(y) = \theta(x \alpha y)$, for all $x, y \in R$ and $\alpha \in \Gamma$.

Proof:

Let $\theta(x)=\theta(x)$ and $\theta(y)=\theta(y)$. Then, by Lemma 4.6, we have the following equality. $\theta(x) \oplus \theta(y) = \theta(x + y) = \theta(\theta(x) + \theta(y))$
$= \theta(\theta(x) + \theta(y)) = \theta(x) \oplus \theta(y)$.
Also, we have an additional equality
$\theta(x) \odot \alpha \odot \theta(y) = \theta(x \alpha y) = (\theta(x) \alpha \theta(y))$
$= \theta(\theta(x) \alpha \theta(y))$
$= \theta(x) \odot \alpha \odot \theta(y)$
Thus, $\oplus$ and $\odot$ are well-defined.
Hence, we can verify that $(R: \theta, \oplus)$ is a commutative semigroup. Now, we deduce that $\theta(x) \odot \alpha \odot (\theta(y) \oplus \theta(z)) = \theta(x) \odot \alpha \odot \theta(y+z)$
$= \theta(x \alpha (y + z))$
$= \theta(x \alpha y + x \alpha z)$
$= \theta(x \alpha y) \oplus \theta(x \alpha z)$
$= (\theta(x) \odot \alpha \odot \theta(y)) \oplus (\theta(x) \odot \alpha \odot \theta(z))$.
Similarly, we can prove that
$(\theta(x) \oplus \theta(y)) \odot \alpha \odot \theta(z) = (\theta(x) \odot \alpha \odot \theta(z)) \oplus (\theta(y) \odot \alpha \odot \theta(z))$,
$\theta(x) \odot (\alpha + \beta) \odot \theta(y) = (\theta(x) \odot \alpha \odot \theta(y)) \oplus (\theta(x) \odot \beta \odot \theta(y))$,
$(\theta(x) \odot \alpha \odot \theta(y)) \odot \beta \odot \theta(z) = \theta(x) \odot \alpha \odot (\theta(y) \odot \beta \odot \theta(z))$. Therefore, $R : \theta$ is a Regular semiring.

Lemma: 4.8

If $\Pi_R : R \rightarrow R : \theta$ is defined by $\Pi_R(x) = \theta(x)$ and 1_Γ is the identity function on Γ, then $(\Pi_R, 1_\Gamma)$: $(R, \Gamma) \rightarrow (R : \theta, \Gamma)$ is an epimorphism.

Proof:

Let $x, y \in R$ and $\alpha \in \Gamma$.
Then, $\Pi_R(x + y) = \theta(x + y) = \theta(x) \oplus \theta(y)$
$= \Pi_R(x) \oplus \Pi_R(y)$.
Also, we have
$\Pi_R(x \alpha y) = \theta(x \alpha y) = \theta(x) \Theta \alpha \Theta \theta(y)$
$= \Pi_R(x) \Theta 1_\Gamma(\alpha) \Theta \Pi_R(y)$.
Clearly, Π_R is surjective and $(\Pi_R, 1_\Gamma)$ is an epimorphism.

CONCLUSION

In this paper we define new concept called Regular semiring, ideals of a Regular Semiring and Commutative Regular semiring. Futher, we can extend this concept to various types of our ideas.

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