CUBIC BI-IDEALS OF NEAR-SUBTRACTION SEMIGROUPS

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Abstract: In this article, the notion of cubic bi-ideals in near-subtraction near-ring has been introduced and some results are discussed.
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1. INTRODUCTION

The notion of subtraction algebra was introduced by Abbott [1] in 1969. Using this notion Schein [12] introduced the concept of subtraction semigroups in 1992. Zelinka [15] studied a special type of subtraction algebra called atomic subtraction algebra. The study of ideals in subtraction algebra was initiated by Jun et al.,[3] who also established some basic properties. Based on near-ring theory, Dheena [2] introduced the near-subtraction semigroups and strongly regular near-subtraction semigroups. K.J.Lee and C.H.Park [8] introduced the notion of a fuzzy ideal in subtraction algebras, and give some conditions for a fuzzy set to be a fuzzy ideal in subtraction algebras. The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [13]. Manikandan [9] studied fuzzy bi-ideals of near-ring and established some of their properties. The purpose of this paper to introduce the notion of cubic bi-ideals in near-subtraction semigroup. We investigate some basic results, examples and properties.

2. PRELIMINARIES

Definition 2.1.

Let S be a near subtraction semigroup, $(S,\bar{\mu})$ be an interval valued fuzzy near sub subtraction semigroup and (S,ν) be a fuzzy near sub subtraction semigroup. A cubic set $A=\langle\bar{\mu},\nu\rangle$ is called a cubic near subtraction subsemigroup of S, if it satisfies the following conditions:

(i) $\bar{\mu}(x-y)\geq min\{\bar{\mu}(x),\bar{\mu}(y)\}$.

(ii) $\nu(x-y)\leq max\{\nu(x),\nu(y)\}$

(iii) $\bar{\mu}(xy)\geq min\{\bar{\mu}(x),\bar{\mu}(y)\}$.

(iv) $\nu(xy)\leq max\{\nu(x),\nu(y)\}\ forall x,y,\in S$

Example 2.2.

Let S= $\{0,a,b,1\}$ in which "" and "$\cdot$" are defined as

-	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

.	0	a	b	1
0	0	0	0	0
a	0	a	0	0
b	0	0	b	b
1	0	0	b	b

Define an interval valued fuzzy set $\overline{\mu}: S \to D[0,1]$ by $\overline{\mu}(0)=[0.9,1], \overline{\mu}(a)=[0.6,0.7], \overline{\mu}(b)=[0.8,0.9]$ and $\overline{\mu}(1)=[0,0.1]$ is an interval valued fuzzy near sub subtraction semigroup of S. Define a fuzzy set $\nu: S \to [0,1]$ by $\nu(0)=0, \nu(a)=0.6, \nu(b)=0.72$ and $\nu(1)=1$ is a fuzzy near subtraction subsemigroup of S.

Definition 2.3.

A cubic subalgebra $\overline{\mu}$ of X is called a cubic bi-ideal of X, if $(\overline{\mu} X \overline{\mu}) \cap (\overline{\mu} X * \overline{\mu}) \subseteq \overline{\mu}$ and $(\nu X \nu) \cup (\nu X * \nu) \supseteq \nu$.

Definition 2.4.

A cubic set $A = \langle \overline{\mu}, \nu \rangle$ of S is called a cubic left (right) ideal of S, if it satisfies the following conditions:

(i) $\overline{\mu}(x-y) \geq min\{\overline{\mu}(x), \overline{\mu}(y)\}$

(ii) $\nu(x) \leq max\{\nu(x-y), \nu(y)\}$

(iii) $\overline{\mu}(xy) \geq min\{\overline{\mu}(x), \overline{\mu}(y)\}$

(iv) $\nu(xy) \leq max\{\nu(x), \nu(y)\}$

(v) $\overline{\mu}(xy) \geq \overline{\mu}(x), [\overline{\mu}(xy) \geq \overline{\mu}(y)]$

(vi) $\nu(xy) \leq \nu(x), [\nu(xy) \leq \nu(y)]$

(vii) $\overline{\mu}(xz - x(y-z)) \geq \overline{\mu}(z)$

(viii) $\nu(xz - x(y-z)) \leq \nu(z) \forall x, y, z \in S$.

Definition 2.5.

Let $A_1 = \langle \overline{\mu}_1, \nu_1 \rangle$ and $A_2 = \langle \overline{\mu}_2, \nu_2 \rangle$ be any two cubic sets of S then from the following cubic sets of S are defined as follows:

$$(A_1 - A_2)(z) = \begin{cases} (\overline{\mu}_1 - \overline{\mu}_2)(z) = \begin{cases} sup_{z=x-y} min\{\overline{\mu}_1(x), \overline{\mu}_2(x)\} \ \forall\, x, y \in S ifz = x - y \\ , otherwise \end{cases} \\ (\nu_1 - \nu_2)(z) = \begin{cases} inf_{z=x-y} max\{\nu_1(x), \nu_2(x)\} \ \forall\, x, y \in S \ ifz = x - y \\ 1, otherwise \end{cases} \end{cases}$$

$$(A_1 \circ A_2)(x) = \begin{cases} \overline{\mu}_1 \circ \overline{\mu}_2)(x) = \begin{cases} sup_{x \le ab} min\{\overline{\mu}_1(a), \overline{\mu}_2(b)\} \ ifx \le ab \\ , otherwise \end{cases} \\ (\nu_1 \circ \nu_2)(x) = \begin{cases} inf_{z \le ab} max\{\nu_1(a), \nu_2(b)\} \ if \ x \le ab \\ 1, otherwise \end{cases} \end{cases}$$

$$(A_1 * A_2)(x') = \begin{cases} \overline{\mu}_1 * \overline{\mu}_2)(x') = \begin{cases} \sup_{x \le ab} \min\{\overline{\mu}_1(x), \overline{\mu}_2(z)\} \ ifx \le ab \\ , otherwise \end{cases} \\ (\nu_1 * \nu_2)(x') = \begin{cases} \inf_{z \le ab} \max\{\nu_1(x), \nu_2(b)\} \ if \ x \le ab \\ 1, otherwise \end{cases} \end{cases}$$

$$(A_1 \text{ ó } A_2)(x) = \begin{cases} (\overline{\mu}_1 \cap \overline{\mu}_2)(x) \\ (\nu_1 \cup \nu_2)(x) \end{cases}$$

Example 2.6.

Let $X = \{0, a, b, c\}$ in which $-$" and $\cdot$" are defined by:

-	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

·	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	b
c	0	0	0	c

Define $A=(\bar{\mu},\nu)$ by $\bar{\mu}:S\to D[0,1]$ and $\nu:X\to[0,1]$ as $\bar{\mu}(0)=[0.9,1], \bar{\mu}(a)=[0.7,0.8], \bar{\mu}(b)=[0.6,0.7]$ and $\bar{\mu}(c)=[0.4,0.5],$ and $\nu(0)=0.1, \nu(a)=0.2, \nu(b)=0.3$ and $\nu(c)=0.5$ Then $\bar{\mu}$ is a fuzzy sub-subtraction semigroup of S. Hence $A=\langle\bar{\mu},\nu\rangle$ is a cubic sub-near-semigroup of S.

Lemma 2.7.

Let $C=(\bar{\mu},\nu)$ be a cubic subset of X. If C is a cubic left ideal of X, then C is a cubic bi-ideal of X.

Proof.

Let $x'\in X$ be such that $x'\leq abc\leq(xz-x(y-z)),$ where a,b,c,x,y and z are in X. Then

$$\begin{aligned}((\bar{\mu}X\bar{\mu})\cap(\bar{\mu}X*\bar{\mu}))(x') &= \min\{(\bar{\mu}X\bar{\mu})(x'),(\bar{\mu}X*\bar{\mu})(x')\}\\ &= \min\{(\sup_{x'\leq abc}\min\{\bar{\mu}(a),X(b),\bar{\mu}(c)\},\\ &\sup_{x'\leq xz-x(y-z)}\min\{(\bar{\mu}X)(x),\bar{\mu}(z)\}\})\\ &= \min\{(\sup\{\bar{\mu}(a),\bar{\mu}(c)\},\sup\{(\bar{\mu}X)(x),\bar{\mu}(z)\}\})\end{aligned}$$

Since $\bar{\mu}X\subseteq X$ and C is a cubic left ideal, then $\bar{\mu}(xz-x(y-z))\geq\bar{\mu}(z)$.

$$\begin{aligned}&\leq\min\{X(a),X(b),X(x),\bar{\mu}(xz-x(y-z))\}\\ &=\min\{\{1,1,1,\bar{\mu}(xz-x(y-z))\}=\bar{\mu}(x').\end{aligned}$$

If x' is not expressible as $x'\leq abc\leq xz-x(y-z)$ then $(\bar{\mu}X\bar{\mu}\cap\bar{\mu}X*\bar{\mu})(x')=0\leq\bar{\mu}(x').$ Thus $\bar{\mu}X\bar{\mu}\cap\bar{\mu}X*\bar{\mu}\subseteq\bar{\mu}.$

$$\begin{aligned}(\nu X\nu)\cup(\nu X*\nu))(x') &= \max\{(\nu X\nu)(x'),(\nu X*\mu)(x')\}\\ &= \max\{(\inf_{x'\leq abc}\max\{\nu(a),X(b),\nu(c)\},\\ &\inf_{x'\leq xz-x(y-z)}\max\{(\nu X)(x),\nu(z)\}\})\\ &\geq\max\{\max(\inf\{\nu(a),\nu(c)\},\inf_{x'\leq xz-x(y-z)}\{\nu(x),\nu(z)\}\})\\ &\geq\max\{\max(\inf\{\nu(ab),\nu(c)\},\inf_{x'\leq xz-x(y-z)}\{\nu(x),\nu(xz-x(y-z)\}\})\\ &\geq\max\{\max(\inf\{\nu(abc),\},\inf_{x'\leq xz-x(y-z)}\{\nu(xz-x(y-z)\}\})\\ &\geq\nu(x')\end{aligned}$$

If x' is not expressible as $x'=abc=xz-x(y-z)$ then $(\nu X\nu\cup\nu X*\nu)(x')=0\geq\nu(x').$ Thus $\nu X\nu\cup\nu X*\nu\supseteq\nu.$ Hence ν is a cubic bi-ideal of X.

Theorem 2.8.

Let A be a cubic subset of X. Then $A=\langle\bar{\mu},\nu\rangle$ is a cubic bi-ideal of X if and only if the level subset $U(A:\bar{t},n)$ is a cubic bi-ideal of X for all $\bar{t}\in D[0,1]$ and $n\in[0,1]$.

Proof.

Assume that $A=\langle\bar{\mu},\nu\rangle$ is a cubic bi-ideal of X. Let $x,y\in U[A:\bar{t},n]$ for all $\bar{t}\in D[0,1]$ and $n\in[0,1]$. Then $\bar{\mu}(x),\bar{\mu}(y)\geq\bar{t}$ and $\nu(x),\nu(y)\leq n,$ since A is a cubic bi-ideal of X, we have $\bar{\mu}(x-y)\geq min\{\bar{\mu}(x),\bar{\mu}(y)\}\geq\bar{t}$ and

$$\nu(x-y)\leq min\{\nu(x),\nu(y)\}\leq n.$$

It follows that $x-y\in U[A:\bar{t},n].$ Let $x'\in X$, $x'\in\bar{\mu}_t X\bar{\mu}_t\cap\bar{\mu}_t X*\bar{\mu}_t$ and $x'\in\nu_n X\nu_n\cup\nu_n X*\nu_n$

If there exist $a_1,b,x_1,z\in U[A:\bar{t},n]$ and $a_2,a,x,x_2,y\in X$ such that $x'\leq ab\leq xz-x(y-z), a\leq a_1a_2$ and $x\leq x_1x_2$. Then $\bar{\mu}(a_1)\geq\bar{t},\bar{\mu}(b)\geq\bar{t},\bar{\mu}(z)\geq\bar{t}$ and $\bar{\mu}(x_1)\geq t$ and $\nu(a_1)\leq n,\nu(b)\leq n,\nu(z)\leq n$, $\nu(x_1)\leq n$. Thus

$$\begin{aligned}\bar{\mu}(x')&\geq\{\bar{\mu}X\bar{\mu}\cap\bar{\mu}X*\bar{\mu})(x')\}\\&=\min\{(\bar{\mu}X\bar{\mu})(x'),(\bar{\mu}X*\bar{\mu})(x')\}\\&=\min\{(\sup_{x'\leq ab}\min\{(\bar{\mu}X)(a),\bar{\mu}(b)\},\sup_{x'\leq xz-x(y-z)}\min\{(\bar{\mu}X)(x),\bar{\mu}(z)\})\}\\&=\min\{\sup_{x'\leq ab}\min\{\sup_{a\leq a_1a_2}\min\{\bar{\mu}(a_1),X(a_2)\},\bar{\mu}(b\},\\&\quad\sup_{x'\leq xz-x(y-z)}\min\{\sup_{x\leq x_1x_2}\min\{\bar{\mu}(x_1),X(x_2)\},\bar{\mu}(z)\}\}\\&=\min\{\bar{\mu}(a_1),\bar{\mu}(b),\bar{\mu}(x_1),\bar{\mu}(z)\}\geq\bar{t}.\end{aligned}$$

$$\begin{aligned}\nu(x')&\leq\{\nu X\nu\cup\nu X*\nu)(x')\}\\&=\leq\max\{\nu X\nu,\nu X*\nu)(x')\}.....(1)\end{aligned}$$

Now

$$\begin{aligned}(\nu X\nu)(x')&=\inf_{x'\leq ab}\max\{\inf_{a\leq a_1a_2}\max\{\nu(a_1),X(a_2)\},\nu(b)\}\}\\&=\inf_{x'\leq ab}\max\{\nu(a_1),\nu(b)\}\}\\&\leq n.....(2)\end{aligned}$$

Again

$$\begin{aligned}(\nu X*\nu)(x')&=\inf_{x'\leq xz-x(y-z)}\max\{\nu X(x),\nu(z)\}\\&=\inf_{x'\leq xz-x(y-z)}\max\{\inf_{x\leq x_1x_2}\max\{\nu(x_1),X(x_2)\},\nu(z)\}\\&=\inf_{x'\leq xz-x(y-z)}\max\{\nu(x_1),X(x_2)\}\\&(sinceX(x_2)=0)\\&\leq n(ifsince\nu(x_1)\leq n,\nu(z)\leq n))......(3)\end{aligned}$$

Using (2) and (3) in (1), we get $\nu(x')\leq n.$

This implies that $\bar{\mu}(x')\geq\bar{t}$ and $\nu(x')\leq n$ so $x'\in U[A:\bar{t},n],$ that is, $\bar{\mu}_t X\bar{\mu}_t\cap\bar{\mu}_t X*\bar{\mu}_t\subseteq\bar{\mu}_t$ and $\nu_n X\nu_n\cup\nu_n X*\nu_n\supseteq\nu_n.$

Hence $A=\langle\bar{\mu},\nu\rangle$ is a bi-ideal of X.

Conversely, assume that $U[A:\bar{t},n]$ is a cubic bi-ideal of X for $t,n\in[0,1]$. Let $x'\in X$. Suppose that $\min\{(\bar{\mu}X\bar{\mu}\cap\bar{\mu}X*\bar{\mu})(x')\}>\bar{\mu}(x')$. Choose $0<t\le 1$ such that $\{(\bar{\mu}X\bar{\mu}\cap\bar{\mu}X*\bar{\mu})(x')\}>\bar{t}>\bar{\mu}(x')$.

This implies that $(\bar{\mu}X\bar{\mu})(x')\ge\bar{t}$ and $(\bar{\mu}X*\bar{\mu})(x')\ge\bar{t}$. Then

$$(\bar{\mu}X\bar{\mu})(x')=\sup_{x'\le ab}\min\{\sup_{a\le a_1a_2}\min\{\bar{\mu}(a_1),X(a_2)\},\bar{\mu}(b)\}\qquad=\min\{\bar{\mu}(a_1),\bar{\mu}(b)\}\ge\bar{t}.$$

$$(\bar{\mu}X*\bar{\mu})(x')=\sup_{x'\le(xz-x(y-z))}\min\{(\bar{\mu}X)(x),\bar{\mu}(z)\}$$

$$=\sup_{x'=xz-x(y-z)}\min\{\sup_{x=x_1x_2}\min\{\bar{\mu}(x_1),X(x_2)\},\bar{\mu}(z)\}$$

$$=\min\{\bar{\mu}(x_1),\bar{\mu}(z)\}\ge\bar{t}.$$

Thus $a_1,b,x,z\in\bar{\mu}_t$. Suppose that $\min\{(\nu X\nu\cup\nu X*\nu)(x')\}<\nu(x')$. Choose $0>n\ge 1$ such that $\{(\nu X\nu\cup\nu X*\nu)(x')\}<n<\nu(x')$.

This implies that $(\nu X\nu)(x')\le n$ or $(\nu X*\nu)(x')\le n$. Then

$$(\nu X\nu)(x')=\inf_{x'\le ab}\max\{\inf_{a\le a_1a_2}\max\{\nu(a_1),X(a_2)\},\nu(b)\}\qquad=\max\{\nu(a_1),\nu(b)\}\le n.$$

$$(\nu X*\nu)(x')=\inf_{x'\le(xz-x(y-z))}\max\{(\nu X)(x),\nu(z)\}$$

$$=\inf_{x'=xz-x(y-z)}\max\{\inf_{x=x_1x_2}\max\{\nu(x_1),X(x_2)\},\nu(z)\}$$

$$=\max\{\nu(x_1),\nu(z)\}\le n.$$

Implies $a_1,b,x,z\in\nu_n$.

Thus $a_1,b,x_1,z\in U[A:\bar{t},n]$. Since $U[A:\bar{t},n]$ is a bi-ideal of X, we have $x'=a_1a_2b\in U[A:\bar{t},n]$ and $x'=x_1x_2z-x_1x_2(y-z)\in U[A:\bar{t},n]$.

So, $x'\in(\bar{\mu}_tX\bar{\mu}_t)\cap(\bar{\mu}_tX*\bar{\mu}_t)$, implying, $x'\in U[A:\bar{t},n]$, since $U[A:\bar{t},n]$ is a bi-ideal of X. Thus $\bar{\mu}(x')\ge\bar{t}$ and $\nu(x')\le n$ which is a contradiction. Therefore $\bar{\mu}X\bar{\mu}\cap\bar{\mu}X*\bar{\mu}\subseteq\bar{\mu}$ and $\nu X\nu\cup\nu X*\nu\supseteq\nu$

Hence $A=\langle\bar{\mu},\nu\rangle$ is a cubic bi-ideal of X.

Lemma 2.9.

Let A and B be two nonempty subsets of X. Then the following are true:

- $f_A\cap f_B=f_{A\cap B}$.
- $f_A\cup f_B=f_{A\cup B}$.
- $f_Af_B=f_{AB}$.
- $f_A*f_B=f_{A*B}$.

Lemma 2.10.

A nonempty subset A of X is a bi-ideal of X if and only if $f_A=(\bar{\mu}_A,\nu_A)$ is a cubic bi-ideal of X.

Proof.

Assume that A is a bi-ideal of X. Let f_A be a cubic subset of X defined by

$$\bar{\mu}_A(x)=\{\bar{1}$$

x A $\bar{0}$otherwise.

$$\nu_A(x) = \{0$$

x A 1otherwise. Let $x, y \in X$. Suppose that $\overline{\mu}_A(x-y) < \min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}$ and $\nu(x-y) > max\{\nu_A(x), \nu_A(y)\}$. Then, $\overline{\mu}_A(x-y) = \bar{0}$ and $\min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\} = 1$. $\nu_A(x-y) = 1$ and $\max\{\nu_A(x), \nu_A(y)\} = 0$. This point out that $x, y \in f_A$ and $x - y \notin f_A$, which is a contradiction to our conjecture. Thus so, f_A is a cubic subalgebra of X. For some $x \in X$, take $\overline{\mu_A}(x) < \min\{\{(\overline{\mu_A} X \overline{\mu_A})(x), (\overline{\mu_A} X * \overline{\mu_A})(x)\}$. and $\nu_A(x) > \max\{\{(\nu_A X \nu_A)(x), (\nu_A X * \nu_A)\}(x)\}$. Then $\overline{\mu_A}(x) = \bar{0}$ and $\min\{(\{X\}\overline{\mu_A})(x), (\overline{\mu_A} X * \overline{\mu_A})(x)\} = \bar{1}$, that is, $(\overline{\mu_A} X \overline{\mu_A})(x) = \bar{1}$ and $\nu_A(x) = 1$ and $\max\{(\{X\}\nu_A)(x), (\nu_A X * \nu_A)(x)\} = 0$, that is, $(\nu X \nu)(x) = 0$ and $(\overline{\mu} X * \overline{\mu})(x) = 1$, $(\nu X * \nu)(x) = 0$. This means that $\overline{\mu}_{AXA}(x) = 1$ and $\overline{\mu}_{AX*A}(x) = 1$, $\nu_{AXA}(x) = 0$ and $\nu_{AX*A}(x) = 0$. Hence by Lemma **Error! Reference source not found.** $\{(\overline{\mu}_{AXA} \cap \overline{\mu}_{AX*A})(x)\} = \{\overline{\mu}_{AXA \cap AX*A}(x)\} = 1$. $\{(\nu_{AXA} \cup \nu_{AX*A})(x)\} = \{\nu_{AXA \cup AX*A}(x)\} = 0$. Thus $x \in AXA \cup AX*A$. This implies that $x \in A$, which is a contradiction. Thus, $\overline{\mu_A}(x) \geq \min\{\{(\overline{\mu_A} X \overline{\mu_A},)(x)\}, (\overline{\mu_A} X * \overline{\mu_A})\}(x)$ and $\nu_A(x) \leq \max\{\{(\nu_A X \nu_A,)(x)\}, (\nu_A X * \nu_A)\}(x)$. So, $f_A = \langle \overline{\mu_A}, \nu_A \rangle$ is a cubic bi-ideal of X.

Conversely, assume that f_A is a cubic bi-ideal of X. Let $x' \in AXA \cap AX*A$. Then $x' \in AXA$ and $x' \in AX*A$. Let $a_1, b, z, x_1 \in A$ and $a_2, x, y, x_2, z \in X$ be such that $x' \leq a_1 a_2 b \leq (xz - x(y-z), x) \leq x_1 x_2$. Now,

$$\begin{aligned}
\overline{\mu_A}(x') &\geq \{(\overline{\mu_A} X \overline{\mu_A} \cap \overline{\mu_A} X * \overline{\mu_A})(x')\} \\
&= \min[\{(\overline{\mu_A} X \overline{\mu_A})(x')\}, \{(\overline{\mu_A} X * \overline{\mu_A})(x')\}] \\
&= \min\{(\sup_{x' \leq ab} \min\{(\overline{\mu_A} X)(a), \overline{\mu_A}(b)\}, \sup_{x' \leq xz - x(y-z)} \min\{(\overline{\mu_A} X)(x), \overline{\mu_A}(z)\})\} \\
&= \min\{(\sup_{x' \leq ab} \min\{\sup_{a \leq a_1 a_2} \min\{\overline{\mu_A}(a_1), X(a_2)\}, \overline{\mu_A}(b)\}, \\
&\qquad \sup_{x' \leq xz - x(y-z)} \min\{\sup_{x \leq x_1 x_2} \min\{\overline{\mu_A}(x_1), X(x_2),\}, \overline{\mu_A}(z)\})\} \\
&= \min\{\{\overline{\mu_A}(a_1), \overline{\mu_A}(b), \overline{\mu_A}(x_1), f_B(z)\}\} = 1.
\end{aligned}$$

$$\begin{aligned}
\nu_A(x') &\leq \{(\nu_A X \nu_A \cup \nu_A X * \nu_A)(x')\} \\
&= \max[\{(\nu_A X \nu_A)(x')\}, \{(\nu_A X * \nu_A)(x')\}] \\
&= \max\{(\inf_{x' \leq ab} \max\{(\nu_A X)(a), \nu_A(b)\}, \inf_{x' \leq xz - x(y-z)} \max\{(\nu_A X)(x), \nu_A(z)\})\} \\
&= \min\{0, 0\} = 0.
\end{aligned}$$

This implies that $\nu_A(x') = 0$ and so $x' \in A$, that is, $AXA \cap AX*A \subseteq A$ and hence A is a bi-ideal of X.

Theorem 2.11.

Let $A = (\overline{\mu}, \nu)$ be a cubic subalgebra of X. If $AXA \subseteq A$ then A is a cubic bi-ideal of X.

Proof:

Assume that A is a Cubic subalgebra of X and $\overline{\mu}X\overline{\mu}\subseteq\overline{\mu}$, Let $x\in X$. Then $(\overline{\mu}X\overline{\mu}\cap\overline{\mu}X*\overline{\mu})(x)=\min\{(\overline{\mu}X\overline{\mu})(x),(\overline{\mu}X*\overline{\mu})(x)\}\le(\overline{\mu}X\overline{\mu})(x)\le\overline{\mu}(x).$

Thus $(\overline{\mu}X\overline{\mu}\cap\overline{\mu}X*\overline{\mu})(x)\subseteq\overline{\mu}$

and

$(\nu X\nu\cap\nu X*\nu)(x)=\max\{(\nu X\nu)(x),(\nu X*\nu)(x)\}\ge(\nu X\nu)(x)\ge\nu(x).$

Thus $(\nu X\nu\cap\nu X*\nu)(x)\supseteq\nu$.

Hence $A=(\overline{\mu},\nu)$ is a cubic bi-ideal of X.

Theorem 2.12.

If X is a zero-symmetric near-subtraction semi-group and $A=(\overline{\mu},\nu)$ be a cubic bi-ideal of X then $\overline{\mu}X\overline{\mu}\subseteq\overline{\mu}$ and $\nu X\nu\supseteq\nu$

Proof:

Let $A=(\overline{\mu},\nu)$ be a cubic bi-ideal of X.

Then $\overline{\mu}X\overline{\mu}\cap\overline{\mu}X*\overline{\mu}\subseteq\overline{\mu}$. Clearly $\overline{\mu}(0)\ge\overline{\mu}(x)$.

Thus $\overline{X}(0)\ge(\overline{X})(x)$ for all $x\in X$. Since X is a zero-symmetric near-subtraction semigroup, $\overline{\mu}X\overline{\mu}\subseteq\overline{\mu}X*\overline{\mu}$. So $\overline{\mu}X\overline{\mu}\cap\overline{\mu}X*\overline{\mu}=\overline{\mu}X\overline{\mu}\subseteq\overline{\mu}$, and $\nu X\nu\cap\nu X*\nu\subseteq\nu$. Clearly $\nu(0)\le\nu(x)$.

Thus $\nu X(0)\le(\nu X)(x)$ for all $x\in X$. Since X is a zero-symmetric near-subtraction semigroup,

$\nu X\nu\subseteq\nu X*\nu$. So $\nu X\nu\cap\nu X*\nu=\nu X\nu\supseteq\nu$, which is the required results.

Theorem 2.13.

Let X be a zero-symmetric near-subtraction semigroup and $A=(\overline{\mu},\nu)$ be a cubic subalgebra of X. Then the following conditions are equivalent:

(1) A is a cubic bi-ideal of X.

(2) $AXA\subseteq A$.

Proof:

The proof is straightforward from Theorem 2.12. and Theorem 2.13.

Theorem 2.14.

Let $A=(\overline{\mu},\nu)$ be a cubic bi-ideal of a zero-symmetric near-subtraction semigroup X. Then $\overline{\mu}(xyz)\ge\min\{\overline{\mu}(x),\overline{\mu}(z)\}$. and $\nu(xyz)\le\max\{\nu(x),\nu(z)\}$.

Proof.

Assume that A is a cubic bi-ideal of zero-symmetric near-subtraction semigroup X. By Theorem (2.13), $\min\{(\overline{\mu}X\overline{\mu})(x)\}\le\overline{\mu}(x)$ and $\max\{(\nu X\nu)(x)\}\ge\nu(x)\ \forall\, x\in X$. Let $x,y,z\in X$. Then

$$\begin{aligned}\overline{\mu}(xyz) &\ge\min\{(\overline{\mu}X\overline{\mu}(xyz)\}\\ &=\min\{\{\sup_{xyz=ab}(\overline{\mu}X)(a),\overline{\mu}(b)\}\}\\ &\ge\min\{(\overline{\mu}X)(xy),\overline{\mu}(b)\}\\ &\ge\min\{\overline{\mu}X(xy),\overline{\mu}(b)\}\\ &\ge\min\{\overline{\mu}(x)X(y),\overline{\mu}(b)\}\\ &=\min\{\overline{\mu}(x),\overline{\mu}(z)\}\end{aligned}$$

Therefore $\overline{\mu}(xyz) \geq \min\{\overline{\mu}(x), \overline{\mu}(z)\}$.

$$\begin{aligned} \nu(xyz) &\leq \max\{(\nu X \nu)(xyz)\} \\ &= \max\{\{\inf_{xyz=ab} \max(\nu X)(a), \nu(b)\}\} \\ &\leq \max\{(\nu X)(xy), \nu(b)\} \\ &\leq \max\{\nu X(xy), \nu(b)\} \\ &\leq \max\{\nu(x)X(y), \nu(b)\} \\ &= \max\{\nu(x), \nu(z)\} \end{aligned}$$

Therefore $\nu(xyz) \leq \max\{\nu(x), \nu(z)\}$.

Theorem 2.15.

Let $A = \langle \overline{\mu}, \nu \rangle$ be a cubic bi-ideal of a zero-symmetric near-subtraction semigroup X. Then the following are equivalent.

(1) $\overline{\mu}(xyz) \geq \min\{\overline{\mu}(x), \overline{\mu}(z)\}$, and $\nu(xyz) \leq \max\{\nu(x), \nu(z)\}$,

(2) $\{(\overline{\mu} X \overline{\mu})(x)\} \subseteq \overline{\mu}(x)$ and $\{(\nu X \nu)(x)\} \supseteq \nu(x)\ \forall\, x \in X$.

Proof.

Let $A = \langle \overline{\mu}, \nu \rangle$ be a cubic bi-ideal of zero-symmetric near-subtraction semigroup X. Let $x' \in X$.

(1) $\Rightarrow$ (2): If there exist $x, y, x_1, x_2 \in X$ such that $x' = xy$ and $x = x_1 x_2$. Then by hypothesis, $\overline{\mu}(x_1 x_2 y) \geq \min\{\overline{\mu}(x_1), \overline{\mu}(y)\}$ and $\nu(x_1 x_2 y) \leq \max\{\nu(x_1), \nu(y)\}$. We have

$$\begin{aligned} \{(\overline{\mu} X \overline{\mu})(x')\} &= \{\sup_{x' \leq xy} \min\{(\overline{\mu} X)(x), \overline{\mu}(y)\}\} \\ &= \{\sup_{x' \leq xy}\{ \sup_{x \leq x_1 x_2} \min\{\overline{\mu}(x_1), X(x_2)\}, \overline{\mu}(y)\}\} \\ &= \{\sup_{x' \leq xy} \min\{ \sup_{x \leq x_1 x_2} \min\{\overline{\mu}(x_1), 1\}, \overline{\mu}(y)\}\} \\ &= \sup_{x' \leq x_1 x_2 y} \min\{\overline{\mu}(x_1), \overline{\mu}(y)\} \\ &\leq \sup_{x' \leq x_1 x_2 y} \overline{\mu}(x_1 x_2 y) = \overline{\mu}(x_1 x_2 y) = \overline{\mu}(x'). \end{aligned}$$

So, $\min\{(\overline{\mu} * \overline{\mu})(x)\} \leq \overline{\mu}(x)$. Also

$$\begin{aligned} \{(\nu X \nu)(x')\} &= \{\inf_{x' \leq xy} \max\{(\nu X)(x), \nu(y)\}\} \\ &= \{\inf_{x' \leq xy} \max\{ \inf_{x x_1 x_2} \max\{\nu(x_1), X(x_2)\}, \nu(y)\}\} \\ &= \{\inf_{x' \leq xy} \max\{ \inf_{x \leq x_1 x_2} \max\{\nu(x_1), 0\}, \nu(y)\}\} \\ &\geq \inf_{x' \leq x_1 x_2 y} \{\nu(x_1), \nu(y)\} \\ &\geq \inf_{x' = x_1 x_2 y} \nu(x_1 x_2 y) = \nu(x_1 x_2 y) = \nu(x'). \end{aligned}$$

So, $\min\{(\overline{\mu}X(\overline{\mu})(x)\} \leq \overline{\mu}(x)$. and $\max\{(\nu*(\nu)(x)\} \geq \nu(x)$. Thus (2) holds.

(2)$\Rightarrow$(1): Assume that $\{(\overline{\mu}X\overline{\mu})(x)\} \leq \overline{\mu}(x)$ and $(\nu X\nu)(x) \geq \nu(x)$. Let $x, y, z, x' \in X$ be such that $x' \leq xyz$. Then

$$\begin{aligned}\overline{\mu}(xyz)(x') &\geq \{(\overline{\mu}X\overline{\mu})(xyz)\} \\ &= \min\{\sup_{x'\leq xyz}\{(\overline{\mu}X)(xy), \overline{\mu}(z)\}\} \\ &\geq \min\{\overline{\mu}(x), X(y), \overline{\mu}(z)\} \\ &= \min\{\overline{\mu}(x), 1, \overline{\mu}(z)\} = \min\{\overline{\mu}(x), \overline{\mu}(z)\}.\end{aligned}$$

and aslo, Assume that $\max\{(\nu X\nu)(x)\} \geq \nu(x)$. Let $x, y, z, x' \in X$ be such that $x' = xyz$. Then

$$\begin{aligned}\nu(xyz) &\leq \{(\nu X\nu)(xyz)\} \\ &= \{\inf_{x'\leq xyz} \max\{(\nu X)(xy), \nu(z)\}\} \\ &\leq \max\{\nu(x), X(y), \nu(z)\} \\ &= \max\{\nu(x), 0, \nu(z)\} = \max\{\nu(x), \nu(z)\}.\end{aligned}$$

Therefore (1) holds.

Theorem 2.16.

Let $A = (\overline{\mu}_A, \nu_A)$ and $B = \overline{\mu}_B, \nu_B$ be any two cubic bi-ideals of X. Then $B \cap A$ is also a cubic bi-ideal of X.

Proof.

Let A and B be any two cubic bi-ideals of X. Let $x, y \in X$.

$$\begin{aligned}(\overline{\mu}_A \cap \overline{\mu}_B)(x-y) &= \min\{\overline{\mu}_A(x-y), \overline{\mu}_B(x-y)\} \\ &\geq \min\{(\min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}, \min\{\overline{\mu}_B(x), \overline{\mu}_B(y)\})\} \\ &= \min\{(\min\{\overline{\mu}_A(x), \overline{\mu}_B(x)\}, \min\{\overline{\mu}_A(y), \overline{\mu}_B(y)\})\} \\ &= \min\{((\overline{\mu}_A \cap \overline{\mu}_B)(x), (\overline{\mu}_A \cap \overline{\mu}_B)(y))\}.\end{aligned}$$

$$\begin{aligned}(\nu_A \cup \nu_B)(x-y) &= \max\{\nu_A(x-y), \nu_B(x-y)\} \\ &\leq \max\{(\max\{\nu_A(x), \nu_A(y)\}, \max\{\nu_B(x), \nu_B(y)\})\} \\ &= \max\{(\max\{\nu_A(x), \nu_B(x)\}, \max\{\nu_A(y), \nu_B(y)\})\} \\ &= \max\{((\overline{\nu}_A \cup \overline{\nu}_B)(x), (\overline{\nu}_A \cup \overline{\nu}_B)(y))\}.\end{aligned}$$

Let $x' \in X$. Choose $a, b, x, y, z \in X$ such that $x' \leq abc \leq xz - x(y-z)$. Since A and B are cubic bi-ideals of X, we have

$$\min\{\{\sup_{x'\leq abc} \min\{\overline{\mu}_A(a), \overline{\mu}_A(c)\}, \sup_{x'\leq xz-x(y-z)} \overline{\mu}_A(z)\}\} \leq \overline{\mu}_A(x) \quad (1)$$

$$\min\{\{\sup_{x'\leq abc} \min\{\overline{\mu}_B(a), \overline{\mu}_B(c)\}, \sup_{x'\leq xz-x(y-z)} \overline{\mu}_B(z)\}\} \leq \overline{\mu}_B(x) \quad (2)$$

$$\max\{\inf_{x\leq abc} \max\{\nu_A(a), \nu_A(b), \sup_{x'\leq xz-x(y-z)} \nu_A(z)\}\} \geq \nu_A(x) \quad (3)$$

$$\max\{\inf_{x\leq abc} \max\{\nu_B(a), \nu_B(b), \sup_{x'\leq xz-x(y-z)} \nu_B(z)\}\} \geq \nu_B(x) \quad (4)$$

Now

$$\min\{\{((\overline{\mu}_A \cap \overline{\mu}_B) X (\overline{\mu}_A \cap \overline{\mu}_B))(x'), ((\overline{\mu}_A \cap \overline{\mu}_B) X * (\overline{\mu}_A \cap \overline{\mu}_B))(x')\}\}$$
$$= \min\{\{\sup_{x' \le abc} \min\{(\overline{\mu}_A \cap \overline{\mu}_B)(a), (\overline{\mu}_A \cap \overline{\mu}_B)(c)\},$$
$$\sup_{x' \le xz - x(y-z)} (\overline{\mu}_A \cap \overline{\mu}_B)(z)\}\}$$
$$= \min\{\sup_{x' \le abc} \min\{\min\{\overline{\mu}_A(a), \overline{\mu}_B(a)\}, \min\{\overline{\mu}_A(c), \overline{\mu}_B(c)\}\},$$
$$\sup_{x' \le xz - x(y-z)} \min\{\overline{\mu}_A(z), \overline{\mu}_B(z)\}\}$$
$$= \min\{\min\{\sup_{x' \le abc} \min\{\overline{\mu}_A(a), \overline{\mu}_B(c)\}, \sup_{x' \le xz - x(y-z)} \{\overline{\mu}_A(z)\}\},$$
$$\min\{\sup_{x' \le abc} \min\{\overline{\mu}_B(a), \overline{\mu}_B(c)\}, \sup_{x' \le xz - x(y-z)} \mu_B(z)\}\}$$
$$\le \min\{\overline{\mu}_A(x), \overline{\mu}_B(x)\}, \text{from(1)and(2)} = (\overline{\mu}_A \cap \overline{\mu}_B)(x).$$

$$\max\{\{((\nu_A \cup \nu_B) X (\nu_A \cup \nu_B))(x'), ((\nu_A \cup \nu_B) X * (\nu_A \cup \nu_B))(x')\}\}$$
$$= \max\{\{\inf_{x' \le abc} \max\{(\nu_A \cup \nu_B)(a), (\nu_A \cup \nu_B)(c)\},$$
$$\inf_{x' \le xz - x(y-z)} (\nu_A \cup \nu_B)(z)\}\}$$
$$= \min\{\inf_{x' \le abc} \max\{\max\{\nu_A(a), \nu_A(a)\}, \max\{\nu_B(c), \nu_B(c)\}\},$$
$$\inf_{x' \le xz - x(y-z)} \max\{\nu_A(z), \nu_B(z)\}\}$$
$$= \min\{\max\{\inf_{x' \le abc} \max\{\nu_A(a), \nu_A(c)\}, \inf_{x' \le xz - x(y-z)} \{\nu_A(z)\}\},$$
$$\min\{\inf_{x' = abc} \max\{\nu_B(a), \nu_B(c)\}, \inf_{x' = xz - x(y-z)} \nu_B(z)\}\}$$
$$\ge \min\{\nu_A(x), \nu_B(x)\} = (\nu_A \cup \nu_B)(x) \text{from(3)and(4)}.$$

Thus $\overline{\mu}_A \cap \overline{\mu}_B$ and $\nu_A \cup \nu_B$ are cubic bi-ideal of X.

3. CONCLUSION:

In this paper cubic bi-ideals in near-subtraction near-ring has been introduced and some results are discussed.

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