DECOMPOSITION OF NEUTROSOPHIC FUZZY MATRICES

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Abstract: In this paper, we study some properties of modal operators in Neutrosophic fuzzy matrix and we introduce a new composition operation and discuss some of its algebraic properties. Finally, we obtain a decomposition of a Neutrosophic fuzzy matrix by using the new composition operation and modal operators.

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1. INTRODUCTION:

In dealing with uncertainties many theories have been recently developed, including the Theory of Probability, Theory of Intuitionistic Fuzzy Sets and Theory of Rough Sets and so on. Although many new techniques have developed as a result of these theories, yet difficulties are still there. The major difficulties arise due to inadequacy of parameters.

The fuzzy set was introduced by Zadeh [14] in 1965 where each element had a degree of membership. The intuitionistic fuzzy set (IFS) was introduced by Atanaasov [1] in 1986 as a generalization of fuzzy set, where besides the degree of membership and the degree of non-membership of each element. It does not handle the indeterminate and inconsistent information which exist in belief system.

In 1995, Smarandache starting from philosophy (when we are fretted to distinguish between absolute truth and relative truth or between absolute falsehood and relative falsehood in logics, and respectively between absolute membership and relative membership or absolute non-membership and relative non-membership in set theory) that time began to use the non-standard analysis. Also, inspired from the sport games (winning, defeating, or tie scores), from votes (pro, contra, null/black votes), from decision making and control theory (making a decision, not making, or hesitating), from accepted/rejected/pending, etc., and guided by the fact that the law of excluded middle did not work any longer in the modern logics. So how to deal with all of them at once is it possible to unit them?. This type of situations well managed by Neutrosophic Set(NS), where indeterminacy is quantified explicitly and truth, indeterminacy, and falsity membership are independent to each other NS provides a more reasonable mathematical framework to deal with indeterminate and inconsistent information. During the last decade, the concept of NS and interval neutrosophic set (INS) have been used in various application such as Medical Diagnosis, Database, Topology, image processing Guo and Sengur [6] and decision making problems Broumi and Smarandache [3].

Smarandache [11] first introduced neutrosophy as a branch of philosophy which studies the origin, nature, and scope of neutralities. Smarandache defined indeterminacy explicitly and state that truth, indeterminacy and falsity membership are independent and lies within $]^{-}0,1^{+}[$, which is the non-standard unit interval and an extension of the standard interval [0,1]. It is generalization of intuitionistic fuzzy sets.

Extremal algebra, in which the addition and multiplication of vectors and matrices is formally replaced by operations of maximum and minimum, or maximum and plus, are a useful tool for approaching problems in many areas, such as System Theory, Graph Theory, Scheduling Theory or knowledge Engineering. Systematic investigation in this direction can be found in [2,4,13].

Kim and Roush [7] introduced the concept of Fuzzy Matrix(FM). FM plays a vital role in various areas in Science and Engendering and solves the problems involving various types of uncertainties [10]. FMs deal only with membership value where as Intuitionistic Fuzzy Matrices(IFMs) deals with both membership and non-membership values.

Dhar et.al, [5] have defined neutrosophic fuzzy matrices and studied square neutrosophic fuzzy matrices. Kavitha et.al, [8]studied the concepts of minimal solution of fuzzy neutrosophic soft matrix. They, also studied on the powers of fuzzy neutrosophic soft matrices in [9]. Uma et.al, [12] introduced two types of fuzzy neutrosophic soft matrices.

In this paper we extend the model operator concept to neutrosophic fuzzy matrix and give some fascinating results in transitive closure of neutrosophic fuzzy matrix.

2. PRELIMINARY

In this section the basic definitions of Neutrosophic Set (NS), , Fuzzy Neutrosophic Matrix (FNM) and fuzzy neutrosophic matrices of type-I are provided.

Definition 2.1.

[11]A neutrosophic set A on the universe of discourse X is defined as $A=\{\langle x,T_A(x),I_A(x),F_A(x)\rangle;x\in X\}$, where $T,I,F:X\rightarrow]^{-}0,1^{+}[$ and

$$^{-}0\leq T_A(x)+I_A(x)+F_A(x)\leq 3^{+}. \qquad (2.1)$$

From philosophical point of view the neutrosophic set takes the value from real standard or non-standard subsets of $]^{-}0,1^{+}[$. But in real life application especially in Scientific and Engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^{-}0,1^{+}[$. Hence we consider the neutrosophic set which takes the value from the subset of $[0,1]$. Therefore we can rewrite equation (2.1) as $0\leq T_A(x)+I_A(x)+F_A(x)\leq 3.$ In short an element a in the neutrosophic set A, can be written as $a=\langle a^T,a^I,a^F\rangle$, where a^T denotes degree of truth, a^I denotes degree of indeterminacy, a^F denotes degree of falsity such that $0\leq a^T+a^I+a^F\leq 3.$

Example 2.1

Assume that the universe of discourse $X=\{x_1,x_2,x_3\}$ where x_1,x_2 and x_3 characterize the quality, reliability, and the price of the objects. It may be further assumed that the values of $\{x_1,x_2,x_3\}$ are in [0,1] and they are obtained from some investigations of some experts. The experts may impose their opinion in three components viz; the degree of goodness, the degree of indeterminacy and the degree of poorness to explain the characteristics of the objects. Suppose A is a Neutrosophic Set (NS) of X, such that $A=\{\langle x_1,0.4,0.5,0.3\rangle,\langle x_2,0.7,0.2,0.4\rangle,\langle x_3,0.8,0.3,0.4\rangle\}$ where for x_1 the degree of goodness of quality is 0.4, degree of indeterminacy of quality is 0.5 and degree of falsity of quality is 0.3 etc.

Definition 2.2.

A neutrosophic matrix is a matrix in which all entries are from neutrosophic set. That is $A=(\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle)$

Definition 2.3.

[12] Let $A=(\langle a_{ij}^{T},a_{ij}^{I},a_{ij}^{F}\rangle),B=(\langle b_{ij}^{T},b_{ij}^{I},b_{ij}^{F}\rangle)\in \mathrm{N}_{m\times n}$. The component wise addition and component wise multiplication is defined as

$$A\oplus B=(sup\{a_{ij}^{T},b_{ij}^{T}\},\ \ sup\{a_{ij}^{I},b_{ij}^{I}\},\ \ inf\{a_{ij}^{F},b_{ij}^{F}\})$$

$$A\otimes B=(inf\{a_{ij}^{T},b_{ij}^{T}\},\ \ inf\{a_{ij}^{I},b_{ij}^{I}\},\ \ sup\{a_{ij}^{F},b_{ij}^{F}\})$$

Definition 2.4.

Let $A\in \mathrm{N}_{m\times n},B\in \mathrm{N}_{n\times p}$, the composition of A and B is defined as

$$A\circ B=\left(\sum_{k=1}^{n}(a_{ik}^{T}\wedge b_{kj}^{T}),\ \sum_{k=1}^{n}(a_{ik}^{I}\wedge b_{kj}^{I}),\ \prod_{k=1}^{n}(a_{ik}^{F}\vee b_{kj}^{F})\right)$$

equivalentlywecanwritethesameas

$$=\left(\bigvee_{k=1}^{n}(a_{ik}^{T}\wedge b_{kj}^{T}),\ \bigvee_{k=1}^{n}(a_{ik}^{I}\wedge b_{kj}^{I}),\ \bigwedge_{k=1}^{n}(a_{ik}^{F}\vee b_{kj}^{F})\right).$$

The product $A\circ B$ is defined if and only if the number of columns of A is same as the number of rows of B. Then A and B are said to be conformable for multiplication. We shall use AB instead of $A\circ B$.

Where $\sum(a_{ik}^{T}\wedge b_{kj}^{T})$ means max-min operation and

$\prod_{k=1}^{n}(a_{ik}^{F}\vee b_{kj}^{F})$ means min-max operation.

Definition 2.5.

Let $\langle x^T,x^I,x^F\rangle,\ \langle y^T,y^I,y^F\rangle\ \in NFS$. *Then we have*

- $\langle x^T,x^I,x^F\rangle\vee\langle y^T,y^I,y^F\rangle=\langle max\{x^T,y^T\},max\{x^I,y^I\},min\{x^F,y^F\}\rangle.$
- $\langle x^T,x^I,x^F\rangle\wedge\langle y^T,y^I,y^F\rangle=\langle min\{x^T,y^T\},min\{x^I,y^I\},max\{x^F,y^F\}\rangle.$
- $\langle x^T,x^I,x^F\rangle^c=\langle x^F,x^I,x^T\rangle.$

Definition 2.6.

Let $\langle x^T,x^I,x^F\rangle,\ \langle y^T,y^I,y^F\rangle\ \in NFS$. *Then*

$$\langle x^T,x^I,x^F\rangle\leftarrow\langle y^T,y^I,y^F\rangle=\begin{cases}\langle 1,1,0\rangle, & \text{if } \langle x^T,x^I,x^F\rangle\geq\langle y^T,y^I,y^F\rangle,\\ \langle x^T,x^I,x^F\rangle, & \text{if } \langle x^T,x^I,x^F\rangle\leq\langle y^T,y^I,y^F\rangle.\end{cases}$$

Here $\langle x^T,x^I,x^F\rangle\geq\langle y^T,y^I,y^F\rangle$ means $x^T\geq y^T,x^I\geq y^I$, and $x^F\leq y^F$

Definition 2.7.

An NFM $J=(\langle 1,1,0\rangle)$ *for all entries is known as the universal matrix and NFM O=(* $\langle 0,0,1\rangle$ *) for all entries is known as Zero matrices. Denote the set of all NFMs of order* $m\times n$ *by* F_{mn} *and square matrix of order n by* F_{n}. *The idenity NFM I=* $\langle \delta_{ij}^{T},\delta_{ij}^{I},\delta_{ij}^{F}\rangle$ *is defined by* $\langle \delta_{ij}^{T},\delta_{ij}^{I},\delta_{ij}^{F}\rangle=\langle 1,1,0\rangle$ *if i=j and* $\langle \delta_{ij}^{T},\delta_{ij}^{I},\delta_{ij}^{F}\rangle=\langle 0,0,1\rangle$ *if* $i\neq j$.

Definition 2.8.

Let $A=(\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle)_{m\times n}$, $B=(\langle b_{ij}^T,b_{ij}^I,b_{ij}^F\rangle)_{m\times n}$ *and* $C=(\langle c_{ij}^T,c_{ij}^I,c_{ij}^F\rangle)_{n\times p}$ *are NFMs. Then*

- $A\vee B=(\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle\vee\langle b_{ij}^T,b_{ij}^I,bij^F\rangle)$.
- $A\wedge B=(\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle\wedge\langle b_{ij}^T,b_{ij}^I,b_{ij}^F\rangle)$.
- AC (max-min composition) $=(\bigvee_K(\langle a_{ij}^T,a_{ij}^I,{}_{ij}^F\rangle\wedge\langle c_{kj}^T,c_{kj}^I,c_{kj}^F\rangle))$.

Definition 2.9.

Let $A=(\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle)_{m\times n}$ *and* $C=(\langle c_{ij}^T,c_{ij}^I,c_{ij}^F\rangle)_{n\times p}$ *are NFMs. Then we have*

- $A\Diamond C$ (min-max composition) $=(\bigwedge_k(\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle\vee\langle c_{ij}^T,c_{ij}^I,c_{ij}^F\rangle))$.
- $A^T=(\langle a_{ji}^T,a_{ji}^I,a_{ji}^F\rangle)$ (Transpose of A).
- $A\leftarrow C=(\bigwedge_k(\langle a_{ik}^T,a_{ik}^I,a_{ik}^F\rangle\leftarrow\langle c_{kj}^T,a_{kj}^I,a_{kj}^F\rangle))$.
- $A\rightarrow C=(\bigwedge_k(\langle a_{ik}^T,a_{ik}^I,a_{ik}^F\rangle\rightarrow\langle c_{kj}^T,c_{kj}^I,c_{kj}^F\rangle))$.
- $A^c=(\langle a_{ij}^F,a_{ij}^I,a_{ij}^T\rangle)$ (Complement of A).

Also we can use $AC=(\left\langle\sum_{K=1}^{n}(a_{ik}^Tc_{kj}^T),\sum_{K=1}^{n}(a_{ik}^Ic_{kj}^I),\prod_{K=1}^{n}(a_{ik}^F+c_{kj}^F)\right\rangle)$.

Also $A^2=AA, A^k=A^{k-1}A$ for max-min composition and $A^{[2]}=A\Diamond A, A^{[k]}=A^{[k-1]}\Diamond A$ for min-max composition.

Definition 2.10.

For any NFM $A\in F_n$,

- A is reflexive if and only if $A\geq I_n$.
- A is symmetric if and only if $A=A^T$.
- A is transitive if and only if $A\geq A^2$.
- A is idempotent if and only if $A=A^2$.
- A is irreflexive if $\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle=\langle 0,0,1\rangle$ for all i=j.
- A is c-transitive if $A\leq A^{[2]}$.

Definition 2.11.

An NFM A is said to be an neutrosophic fuzzy equivalence matrix if it satisfy reflexivity, symmetry and transitivity.

Proposition 2.1.

$(A\circ B)^c=A^c+B^c$ for A,B $\in F_{mn}$.

Proposition 2.2.

$(A+B)^c=A^c\circ B^c$ for A,B $\in F_{mn}$.

Definition 2.12.

For an NFM A, define $WA=(\langle a_{ij}^T,a_{ij}^I,1-a_{ij}^T\rangle),\Diamond A=(\langle 1-a_{ij}^F,a_{ij}^I,a_{ij}^F\rangle)$.

Lemma 2.1.

$$1-\prod_{k=1}^{n}(a_{ik}+b_{kj})=\sum_{k=1}^{n}(1-a_{ik})(1-b_{kj}) \text{ for all } i,j,a_{ij},b_{ij}\in[0,1].$$

Lemma 2.2.

$$1-\prod_{k=1}^{n}(a_{ik}b_{kj})=\prod_{k=1}^{n}((1-a_{ik})+(1-b_{ij})) \text{ for all } i,j,a_{ij},b_{ij}\in[0,1].$$

3. MODAL OPERATORS IN NFM

Throughout this section, matrices means Neutrosophic fuzzy matrices. In this section, some results about model operators are proved and the definitions of transitive and c-transitive of a NFM are given.

Lemma 3.1.

For any two NFMs A and B,

$$W(\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle\leftarrow\langle b_{ij}^T,b_{ij}^I,b_{ij}^F\rangle)=W(\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle)\leftarrow W(\langle b_{ij}^T,b_{ij}^I,b_{ij}^F\rangle). \quad (3.1)$$

Proof:

- If $\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle\geq\langle b_{ij}^T,b_{ij}^I,b_{ij}^F\rangle$, then

$$W(\langle a_{ik}^T,a_{ik}^I,a_{ik}^F\rangle\leftarrow\langle b_{kj}^T,b_{kj}^I,b_{kj}^F\rangle)=W(\langle 1,1,0\rangle)=\langle 1,1,0\rangle. \quad (3.2)$$

Since, $\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle\geq\langle b_{ij}^T,b_{ij}^I,b_{ij}^F\rangle$, $a_{ij}^T\geq b_{ij}^T$, $a_{ij}^I\geq b_{ij}^I$ and $a_{ij}^F\leq b_{ij}^F$. Therefore, $1-a_{ij}^T\leq 1-b_{ij}^T$ and $\langle a_{ij}^T,a_{ij}^I,1-a_{ij}^T\rangle\geq\langle b_{ij}^T,b_{ij}^I,1-b_{ij}^T\rangle$, so $W(\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle)\geq W(\langle b_{ij}^T,b_{ij}^I,b_{ij}^F\rangle)$. Thus

$$W\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle\leftarrow W\langle b_{ij}^T,b_{ij}^I,b_{ij}^F\rangle=\langle 1,1,0\rangle. \quad (3.3)$$

From (3.2) and (3.3), (3.1) holds.

- If $\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle\leq\langle b_{ij}^T,b_{ij}^I,b_{ij}^F\rangle$, then

$$W(\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle\leftarrow\langle b_{ij}^T,b_{ij}^I,b_{ij}^F\rangle)=W\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle=\langle a_{ij}^T,a_{ij}^I,1-a_{ij}^T\rangle. \quad (3.4)$$

$$W\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle\leftarrow W\langle b_{ij}^T,b_{ij}^I,b_{ij}^F\rangle=\langle a_{ij}^T,a_{ij}^I,1-a_{ij}^T\rangle\leftarrow\langle b_{ij}^T,b_{ij}^I,1-b_{ij}^T\rangle=\langle a_{ij}^T,a_{ij}^I,1-a_{ij}^T\rangle. \quad (3.5)$$

Clearly, from (3.4) and (3.5),(3.1) holds.

Lemma 3.2.

For any two NFMs A and B,

$$\Diamond(\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle\leftarrow\langle b_{ij}^T,b_{ij}^I,b_{ij}^F\rangle)=\Diamond(\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle\leftarrow\Diamond\langle b_{ij}^T,b_{ij}^I,b_{ij}^F\rangle). \quad (3.6)$$

Proof:

- If $\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle\geq\langle b_{ij}^T,b_{ij}^I,b_{ij}^F\rangle$, then

$$\Diamond(\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle\leftarrow\langle b_{ij}^T,b_{ij}^I,b_{ij}^F\rangle)=\Diamond(\langle 1,1,0\rangle)=\langle 1,1,0\rangle. \quad (3.7)$$

Since, $\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle\geq\langle b_{ij}^T,b_{ij}^I,b_{ij}^F\rangle$, $a_{ij}^T\geq b_{ij}^T$, $a_{ij}^I\geq b_{ij}^I$ and $a_{ij}^F\leq b_{ij}^F$ .Therefore $1-a_{ij}^T\leq 1-b_{ij}^T$ and $\langle a_{ij}^T,a_{ij}^I,1-a_{ij}^T\rangle\geq\langle b_{ij}^T,b_{ij}^I,1-b_{ij}^T\rangle$, So $\Diamond(\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle)\geq\Diamond(\langle b_{ij}^T,b_{ij}^I,b_{ij}^F\rangle)$. Thus,

$$\Diamond\langle a_{ij}^T,a_{ij}^I,a_{ij}^F\rangle\leftarrow\Diamond\langle b_{ij}^T,b_{ij}^I,b_{ij}^F\rangle=\langle 1,1,0\rangle \quad (3.8)$$

From (3.7) and (3.8), (3.6) holds,

• If $\langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle < \langle b_{ij}^T, b_{ij}^I, b_{ij}^F \rangle$. Then

$$\Diamond(\langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle \leftarrow \langle b_{ij}^T, b_{ij}^I, b_{ij}^F \rangle) = \Diamond\langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle = \langle 1-a_{ij}^F, a_{ij}^I, a_{ij}^F \rangle. \tag{3.9}$$

$$\Diamond\langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle \leftarrow \Diamond\langle b_{ij}^T, b_{ij}^I, b_{ij}^F \rangle = \langle 1-a_{ij}^F, a_{ij}^I, a_{ij}^F \rangle \leftarrow \langle 1-b_{ij}^F, b_{ij}^I, b_{ij}^F \rangle = \langle 1-a_{ij}^F, a_{ij}^I, a_{ij}^F \rangle. \tag{3.10}$$

Clearly from (3.9) and (3.10),(3.6) holds.

Lemma 3.3.

A is reflexive matrix if and only if WA is reflexive matrix

Proof:

A is reflexive $\Leftrightarrow A \geq I \Leftrightarrow \langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle \geq \langle \delta_{ij}^T, \delta_{ij}^I, \delta_{ij}^F \rangle$ for all i,j. $\Leftrightarrow \langle a_{ij}^T, a_{ij}^I, 1-a_{ij}^T \rangle \geq \langle \delta_{ij}^T, \delta_{ij}^I, 1-\delta_{ij}^T \rangle$ for all i, j. $\Leftrightarrow WA \geq WI \Leftrightarrow WA$ is reflexive. In dual way we can prove the following lemma.

Lemma 3.4.

A is reflexive matrix if and only if $\Diamond A$ is reflexive matrix.

Lemma 3.5.

A is reflexive if and only if WA^c is reflexive.

Proof: It is evident that if A is reflexive if and only if A^c is reflexive and so WA^c.
Similarly, $\Diamond A^c$ is irreflexive if and only if A is reflexive.

Lemma 3.6.

A is symmetric matrix if and only if WA is symmetric matrix and so WA^c.

Proof: A is symmetric $\Leftrightarrow \langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle = \langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle$ for all i, j $\Leftrightarrow \langle a_{ij}^T, a_{ij}^I, 1-a_{ij}^T \rangle = \langle a_{ij}^T, a_{ij}^I, 1-a_{ij}^T \rangle \Leftrightarrow WA = (WA)^T$. Thus A is symmetric if and only if WA is symmetric. Similarly we can prove the following lemma.

Lemma 3.7.

A is symmetric matrix if and only if $\Diamond A$ is reflexive matrix.

Lemma 3.8.

A is transitive matrix if and only if WA is transitive matrix.

Proof:

A is transitive $\Leftrightarrow A \geq A^2 \Leftrightarrow \langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle \geq \left\langle \sum_{k=1}^{n}(a_{ik}^T a_{kj}^T), \sum_{k=1}^{n}(a_{ik}^I a_{kj}^I), \prod_{k=1}^{n}(a_{ik}^F + a_{kj}^F) \right\rangle$ for all i,j

$$\Leftrightarrow a_{ij}^T \geq \sum_{k=1}^{n}(a_{ik}^T, a_{kj}^T), a_{ij}^I \geq \sum_{k=1}^{n}(a_{ik}^I a_{kj}^I), a_{ij}^F \leq \prod_{k=1}^{n}(a_{ik}^F + a_{kj}^F) \Leftrightarrow a_{ij}^T \geq \sum_{k=1}^{n}(a_{ik}^T a_{kj}^T), 1-a_{ij}^T \leq 1-\sum_{k=1}^{n}(a_{ik}^T a_{kj}^T) \Leftrightarrow \langle a_{ij}^T, 1-a_{ij}^T \rangle \geq$$

by lemma 2.2 Similarly, we can prove the following lemma.

Lemma 3.9.

A is transitive matrix if and only if $\Diamond A$ is transitive matrix.

Lemma 3.10.

A is idempotent matrix if and only if WA is idempotent matrix.

Proof:

$$\text{A idempotent } \Leftrightarrow A = A^2 \Leftrightarrow \left\langle a_{ij}^T, a_{ij}^I, a_{ij}^F \right\rangle = \left\langle \sum_{k=1}^{n}(a_{ik}^T a_{kj}^T), \sum_{k=1}^{n}(a_{ik}^I a_{kj}^I), \prod_{k=1}^{n}(a_{ik}^F + a_{kj}^F) \right\rangle \text{ for all i,j.}$$

$$\Leftrightarrow \left\langle a_{ij}^T, a_{ij}^I, 1-a_{ij}^T \right\rangle = \left\langle \sum_{k=1}^{n}(a_{ik}^T a_{kj}^T), \sum_{k=1}^{n}(a_{ik}^T a_{kj}^T), 1-\sum_{k=1}^{n}(a_{ik}^T a_{kj}^T) \right\rangle \Leftrightarrow \left\langle \sum_{k=1}^{n}(a_{ik}^T a_{kj}^T), \sum_{k=1}^{n}(a_{ik}^T a_{kj}^T), \prod_{k=1}^{n}((1-a_{ik}^T)+(1-a_{kj}^T)) \right\rangle$$

by Lemma 2.2 $\Leftrightarrow WA = (WA)^2$. Thus A is idempotent $\Leftrightarrow WA$ is idempotent. The following lemma is trivial from the above.

Lemma 3.11.

A is an idempotent matrix if and only if $\Diamond A$ is an idempotent matrix.

Remark 3.1.

If A is an neutrosophic fuzzy equivalence matrix, then WA and $\Diamond A$ are also neutrosophic fuzzy equivalence matrices.

Definition 3.1.

Let A $\in$ F the transitive closure and c-transitive closure of A is defined by $A^{\infty} = A \vee A^2 \vee A^3 \vee \ldots \vee A^n$ and $A_{\infty} = A^c \wedge (A^c)^{[2]} \wedge \ldots \wedge (A^c)^{[n]}$ respectively.

Theorem 3.1.

For $A \in \mathrm{F}_n$, $A_{\infty} = (A^{\infty})^c$.

Proof:

By definition 3.1, $(A^{\infty})^c = (A \vee A^2 \vee A^3 \vee \ldots \vee A^n)^c = (A^c \wedge (A^2)^c \wedge \ldots \wedge (A^n)^c)$.

first let us prove $(A^2)^c = (A^c)^{[2]}$. We know that $A^2 = \left\langle \sum_{k=1}^{n}(a_{ik}^T a_{kj}^T), \sum_{k=1}^{n}(a_{ik}^I a_{kj}^I), \prod_{k=1}^{n}(a_{ik}^F + a_{kj}^F) \right\rangle$ and so

$$(A^2)^c = \left\langle \prod_{k=1}^{n}(a_{ik}^F + a_{kj}^F), \sum_{k=1}^{n}(a_{ik}^I a_{kj}^I), \sum_{k=1}^{n}(a_{ik}^T a_{kj}^T) \right\rangle. \tag{3.11}$$

Also $A^c = \left\langle a_{ij}^F, a_{ij}^I, a_{ij}^T \right\rangle$ gives by the definition of $A^[2]$,

$$(A^c)^2 = \langle \prod_{k=1}^{n}(a_{ik}^F + a_{kj}^F), \sum_{k=1}^{n}(a_{ik}^I a_{kj}^I), \sum_{k=1}^{n}(a_{ik}^T a_{kj}^T) \tag{3.12}$$

Thus by (3.11) and (3.12) $(A^2)^c = (A^c)^{[2]}$, so in general $(A^n)^c = (A^c)^{[n]}$. By Definition 3.1, $(A^{\infty})^c = (A \vee A^2 \vee A^3 \vee \ldots \vee A^n)^c = (A \vee A^2 \vee A^3 \vee \ldots \vee A^n)^c = (A^c \wedge (A^2)^c \wedge \ldots \wedge A^n)^c = (A^c) \wedge (A^c)^2 \wedge \ldots \wedge (A^c)^n)$.

Lemma 3.12.

A is transitive if and only if A^c is c-transitive and so WA^c is transitive.

Proof:

It is evident from the definition of transitive and c-transitive.

Lemma 3.13.

If A is reflexive NFM, then

- A^T is reflexive.
- $A \vee B$ is reflexive.

- $A \wedge B$ is reflexive if and only if B is reflexive.

Proof:

(i) and (ii) are obvious from the definition of reflexive. (iii) If B is not reflexive, then $\langle b_{ii}^T, b_{ii}^I, b_{ii}^F \rangle \neq \langle 1,1,0 \rangle$ for at least one i, that is $\langle b_{ii}^T, b_{ii}^I, b_{ii}^F \rangle < \langle 1,1,0 \rangle$. Thus $\langle a_{ii}^T, a_{ii}^I, a_{ii}^F \rangle \wedge \langle b_{ii}^T, b_{ii}^I, b_{ii}^F \rangle < \langle 1,1,0 \rangle$. Therefore the condition B is reflexive is necessary, the sufficient part is trivial.

Theorem 3.2.

If $A, B \in \mathrm{F}_n$ where A is reflexive and symmetric, B is reflexive, symmetric and transitive and $A \leq B$, then $A^\infty \leq B$.

Proof:

For $A = (\langle a_{ij}^T, a_{ij}^I, a_{ij}^F \rangle)$, $B = (\langle b_{ij}^T, b_{ij}^I, b_{ij}^F \rangle)$, $AB = (\left\langle \sum_{k=1}^{n}(a_{ik}^T b_{kj}^T), \sum_{k=1}^{n}(a_{ik}^I b_{kj}^I), \prod_{k=1}^{n}(a_{ik}^F + b_{kj}^F) \right\rangle)$

and each

$$\left\langle \sum_{k=1}^{n}(a_{ik}^T b_{kj}^T), \sum_{k=1}^{n}(a_{ik}^I b_{kj}^I), \prod_{k=1}^{n}(a_{ik}^F + b_{kj}^F) \right\rangle = \begin{cases} \langle 1,1,0 \rangle, & \text{if } i = j, \\ \langle b_{ij}^T, b_{ij}^I, b_{ij}^F \rangle, & \text{if } i \neq j. \end{cases}$$

Thus $AB = B$ implies $AA \leq AB = B$. That is $A^2 \leq B$. Continuing in this way, we have $A^3 \leq B, A^4 \leq B \ldots$ and also $A \vee A^2 \vee A^3 \vee \ldots \vee A^n \leq B$ and hence $A^\infty \leq B$.

Lemma 3.14.

If A^∞ is the transitive closure of A, then the transitive closure of WA is WA^∞

Proof:

Now

$WA^\infty = W[A \vee A^2 \vee \ldots \vee A^n] = WA \vee WA^2 \vee \ldots W \vee A^n = WA \vee (WA)^2 \vee \ldots \vee (WA)^n = (WA)^\infty$.

Similarly, the following results are also true.

- $WA_\infty = (WA)_\infty$.
- $\Diamond A^\infty = (\Diamond A)^\infty$.
- $\Diamond A_\infty = (\Diamond A)_\infty$.

Lemma 3.15.

For an NFM $A \in \mathrm{F}_n, [(WA)^c]^\infty = [(WA)_\infty]^c$.

Proof:

As we know $(WA)^c = \Diamond A^c, [(WA)^c]^\infty = [(\Diamond A^c]^\infty = \Diamond A^c \vee (\Diamond A^c)^2 \ldots \vee (\Diamond A^c)^n$.

$$(\Diamond A^c)^2 = (\left\langle \sum_{k=1}^{n}(a_{ik}^F a_{kj}^F), \sum_{k=1}^{n}(a_{ik}^I a_{kj}^I), \prod_{k=1}^{n}(1 - a_{ik}^F)(1 - a_{kj}^f) \right\rangle).$$

$= (\left\langle \prod_{k=1}^{n}(a_{ik}^F + a_{kj}^F), \prod_{k=1}^{n}(a_{ik}^I + a_{kj}^I), \prod_{k=1}^{n}(1 - a_{ik}^F) + (1 - a_{kj}^F) \right\rangle)$ By definition,

$A^2 = (\left\langle \prod_{k=1}^{n}(a_{ik}^T + a_{kj}^T), \prod_{k=1}^{n}(a_{ik}^I + a_{kj}^I), \sum_{k=1}^{n}(a_{ik}^F a_{kj}^F) \right\rangle)$

$WA^2 = (\left\langle \prod k = 1^n (a_{ik}^T + a_{kj}^T), \prod_{k=1}^{n}(a_{ik}^I + a_{kj}^I), 1 - \prod_{k=1}^{n}(a_{ik}^F + a_{kj}^F) \right\rangle)$ Which yields,

$(\mathrm{W}A^2)^c = (\left\langle 1-\prod_{k=1}^{n}(a_{ik}^F + a_{kj}^F), \prod_{k=1}^{n}(a_{ik}^I + a_{kj}^I), \prod k = 1^n (a_{ik}^T + a_{kj}^T) \right\rangle)$ Therefore, $(\Diamond A^c)^2 = (\mathrm{W}A^2)^c$, so in general $(\Diamond A^c)^n = (\mathrm{W}A^n)^c$

$$\left[(\mathrm{W}A)^c\right]^\infty = \left[\Diamond A^c\right]^\infty = \Diamond A^c \vee (\Diamond A^c)^2 \ldots \vee (\Diamond A^c)^n \ = (\mathrm{W}A)^c \vee (\mathrm{W}A^2)^c \vee \ldots \vee (\mathrm{W}A^n)^c$$

$= (\mathrm{W}A \wedge \mathrm{W}A^2 \wedge \ldots \wedge \mathrm{W}A^n)^c = (\mathrm{W}A_\infty)^c$.

In dual fashion, one can prove the following lemma.

Lemma 3.16.

For an NFM $A \in \mathrm{F}_n, ((\Diamond A)_\infty)^c = ((\Diamond A)^c)^\infty$.

Definition 3.2.

For any two elements $\left\langle x^T, x^I, x^F \right\rangle, \left\langle y^T, y^I, y^F \right\rangle \in NFS$, we introduce the operation $'\wedge_{m'}$ as $\left\langle x^T, x^I, x^F \right\rangle \wedge_m \left\langle y^T, y^I, y^F \right\rangle = \left\langle min\left\{x^T, y^T\right\}, min\left\{x^I, y^I\right\}, min\left\{x^F, y^F\right\} \right\rangle$. Using this definition the following lemmas are trivial.

Lemma 3.17.

The operation $\wedge_m$ is commutative on NFSs.

Lemma 3.18.

The operation $\wedge_m$ is associative on NFS.

Lemma 3.19.

The operation $\wedge_m$ is distributive over addition in NFSs.

Proof:

For any $\left\langle x^T, x^I, x^F \right\rangle, \left\langle y^T, y^I, y^F \right\rangle, \left\langle z^T, z^I, z^F \right\rangle \in NFSs$

$$\left(\left\langle x^T, x^I, x^F \right\rangle + \left\langle y^T, y^I, y^F \right\rangle\right) \wedge_m \left\langle z^T, z^I, z^F \right\rangle = \left\langle max\left\{x^T, y^T\right\}, max\left\{x^I, y^I\right\}, min\left\{x^F, y^F\right\} \right\rangle \wedge_m \left\langle z^T, z^I, z^F \right\rangle$$

$$= \left\{min\left\{x^T, y^T\right\}, z^T\right\}, min\left\{max\left\{x^I, y^I\right\}, z^I\right\}, min\left\{min\left\{x^F, y^F\right\} z^F\right\} \qquad (3.13)$$

case(1)

If $\left\langle x^T, y^I, z^F \right\rangle \geq \left\langle y^T, y^I, y^F \right\rangle$ and $\left\langle x^T, x^I, x^F \right\rangle \geq \left\langle y^T, y^I, y^F \right\rangle$ then,RHS of 14 is $\left\langle z^T, x^F \right\rangle$ now consider, $(\left\langle x^T, x^I, x^F \right\rangle \wedge_m \left\langle z^T, z^I, z^F \right\rangle) + (\left\langle y^T, y^I, y^F \right\rangle \wedge_m \left\langle z^T, z^I, z^F \right\rangle)$

$$= \left\langle z^T, z^I, z^F \right\rangle + \begin{cases} \left\langle z^t, z^I, y^F \right\rangle, & \text{if } \left\langle z^T, z^I, z^F \right\rangle \leq \left\langle y^T, y^I, y^F \right\rangle, \\ \left\langle y^T, y^I, z^F \right\rangle, & \text{if } \left\langle y^T, y^I, y^F \right\rangle \leq \left\langle z^T, z^I, z^F \right\rangle \end{cases}$$

$$= \left\langle z^t, z^I, x^F \right\rangle \qquad (3.14)$$

In this case, it is distributive.

case(2)

If $\left\langle x^T, x^I, x^F \right\rangle \leq \left\langle y^T, y^I, y^f \right\rangle$ and $\left\langle x^T, x^I, x^F \right\rangle \leq \left\langle z^T, z^I, z^F \right\rangle$ then the left hand side of above equation 15 reduces to $\left\langle y^T, y^I, y^F \right\rangle \wedge_m \left\langle z^T, z^I, z^F \right\rangle$.

sub case (2.1)

If $\langle z^T, z^I, z^F\rangle \le \langle y^T, y^I, y^F\rangle$ then $\langle z^T, z^I, z^F\rangle \wedge_m \langle y^T, y^I, y^F\rangle = \langle z^T, z^I, y^F\rangle$ Now, $\langle x^T, x^I, x^F\rangle \wedge_m \langle z^T, z^I, z^F\rangle + (\langle x^T, x^I, x^F\rangle \wedge_m \langle z^T, z^I, z^F\rangle) = \langle x^T, x^I, z^F\rangle + \langle z^T, z^I, y^F\rangle = \langle z^T, z^I, y^F\rangle$

Thus distributivity holds,

sub case (2.2)

If $\langle z^T, z^I, z^F\rangle \ge \langle y^T, y^I, y^F\rangle$ then L.H.S of equation 15 becomes $\langle y^T, y^I, z^F\rangle$ and R.H.S of equation 15 becomes , $\langle x^T, x^I, x^F\rangle \wedge_m \langle z^T, z^I, z^F\rangle + (\langle x^T, x^I, x^F\rangle \wedge_m \langle z^T, z^I, z^F\rangle) = \langle x^T, x^I, z^F\rangle + \langle y^T, y^I, z^F\rangle = \langle y^T, y^I, z^F\rangle$

Thus it is distributive in this case also.

case(3)

If $\langle y^T, y^I, y^F\rangle \le \langle x^T, x^I, x^F\rangle \le \langle z^T, z^I, z^F\rangle$ then L.H.S becomes, $(\langle x^T, x^I, x^F\rangle + \langle y^T, y^I, y^F\rangle) \wedge_m \langle z^T, z^I, z^F\rangle = \langle x^T, x^I, x^F\rangle \wedge_m \langle z^T, z^I, z^F\rangle = \langle x^T, x^I, z^F\rangle$ Also, $(\langle x^T, x^I, x^F\rangle \wedge_m \langle z^T, z^I, z^F\rangle) + (\langle z^T, z^I, z^F\rangle \wedge_m \langle z^T, z^I, z^F\rangle) = \langle x^T, x^I, y^F\rangle + \langle y^T, y^I, z^F\rangle = \langle x^T, x^I, z^F\rangle$

so, it is distributive in this case too

case (4)

If $\langle z^T, z^I, z^F\rangle \ge \langle x^T, x^I, x^F\rangle \ge \langle y^T, y^I, y^F\rangle$ then the L.H.S reduces to $\langle y^T, y^I, y^F\rangle \wedge_m \langle z^T, z^I, z^F\rangle = \langle z^T, z^I, y^F\rangle$ and $(\langle x^T, x^I, x^F\rangle \wedge_m \langle z^T, z^I, z^F\rangle) + (\langle x^T, x^I, x^F\rangle \wedge_m \langle z^T, z^I, z^F\rangle) = \langle z^T, z^I, x^F\rangle + \langle z^T, z^I, y^F\rangle = \langle z^T, z^I, z^F\rangle$

Thus distributivity holds for all cases.

Definition 3.3.

For any two elements $\langle x^T, x^I, x^F\rangle, \langle y^T, y^I, y^F\rangle \in NFS$, we define the inequality $'\le'$ as $\langle x^T, x^I, x^F\rangle \le \langle y^T, y^I, y^F\rangle$ means $x \le y$, $x^I \le y^I$ and $x^F \le y^F$.

Remark 3.2.

The elements in the set $\{\langle y^T, y^I, y^F\rangle \in NFS \mid \langle x^T, x^I, x^F\rangle \le \langle y^T, y^I, y^F\rangle\}$ are identity element of $\langle x^T, x^I, x^F\rangle$ with respect to $'\wedge_{m'}$. That is we have multiple identity element.

Remark 3.3.

Any NFMs A can be decomposed into two Neutrosophic fuzzy matrices WA and $\Diamond A$ by means of $\wedge_m$. That is $A = (WA) \wedge_m (\Diamond A)$.

Remark 3.4.

For any two NFMs A and B, $(A \vee B) \wedge_m (A \wedge B) = (A \wedge_m B)$.

4. CONCLUSION:

In this paper, we introduced modal operators and a new composition operation in Neutrosophic fuzzy matrix. Further some of its algebraic properties are investigated.

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