STABILITY OF RECIPROCAL DIFFERENCE AND ADJOINT FUNCTIONAL EQUATION IN MULTI-BANACH SPACES: A FIXED POINT METHOD

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ABSTRACT. In this paper, we present the Hyers-Ulam stability of reciprocal difference and adjoint functional equations in Multi-Banach Space

1. INTRODUCTION

The question under what conditions an approximate solution to an equation can be replaced by an exact solution to it (or conversely) and what error we thus commit seems to be very natural. The theory of Ulam (often also called the Hyers-Ulam) type stability provides some convenient tools to investigate such issues. Let us only mention that the study of such stability has been motivated by a problem raised by S. Ulam in 1940 and a solution to it given by Hyers in [3]. For some updated information and further references concerning that type of stability we refer to ([1], [2]).

In 2013, K. Ravi, J.M. Rassias, B.V. Senthil Kumar [8], we investigated the generalized Hyers-Ulam stability of reciprocal difference and adjoint functional equations in paranormed spaces by direct and fixed point methods.

In 2016, A. Bodaghi, P. Narasimman, J.M. Rassias and K. Ravi [2], we introduced a new generalized reciprocal functional equation in Non-Archimedean fields.

In 2017, Sang Og Kim, B.V. Senthil Kumar and Abasalt Bodaghi [12], we investigated the generalized Hyers-Ulam Stability of a form of reciprocal-cubic and reciprocal-quartic functional equations in non-Archimedean fields.

Recently, R. Murali and A. Antony Raj [7], we investigated the Ulam -Hyers Stability of Nonadecic functional equation in Multi-Banach Spaces. John Michael Rassias, R. Murali, Matina John Rassias and A. Antony Raj [5], we estabilished the general solution, stability and non-stability of Quattuorvigintic functional equation in Multi-Banach Spaces.

2.PRILIMINARIES

In this section, we recall basic facts concerning Multi-Banach spaces and fundamental results of fixed point theory. Let $(\rho, \|\cdot\|)$ be a complex normed space, and let $k \in N$. We denote by ρ^k the linear space $\rho \oplus \rho \oplus \rho \oplus \ldots \oplus \rho$ consisting of k-$(x_1, \ldots, x_k)$ where $(x_1, \ldots x_k) \in \rho$. The linear operations on ρ^k are defined coordinate wise. The zero element of either ρ or ρ^k is denoted by 0. We denote by N_k the set $\{1,2,\ldots k\}$ and by Ψ_k the group of permutations on k symbols.

Definition 2.1. [4] A Multi-norm on $\{\rho^k : k \in N\}$ is a sequence $(\|.\|) = (\|.\|_k : k \in N)$ such that $\|.\|_k$ is a norm on ρ^k for each $k \in N, \|x\|_1 = \|x\|$ for each $x \in \rho$, and the following axioms are satisfied for each $k \in N$ with $k \geq 2$:

(1) $\|(x_{\sigma(1)},\ldots x_{\sigma(k)})\|_k = \|(x_1,\ldots x_k)\|_k$, for $\sigma \in \Psi_k, x_1,\ldots x_k \in \rho$;

(2) $\|(\alpha_1 x_1,\ldots \alpha_k x_k)\|_k \leq (\max_{i\in N_k} |\alpha_i|)\|(x_1,\ldots x_k)\|_k$,
for $\alpha_1,\ldots\alpha_k \in C, x_1,\ldots x_k \in \rho$;

(3) $\|(x_1,\ldots x_{k-1},0)\|_k = \|(x_1,\ldots x_{k-1})\|_{k-1}$, for $x_1,\ldots x_{k-1} \in \rho$;

(4) $\|(x_1,\ldots x_{k-1},x_{k-1})\|_k = \|(x_1,\ldots x_{k-1})\|_{k-1}$, for $x_1,\ldots x_{k-1} \in \rho$.

In this case, we say that $((\rho^k,\|.\|_k): k \in N)$ is a multi normed space . Suppose that $((\rho^k,\|.\|_k): k \in N)$ is a multi normed space, and take $k \in N$. We need the following two property of a multi norms .

(a) $\|(x,\ldots x)\|_k = \|x\|, \forall x \in \rho$,

(b) $\max_{i\in N_K} \|x_i\| \leq \|(x,\ldots x)\|_k \leq \sum \|x_i\| \leq k \max_{i\in N_K} \|x_i\| \; \forall x_1,\ldots x_k \in \rho$

It is follows from (b) that if $(\rho,\|.\|)$ is a banach space, $(\rho^k,\|.\|_k)$ a multi banach space for each $k \in N$ in this case $((\rho^k,\|.\|_k): k \in N)$ is a multi banach space.

Theorem 2.2. [10] Let $(\mathcal{X}, d)$ be a complete generalized metric space and let $\mathcal{J}: \mathcal{X} \to \mathcal{X}$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in \mathcal{X}$, either

$$d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(i) $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty$ for all $n \geq n_0$;

(ii) The sequence $\{\mathcal{J}^n\}$ is a convergent to a fixed point y^* of $\mathcal{J}$;

(iii) y^* is the unique fixed of T in the set $Y = \{y \in \mathcal{X}: d(\mathcal{J}^{n_0} x, y) < \infty\}$;

(iv) $d(y, y^*) \leq \frac{1}{1-L} d(y, \mathcal{J}y)$ for all $y \in Y$.

For convenience, we take

$$R_1 f(x,y) = f\left(\frac{x+y}{2}\right) - f(x+y) - \frac{f(x)f(y)}{f(x)+f(y)}$$ and

$$R_2 f(x,y) = f\left(\frac{x+y}{2}\right) + f(x+y) - \frac{3f(x)f(y)}{f(x)+f(y)}$$

In this paper, we establish the generalized Hyers-Ulam stability of the functional equations

$$R_j f(x,y) = 0 \quad for \;\; \text{j=1,2} \tag{2.1}$$

In Multi-Banach spaces. Throughout this paper, X be a linear space and $(Y^n, ||.||n \in N)$ be a multi banach space.

Theorem 2.3. suppose that the mapping $f: X \to Y$ satisfies the inequality

$$||R_j f(x_1, y_1), \dots R_j f(x_k, y_k)| \leq \Psi(x_1, y_1, \dots x_k, y_k) \tag{2.2}$$

for all $x_i y_i \in X \;\; \forall i = 1,2,\dots k$ and $j = 1,2$ where $\Psi: X^{2k} \to [0, \infty)$ is a given function. If there exists $L < 1$ such that

$$\Psi(x_1, y_1, \dots x_k, y_k) \leq \frac{1}{2} L \Psi\left(\frac{x_1}{2}, \frac{y_1}{2}, \dots \frac{x_k}{2}, \frac{y_k}{2}\right) \tag{2.3}$$

for all $x, y \in X$, then there exist a unique reciprocal mapping $r: X \to Y$ such that

$$||r(x_1) - f(x_1), \dots r(x_k) - f(x_k)|| \leq \frac{1}{1-L} \Psi\left(\frac{x_1}{2}, \frac{y_1}{2}, \dots \frac{x_k}{2}, \frac{y_k}{2}\right) \tag{2.4}$$

Proof.

Replacing (x_i, y_i) by (x_i, x_i) in (2.2) and multiplying by 2 we get

$$||f(x_1) - 2f(2x_1), \dots f(x_k) - 2f(2x_k)|| \leq 2\Psi(x_1, y_1, \dots x_k, y_k)$$

$$\leq L\Psi\left(\frac{x_1}{2}, \frac{y_1}{2}, \dots \frac{x_k}{2}, \frac{y_k}{2}\right) \tag{2.5}$$

$\forall x_i \in x$ where $i = 1,2, \dots k$.

Define a set S by S={$h: X \to Y | h$ is a function} and introduce the generalized metric d on S as follows:

$$d(g,h) = \inf\{C \in R_+ : ||g(x_1) - h(x_1), \dots g(x_k) - h(x_k)|| \leq C\Psi(x_1, x_1, \dots x_k, x_k)\} \tag{2.6}$$

$\forall x_i \in X$ where $i = 1,2,\dots k$. Where as usual, $\inf \Phi = +\infty$. It is easy to show that (S,d) is complete.

Define a mapping $\sigma: S \to S$ by

$\sigma h(x) = 2h(2x) \quad \forall x \in X, h \in S.$

We claim that σ is strictly contractive on S, for every given $g, h \in S$, let $C_{gh} \in [0, \infty]$ be an arbitrary constant with $d(g,h) \leq C_{gh}$. Hence $\| g(x_1) - h(x_1), \dots g(x_k) - h(x_k) \|_k \leq C_{gh} \Psi(x_1, x_1, \dots x_k, x_k)$

$\| 2g(x_1) - 2h(x_1), \dots 2g(x_k) - 2h(x_k) \|_k \leq 2C_{gh} \Psi(x_1, x_1, \dots x_k, x_k)$

$$\| 2g(2x_1) - 2h(2x_1), \dots 2g(2x_k) - 2h(2x_k) \|_k \leq C_{gh} L \Psi(x_1, x_1, \dots x_k, x_k) \tag{2.7}$$

$\forall x_i \in X$ where $i = 1,2,\dots k.$

$\Rightarrow d(\sigma g, \sigma h) \leq L C_{gh}$

Therefore we see that $d(\sigma g,\sigma h)\le Ld(g,h)\forall_{g,h}\in S$, that is σ is strictly contractive mapping of S with Lipschitz constant L. Hence (5) implice that $d(f,\sigma f)\le 1$. Hence by applying the fixed point alternative theorem 2.2, there exist a function $r: X\to Y$ satisfying the followings:

(1) r is a fixed point of σ,that is

$$r(2x)=\frac{1}{2}r(x) \qquad \forall x\in X. \tag{2.8}$$

The mapping r is a unique fixed point of σ is the set $\mu=\{g\in s: d(f,g)\le\infty\}$. This implies that r is the unique mapping satisfying (2.8), such that there exist $C\in(0,\infty)$ satisfying

$$||r(x_1)-f(x_1),\dots r(x_k)-f(x_k)||\le C\Psi(x_1,\mathrm{y}_1,\dots \mathrm{x}_\mathrm{k},\mathrm{y}_\mathrm{k}) \;\; \forall x_i\in X, i=1,2,\dots K$$

(2) $d(\sigma^n f,r)\to 0$ as $n\to\infty$. Thus we have

$$\lim_{n\to\infty}2^n f(2^n x)=r(x) \qquad \forall x\in X. \tag{2.9}$$

(3) $d(r,f)\le\frac{1}{1-L}d(r,\sigma f)$ which implies $d(r,f)\le\frac{1}{1-L}$ thus the inequality (2.4) holds. Hence (2.3), (2.4) and (2.9) we have

$$||R_j r(x,y)||=\lim_{n\to\infty}2^n||R_j f(2^n x,2^n y)||\le\lim_{n\to\infty}2^n\frac{L^n}{2^n}\Psi(x,y)=0$$

$\forall x,y\in X$.and $R_j r(x,y)=0 \;\; \forall j=1,2.$ (2.10)

Hence r is a solution of functional equation (2.1). By Theorem 2.1 in [9] $r: X\to Y$ is a reciprocal mapping. Next, we show that r is a unique reciprocal mapping (2.1), for $j=1,2$ and (2.4). Suppose, let $R: X\to Y$ be another reciprocal mapping satisfying (2.1), for $j=1,2$ and (2.4). Then from (2.1), for $j=1,2$, we have that R is a fixed point of σ. since

$d(f,R)\le\infty$.

We have $R\in s^*=\{g\in s|d(f,g)\le\infty\}$.

From Theorem 2.2 (iii) and since both r and R are fixed points of σ we have $r=R$.

Therefore r is unique. Hence, there exist a unique reciprocal mapping $r: X\to Y$

Satisfying (2.1) for $j=1,2$ and (2.4), which complete the proof of the Theorem.

Corollary 2.4. Let X be a linear space, and let $\left(Y^n,\|.\|_n\right)$ be a multi-banach space. Let $\alpha<-1$ $c_1\ge 0$ such that satisfying the inequality $\left\|R_j f(x_1,y_1),\dots R_j f(x_k,y_k)\right\|_k\le c_1\left(\|x_1\|^\alpha+\|y_1\|^\alpha,\dots\|x_k\|^\alpha.\|y_k\|^\alpha\right)$ for all $x_i,y_i\in X$ $i=1,2,\dots\dots k$ and $j=1,2$. Then there is a unique reciprocal mapping $r: X\to Y$ satisfying (2.1), for $j=1,2$ and $\left\|f(x_1)-r(x_1),\dots f(x_k)-r(x_k)\right\|\le\frac{4c_1}{1-2^{n+1}}\left(\|x_1\|^\alpha,\dots\|x_k\|^\alpha\right)$

for all $x_i\in X$, where $i=1,2,3,\dots k$.

Proof.

The Proof follows from above theorem taking

$$\Psi(x_1,y_1,...x_k,y_k)=c_1\left(\|x_1\|^{\alpha}+\|y_1\|^{\alpha},...\|x_k\|^{\alpha}+\|y_k\|^{\alpha}\right)$$

$x_i,y_i \in X$ $i=1,2,.......k$ and $L=2^{n+1}$, we get the desired result.

Corollary 2.5. Let x be a linear space, let $\left(Y^n,\|.\|_n\right)$ be a multi-banach space. Let $f:X\to Y$ be a linear mapping and let there exists a real numbers a, b such that $l=a+b<-1$. Then there exists $c_2\geq 0$, such that satisfying the inequality

$$\left\|R_j f(x_1,y_1),...R_j f(x_k,y_k)\right\|_k \leq c_2\left(\|x_1\|^a.\|y_1\|^b,...\|x_k\|^a.\|y_k\|^b\right)$$

For all $x_i,y_i\in X$ where $i=1,2,.......k$ for $j=1,2$. Then there exists a unique reciprocal mapping $r:X\to Y$ satisfying (2.1), for $j=1,2$

$$\|f(x_1)-r(x_1),...f(x_k)-r(x_k)\|\leq\frac{2c_1}{1-2^{l+1}}\left(\|x_1\|^l,....\|x_k\|^l\right)$$

$x_i\in X$, where $i=1,2,3,....k$.

Proof.

The Proof follows from above theorem taking

$$\Psi(x_1,y_1,...x_k,y_k)=c_1\left(\|x_1\|^a+\|y_1\|^b,...\|x_k\|^a+\|y-k\|^b\right)$$

$x_i,y_i\in X$ $i=1,2,.......k$ and $L=2^{l+1}$

Corollary 2.6 Let x be a linear space, let $\left(Y^n,\|.\|_n\right)$ be a multi-banach space. Let $c_3\geq 0$ and p,q be real numbers such that $\lambda=p+q<-1$ and $f:X\to Y$ be a mapping satisfying the inequality

$$\left\|R_j f(x_1,y_1),...R_j f(x_k,y_k)\right\|\leq c_3\left(\|x_1\|^p.\|y_1\|^q+\left(\|x_1\|^{p+q}.\|y_1\|^{p+q}\right),...\|x_k\|^p.\|y_k\|^q+\left(\|x_k\|^{p+q}.\|y_k\|^{p+q}\right)\right)$$ for all $x_i,y_i\in X$ where $i=1,2,.......k$ for $j=1,2$. Then there exists a unique reciprocal mapping $r:X\to Y$ satisfying (2.1), for $j=1,2$

$$\|f(x_1)-r(x_1),...f(x_k)-r(x_k)\|\leq\frac{6c_3}{1-2^{\lambda+1}}\left(\|x_1\|^{\lambda},....\|x_k\|^{\lambda}\right)$$

$x_i\in X$, where $i=1,2,3,....k$.

Proof: By choosing

$$\Psi(x_1, y_1, \ldots x_k, y_k) = c_3\left(\|x_1\|^p.\|y_1\|^q + \left(\|x_1\|^{p+q}.\|y_1\|^{p+q}\right), \ldots \|x_k\|^p.\|y_k\|^q + \left(\|x_k\|^{p+q}.\|y_k\|^{p+q}\right)\right) \qquad x_i \in X, \quad \text{where}$$

$i = 1, 2, 3, \ldots k$ and taking $L = 2^{\lambda+1}$

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