HYERS-ULAM STABILITY Of QUARTIC FUNCTIONAL EQUATION IN MODULAR SPACES

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Abstract: In this paper, we study the Hyers-Ulam stability of a quartic functional equation of the form

$$f(2u+v)+f(2u-v)=4f(u+v)+4f(u-v)+24f(u)-6f(v),$$

in modular space by using direct method.

Keywords: Hyers-Ulam stability, Quartic functional equation, Fatou property, Δ_2-condition.

MSC: 26D10; 39B52; 39B72; 39B82; 47H09

INTRODUCTION AND PRILIMINARIES

The theory of stability is an important branch of the qualitative theory of functional equations. The concept of stability of various functional equation arises when one can replace a functional equation by inequality which acts as a perturbation of the equation. The stability problem of functional equation was raised by S.M. Ulam [12] about seventy eight years ago. Since then, this question has attracted the attention of many researches. Note that the affirmative solution to this question was given in the next year by D.H. Hyers [5] in 1941. In this year 1950, T. Akoi [1] generalized Hyers theorem for additive mappings. The results of Hyers theorem for additive mappings. The result of Hyers was generalized independently by Th.M. Rassias [11] for linear mapping by considering an unbounded Cauchy difference. In 1944, a further generalization of Th.M. Rassias theorem was obtained by P. Gavruta [4].

After that, the stability problem of several functional equations have been extensively investigated by a number of mathematicians, and there are many interesting results concerning this problem [2,3,6,7,8,9,10,13]. The theory of modulars on linear spaces and related theory of modular linear spaces have been established by Nakano in 1950 [19]. Since then, these have been thoroughly developed by several mathematicians, for example, Amemya [22], Koshi [21], Musielak [20]. Upto now, the theory of modulars and modular spaces is widely applied in the study of interpolation theory and various Orlicz spaces. First of all we introduce the usual terminologies, notation, definitions, and properties on the theory of modular spaces.

Definition 1.1.[18] Let U be a linear space over a field K. We say that a generalized functional $\rho: U \to [0,\infty)$ is modular if for any $u,v \in V$ which satisfies the following condition

$\rho(u)=0$ if and only if $u=0$.

1. $\rho(\alpha u)=\rho(u)$ for all scalars $|\alpha|=1$.
2. $\rho(\alpha u+\beta v)\leq\rho(u)+\rho(v)$ for all scalars $\alpha,\beta\geq 0$, with $\alpha+\beta=1$.

If (3) is replaced by

3. $\rho(\alpha u+\beta v)\leq\alpha\rho(u)+\beta\rho(v)$ for all scalar $\alpha,\beta\geq 0$, the function ρ is convex modular.

A modular ρ defines the following vector space,

$$U_\rho=\{u\in U;\rho(\lambda u)\to 0 \text{ as } \lambda\to 0\}$$

And we say that U_ρ is a modular space.

Definition 1.2. [18] Let U_ρ be a modular space and let u_n be the sequence in U_ρ. Then

1. u_n is ρ-convergent to a point $u\in U_\rho$ and $u_n\xrightarrow{\rho}u$ if $\rho(u_n-u)\to 0$ as $n\to\infty$.
2. u_n is called ρ-cauchy sequence if for any $\varepsilon>0$ one has $\rho(u_n-u_m)<\varepsilon$ for sufficiently large $m,n\in N$.
3. A subset $K\subseteq U_\rho$ is called ρ complete if any ρ Cauchy sequence is ρ-convergent to a point K.

Definition 1.3 [18] The modular ρ has the Fatou property if and only if $\rho(u)\leq\liminf_{n\to\infty}\rho(u_n)$ whenever the sequence $\{u_n\}$ is ρ-convergent to u in modular space U_ρ.

Definition1.4 [18] A modular ρ is convex and satisfy the Δ_2-condition if there exists $k>0$ such that $\rho(2u)\leq k\rho(u)$ for all $u\in U_\rho$.

Remark 1.5 [18] Suppose that ρ is convex and satisfies the Δ_2-condition with Δ_2 constant $k>0$. If $k<2$, then $\rho(u)\leq k\rho\left(\frac{u}{2}\right)\leq\frac{k}{2}\rho(x)$, thus $\rho=0$. Therefore, we must have Δ_2 constant $k\geq 2$, if ρ is convex modular.

The motivation of the above facts, the author are very interested in proving the Ulam-Hyers Stability of a Quartic functional equation of the form

$$f(2u+v)+f(2u-v)=4f(u+v)+4f(u-v)+24f(u)-6f(v) \tag{1.1}$$

In modular spaces by using direct method.

2. HYERS-ULAM STABILITY OF QUARTIC FUNCTIONAL EQUATION (1.1) BY USING Δ_2-CONDITION

In this section, we are going to prove the Hyers-Ulam stability of a quartic functional equation in modular spaces without using Fatou's property. So we are assuming that V be linear space and U_ρ is a ρ-complete modular spaces.

Theorem 2.1. Assume that U_ρ satisfies the Δ_2-condition and there exists a mapping $f:V\to U_\rho$ satisfies the inequality

$$\rho\big(f(2u+v)+f(2u-v)-4f(u+v)-4f(u-v)-24f(u)+6f(v)\big)\le\phi(u,v) \qquad \textbf{(2.1)}$$

and

$$\lim_{n\to\infty}k^{4n}\left(\frac{1}{2}\right)\phi\left(\frac{u}{2^n},0\right)=0$$

$$\sum_{i=1}^{\infty}\left(\frac{k^5}{4}\right)\left(\frac{1}{2}\right)\phi\left(\frac{u}{2^i},0\right)<\infty$$

for all $u,v\in V$, then there exists a unique cubic mapping $D:V\to U_\rho$ defined by $D(u)=\lim_{n\to\infty}2^{4n}f\left(\frac{u}{2^n}\right)$

and $\rho\big(f(u)-D(u)\big)\le\frac{1}{2k^3}\sum_{1=0}^{\infty}\left(\frac{k^5}{4}\right)^i\frac{1}{2}\phi\left(\frac{u}{2^i},0\right)$, for all $u,v\in V$ (2.2)

Proof. Let us substitute $v=0$ in (2.1) and dividing it by 2, we get

$$\rho(f(2u)-16f(u))\le\frac{1}{2}\phi(u,0) \text{ for all } u\in V \qquad (2.3)$$

Now, by using the Δ_2-condition and $\sum_{i=1}^{n}\frac{1}{2^i}\le 1$, we can prove the following inequality

$$\rho\left(f(u)-2^{4n}f\left(\frac{u}{2^n}\right)\right)\le\rho\left(\sum_{i=1}^{n}\frac{1}{2^i}\left(2^{5i-4}f\left(\frac{u}{2^{i-1}}\right)-2^{5i}f\left(\frac{u}{2^i}\right)\right)\right)$$

$$\le\frac{1}{k^4}\sum_{i=1}^{n}\left(\frac{k^5}{4}\right)\frac{1}{2}\phi(u,0), \qquad \text{for all } u\in V.$$

Let us consider $m,n\in N$ with $n\ge m$, we get

$$\rho\left(2^{4m}f\left(\frac{u}{2^m}\right)-2^{4(n+m)}f\left(\frac{u}{2^{n+m}}\right)\right)\le\rho\left(2^{4m}\left(f\left(\frac{u}{2^m}\right)-2^{4n}f\left(\frac{u}{2^{n+m}}\right)\right)\right)$$

$$\le\frac{2^m}{k^{m-4}}\sum_{i=m+1}^{n+m}\left(\frac{k^5}{4}\right)^i\frac{1}{2}\phi\left(\frac{u}{2^i},0\right), \quad \text{for all } u\in V.$$

as $n\to\infty$, we get the sequence $\left\{2^{4n}f\left(\frac{u}{2^n}\right)\right\}$ is a ρ-Cauchy sequence in U_ρ and ρ-convergent senquence in U_ρ thus U_ρ is ρ-complete for all $u\in V$. Now let us define a mapping $D:V\to U_\rho$ and satisfies

$$D(u)=\rho.\lim_{n\to\infty}2^{4n}f\left(\frac{u}{2^n}\right)$$

$$\lim_{n\to\infty}\rho\left(2^{4n}f\left(\frac{u}{2^n}\right)-D(u)\right)=0 \qquad (2.4)$$

By using Δ_2-condition, without using Fatou property, we can write the following inequality as

$$\rho\big(f(u)-D(u)\big)\le\frac{1}{2}\rho\left(2f(u)-2.2^{4n}f\left(\frac{u}{2^n}\right)\right)+\frac{1}{2}\rho\left(2\left(2^{4n}f\left(\frac{u}{2^n}\right)-D(u)\right)\right)$$

$$\le\frac{1}{2}\rho\left(2\left(f(u)-2^{4n}f\left(\frac{u}{2^n}\right)\right)\right)+\frac{1}{2}\rho\left(2\left(2^{4n}f\left(\frac{u}{2^n}\right)-D(u)\right)\right)$$

$$\rho(f(u)-D(u))\le\frac{k}{2}\rho\left(f(u)-2^{2n}f\left(\frac{u}{2^n}\right)\right)+\frac{k}{2}\rho\left(2^{2n}f\left(\frac{u}{2^n}\right)-2D(u)\right) \qquad (2.5)$$

as $n\to\infty$ we get

$$\rho\big(f(u)-D(u)\big)\le\frac{1}{2k^3}\sum_{i=0}^{\infty}\left(\frac{k^5}{4}\right)^i\frac{1}{2}\phi\left(\frac{u}{2^i},0\right),$$

which satisfies the equation (2.2). Now, we claim that D is quartic mapping and replace (u,v) by $(2^{-n}u,2^{-n}v)$ in equation (2.1). Then we get

$$\rho\left(2^{4n}f\left(\frac{2u+v}{2^n}\right)+2^{4n}f\left(\frac{2u-v}{2^n}\right)-4.2^{4n}f\left(\frac{u+v}{2^n}\right)-4.2^{4n}f\left(\frac{u-v}{2^n}\right)\right.$$

$$\left.-24.2^{4n}f\left(\frac{u}{2^n}\right)-6.2^{4n}f\left(\frac{v}{2^n}\right)\right)\le\frac{1}{2}\phi\left(\frac{u}{2^n}\right)$$

which tends to zero as $n\to\infty$ for all $u,v\in V$. Thus, by using the convexity of ρ and using the equation (2.5) we can write,

$$\rho\left(\frac{1}{41}D(2u+v)+\frac{1}{41}D(2u-v)-\frac{1}{41}4.D(u+v)-\frac{1}{41}4.D(u-v)-\frac{1}{41}24.D(u)-\frac{1}{41}6.D(v)\right)$$

$$\le\frac{1}{41}\rho\left(D(2u+v)-2^{4n}f\left(\frac{2u+v}{2^n}\right)\right)+\frac{1}{41}\rho\left(D(2u-v)-2^{4n}f\left(\frac{2u-v}{2^n}\right)\right)$$

$$\frac{4}{41}\rho\left(D(u+v)-2^{4n}f\left(\frac{u+v}{2^n}\right)\right)+\frac{4}{41}\rho\left(D(u-v)-2^{4n}f\left(\frac{u-v}{2^n}\right)\right)$$

$$\frac{24}{41}\rho\left(D(u)-2^{4n}f\left(\frac{u}{2^n}\right)\right)+\frac{6}{41}\rho\left(D(u)-2^{4n}f\left(\frac{u}{2^n}\right)\right)$$

$$\rho\left(2^{4n}f\left(\frac{2u+v}{2^n}\right)+2^{4n}f\left(\frac{2u-v}{2^n}\right)-4.2^{4n}f\left(\frac{u+v}{2^n}\right)-4.2^{4n}f\left(\frac{u-v}{2^n}\right)\right)$$

$$-24.2^{4n}f\left(\frac{u}{2^n}\right)-6.2^{4n}f\left(\frac{v}{2^n}\right)\leq\frac{1}{2}\phi\left(\frac{u}{2^n}\right)$$

for all $u,v\in V$. If $n\to\infty$, then we get, D is quartic.

To show the uniqueness of the theorem, let us assume that the mapping $D':V\to U_\rho$ satisfies

$$\rho\left(f(u)-D'(u)\right)\leq\frac{1}{2k^3}\sum_{i=0}^{\infty}\left(\frac{k^5}{4}\right)^i\frac{1}{2}\phi\left(\frac{u}{2^i},0\right),\ \text{for all } u\in V \tag{2.6}$$

If D and D' are quartic mapping then we can write $D(2^{-n}u)=2^{-4n}D(u)$. Similarly we have,

$D'(2^{-n}u)=2^{-4n}D'(u)$.

$$\rho\left(D(u)-D'(u)\right)\leq\frac{1}{2}\left(2.2^{4n}D\left(\frac{u}{2^n}\right)-2.2^{4n}f\left(\frac{u}{2^n}\right)\right)+\frac{1}{2}\left(2.2^{4n}f\left(\frac{u}{2^n}\right)-2.2^{4n}D'\left(\frac{u}{2^n}\right)\right)$$

$$\leq\left(\frac{k^{4n+1}}{2}\right)\left(\frac{2^n}{k^{5n}}\right)\sum_{i=1}^{\infty}\left(\frac{k^5}{4}\right)^i\frac{1}{2}\phi\left(\frac{u}{2^i},0\right),\quad\text{for all } u\in V$$

as $n\to\infty$ we obtain the uniqueness of the theorem

Corollary 2.2. Let us assume that V be a normed space with $\|.\|$ and U_ρ satisfies Δ_2-condition. For a given real number $\theta\geq 0$ and $p\geq\log\frac{k^4}{2}$, if $f:V\to U_\rho$ is a mapping such that

$$\rho\left(f(2u+v)+f(2u-v)-4f(u+v)-4f(u-v)-24f(u)+6f(v)\right)\leq\theta\left(\|u\|^p+\|v\|^p\right) \tag{2.7}$$

for all $u,v\in U_\rho$ which is a unique quartic mapping and $D:V\to U_\rho$

satisfies the equation $\rho\left(f(u)-D(u)\right)\leq\dfrac{k\theta}{2^{p+3}-4k^5}\|u\|^p$ for all $u\in V$.

Proof. Let us consider,

$$\rho\left(f(u)-D(u)\right)\leq\frac{1}{2k^3}\sum_{i=0}^{\infty}\left(\frac{k^5}{4}\right)^i\frac{1}{2}\phi\left(\frac{u}{2^i},0\right),$$

$$\frac{1}{2}\phi\left(\frac{u}{2^i},0\right)=\frac{\theta\|u\|^p}{2^{ip}}.$$

Now, $$\rho(f(u)-D(u))\leq \frac{1}{2k^3}\sum_{i=0}^{\infty}\left(\frac{k^5}{4}\right)^i \frac{1}{2}\phi\left(\frac{u}{2^i},0\right)$$
$$\leq \frac{1}{2k^3}\sum_{i=0}^{\infty}\left(\frac{k^5}{4}\right)^i \frac{1}{2}\frac{\theta\|u\|^p}{2^{ip}}$$
$$\leq \frac{k\theta}{2^{p+3}-4k^4}\|u\|^p .$$

This completes the proof.

3. HYERS-ULAM STABILITY OF QUARTIC FUNCTIONAL EQUATION (1.1) BY USING FATOU PROPERTY

In this section, we are going to prove the Hyers-Ulam stability of quartic functional equation in the modular spaces without using Δ_2-condition. So we are assuming that V be linear space and U_ρ is ρ-complete modular space.

Theorem 3.1. If there exists a mapping $f:V\to U_\rho$ satisfies the inequality

$$\rho\left(f(2u+v)+f(2u-v)-4f(u+v)-4f(u-v)-24f(u)+6f(v)\right)\leq\phi(u,v) \qquad (3.1)$$

and $\phi:V\times V\to[0,\infty)$ is a mapping such that

$$\lim_{n\to\infty}\frac{1}{2}\frac{\phi(2^n u,0)}{2^{4n}}=0, \qquad \sum_{i=0}^{\infty}\frac{1}{2}\frac{\phi(2^i u,0)}{2^{4i}}=0$$

for all $u,v\in V$. Then there exists a unique quartic mapping $D:V\to U_\rho$ such that

$$\rho(f(2u)-3f(0)-D(u))\leq\frac{1}{2^4}\sum_{i=0}^{\infty}\frac{1}{2}\frac{\phi(2^i u,0)}{2^{4i}} \text{ for all } u\in V. \qquad (3.2)$$

Proof. Let us substitute $v=0$ in (2.1) and dividing it by 2, we get

$$\rho(f(2u)-16f(u))\leq\frac{1}{2}\phi(u,0) \text{ for all } u\in V \qquad (3.3)$$

If we consider, $\tilde f(u)=f(u)+\frac{f(0)}{5}$. Then we get, $\rho\left(\tilde f(2u)-16\tilde f(u)\right)\leq\frac{1}{2}\phi(u,0)$.

Now by using the convexity of ρ, we have

$$\rho\left(\tilde f(u)-\frac{\tilde f(2^n u)}{2^{4n}}\right)\leq\rho\left(\sum_{i=0}^{n-1}\frac{\left(2^4\tilde f(2^i u)-\tilde f(2^{i+1}u)\right)}{2^{4i+4}}\right)$$

$$\leq \frac{1}{2^4}\sum_{i=0}^{\infty}\frac{1}{2}\frac{\phi(2^i u,0)}{2^{4i}} \quad \text{for all } u\in V \tag{3.3}$$

Now,
$$\rho\left(\frac{f(2^n u)}{2^{4n}}-\frac{f(2^m v)}{2^{4m}}\right)\leq \frac{1}{2^{4m}}\left(\rho\left(\frac{f(2^{n-m}2^m.u)}{2^{4n-4m}}-f(2^m u)\right)\right)$$

$$\leq \frac{1}{2^{4m}}\sum_{i=m}^{n-m-1}\frac{1}{2^{4i+4}}\phi\left(2^i.2^m u,0\right)$$

$$\leq \frac{1}{2^4}\sum_{i=m}^{n-1}\frac{1}{2^{4i}}\frac{1}{2}\phi\left(2^i u,0\right),$$

for all $u\in V$ and $m,n\in N$ with $n>m.$ the sequence $\left\{2^{4n} f\left(\frac{u}{2^n}\right)\right\}$ is a ρ-Cauchy sequence in U_ρ and ρ-convergent senquence in U_ρ thus U_ρ is ρ-complete for all $u\in V$. Now let us define a mapping $D:V\to U_\rho$ and satisfies

$$D(u)=\rho.\lim_{n\to\infty}2^{4n} f\left(\frac{u}{2^n}\right)$$

$$\lim_{n\to\infty}\rho\left(2^{4n} f\left(\frac{u}{2^n}\right)-D(u)\right)=0 \quad \text{for all } u\in V \tag{3.4}$$

Then by using Fatou property we have,

$$\rho\left(\tilde{f}(u)-D(u)\right)\leq \lim_{n\to\infty}\rho\left(\tilde{f}(u)-\frac{f(2^n u)}{2^{4n}}\right)\leq \frac{1}{2^4}\sum_{i=0}^{\infty}\frac{1}{2}\frac{\phi(2^i u,0)}{2^{4i}} \tag{3.5}$$

holds for all $u\in V.$ Now,

$$\rho\left(\tilde{f}(u)-D(u)\right)\leq \rho\left(\sum_{i=0}^{n-1}\left(\frac{2^4\tilde{f}(2^i u)-\tilde{f}(2^{i+1}u)}{2^{4i+4}}\right)+\frac{\tilde{f}(2^n u)}{2^{4n}}-\frac{D(2u)}{2^4}\right)$$

$$\leq \sum_{i=0}^{n-1}\frac{1}{2^{4i+4}}\rho\left(2^4\tilde{f}(2^i u)-\tilde{f}(2^{i+1}u)\right)+\frac{1}{2^{4n}}\rho\left(\frac{\tilde{f}(2^{n-1}.2u)}{2^{4n-4}}-D(2u)\right)$$

$$\leq \frac{1}{2^4}\sum_{i=0}^{\infty}\frac{1}{2}\frac{\phi(2^i u,0)}{2^{4i}}+\frac{1}{2^4}\rho\left(\frac{\tilde{f}(2^{n-1}.2u)}{2^{4n-4}}-D(2u)\right), \text{ for all } u\in V$$

and all positive integer $n>1$ as $n\to\infty$ we get

$$\rho\left(\tilde{f}(u)-D(u)\right)\leq \frac{1}{2^4}\sum_{i=0}^{\infty}\frac{1}{2}\frac{\phi(2^i u,0)}{2^{4i}} \tag{3.6}$$

To show the uniqueness of the theorem, let us consider

$$\rho\big(D(u)-D'(u)\big)\le\rho\left(\frac{D(2^nu)}{2^{4n}}-\frac{\tilde f(2^nu)}{2^{4n}}\right)+\rho\left(\frac{\tilde f(2^nu)}{2^{4n}}-\frac{D'(2^nu)}{2^{4n}}\right)$$

$$\le\frac{1}{2^4}\sum_{i=0}^{\infty}\frac{1}{2}\frac{\phi(2^iu,0)}{2^{4i}}\quad\text{for all } u\in V$$

as $n\to\infty$ we obtain the uniqueness of the theorem

Corollary 3.2. Assume that U_ρ satisfies the Fatou property and V be normed space with $\|.\|$. For a given real number $\theta\ge 0$, $\varepsilon>0$, $p\in(-\infty,1)$ and there exists a mapping $f:V\to U_\rho$ satisfies the inequality

$$\rho\big(f(2u+v)+f(2u-v)-4f(u+v)-4f(u-v)-24f(u)+6f(v)\big)\le\theta\left(\|u\|^p+\|v\|^p\right)+\varepsilon \qquad (3.7)$$

Then there exists a unique quartic mapping such that $D:V\to U_\rho$ which satisfies the inequality

$$\rho(f(2u)-3f(0)-D(u))\le\frac{\theta}{2^5-2^{p+1}}\|u\|^p+\varepsilon \qquad (3.8)$$

for all $u\in V$. where $u\ne 0$.

Proof. Consider,

$$\rho(f(2u)-3f(0)-D(u))\le\frac{1}{2^4}\sum_{i=0}^{\infty}\frac{1}{2}\frac{\phi(2^iu,0)}{2^{4i}}$$

$$\frac{1}{2}\phi\left(\frac{u}{2^i},0\right)=\frac{\theta\|u\|^p}{2^{ip}}+\varepsilon$$

$$\rho(f(2u)-3f(0)-D(u))\le\frac{1}{2}\frac{1}{2^4}\sum_{i=0}^{\infty}\frac{1}{2^{4i}}\theta\|u\|^p\frac{1}{2^{ip}}+\varepsilon$$

$$\le\frac{\theta}{2^5-2^{p+1}}\|u\|^p+\varepsilon.$$

This completes the proof.

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