Oscillation of Third Order Nonlinear Neutral Type Difference equations

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Abstract : This paper is concerned with the oscillatory behavior of solutions of third order difference equations with neutral term. Sufficient conditions guarantee that every solution of

$$\Delta[a_1(n)\Delta[a_2(n)\Delta z(n)]]+q(n)f(x(\sigma(n)))=0, n\geq n_0>0$$

is oscillatory.

IndexTerms **-.Oscillation, Third order, Difference Equation, Neutral type.**

I. INTRODUCTION

Last few decades have seen rapid growth in the study of the qualitative theory of difference equations [1].In the recent years , researchers have paid great attention to the oscillation theory of third-order differential and difference equations [2,5,3,4,6].This papers considers the following form of third order neutral difference equations

$$\Delta[a_1(n)\Delta[a_2(n)\Delta z(n)]]+q(n)f(x(\sigma(n)))=0, n\geq n_0>0 \qquad (1)$$

Where $z(n)=[x(n)+p(n)x^{\alpha}(\tau(n))]$ for the oscillation of its solutions .Assume that the following Conditions holds,

(H_1) $0<\alpha\leq 1$ is the ratio of odd positive integers.

(H_2) $a_1(n), a_2(n), p(n), q(n)$ are positive sequences.

(H_3) $\tau(n), \sigma(n)$ are all positive and $\tau(n)\geq n; \sigma(n)\geq n$;

(H_4) f is non decreasing such that $uf(u)\geq k>0$ for $u\neq 0$ and $\lim\limits_{n\to+\infty}\sigma(n)=\lim\limits_{n\to+\infty}\delta(n)=\infty.$

This paper considers the following two cases:

(A) $\sum\limits_{n=n_0}^{\infty}\frac{1}{a_1(n)}=\infty; \sum\limits_{n=n_0}^{\infty}\frac{1}{a_2(n)}=\infty.$

(B) $\sum\limits_{n=n_0}^{\infty}\frac{1}{a_1(n)}<\infty; \sum\limits_{n=n_0}^{\infty}\frac{1}{a_2(n)}=\infty.$

A Solution $x(n)$ of (1) is said to be oscillatory if the terms $x(n)$ of the solution are not eventually positive or eventually negative .Otherwise the solution is called non oscillatory.

II. MAIN RESULT

Theorem2.1. Let $0\leq p(n)\leq p_1\leq 1$. Assume $(H_1)-(H_4)$ and (A) holds if there exists a positive function ϕ , such that for all sufficiently large $n_3>n_2>n_1\geq n_0$, we have

$$\lim_{n\to\infty} Sup\sum_{m=n_3}^{n-1}\left[kq(m)\phi(m)p_*(m)\Theta(m)-\frac{a_1(m)\Delta^2\phi(m)}{4\phi(m)}\right]=\infty \qquad (2)$$

hold for all constants $M>0$.Then any solution $x(n)$ of (1) is oscillatory.

Proof . Suppose that $x(n)$ is solution of (1).By condition (A),there exists a possible case

$z(n)>0, \Delta z(n)>0, \Delta[a_2(n)\Delta z(n)]>0, \Delta[a_1(n)\Delta[a_1(n)\Delta z(n)]\ ]<0$, for $n>n_1, n_1$ is large enough. Also,

$$\Delta[a_1(n)\Delta[a_2(n)\Delta z(n)]\ ]=-q(n)f(x(\sigma(n)))$$

$$\le -kq(n)x(\sigma(n))$$

Assume that $z(n)$ satisfying the case, there exists $n \ge n_1$ such that $z(n) > 0$; $z(\sigma(n)) > 0$ $\Delta z(n) > 0$, then $z(n)$ is monotonically increasing and there exists a constant $M > 0$ such that $z(n) \ge M$. Now by definition of $z(n)$ we have

$$x(n) = z(n) - p(n)x^{\alpha}(\tau(n)) \ge z(n) - p(n)z^{\alpha}(\sigma(n)) \ge \left[1 - \frac{p(\sigma(n))}{M^{1-\alpha}}\right] z(n) \qquad (3)$$

$$x(n) \ge p_*(n)z(n)$$

Where $p_*(n) = \left[1 - \frac{p(n)}{M^{1-\alpha}}\right]$.Since , $\Delta[a_1(n)\Delta[a_2(n)\Delta z(n)]] < 0$, then $a_1(n)\Delta[a_2(n)\Delta z(n)]$ is decreasing ,so

$$a_2(n)\Delta z(n) \ge \sum_{m=n_1}^{n-1} \frac{a_1(m)\Delta[a_2(m)\Delta z(m)]}{a_1(m)}$$

III.

$$\frac{a_2(n)\Delta z(n)}{\sum_{m=n_1}^{n-1} \frac{1}{a_1(m)}} \ge a_1(n)\Delta[a_2(n)\Delta z(n)] \qquad (4)$$

We have that

$$\Delta\left[\frac{a_2(n)\Delta z(n)}{\sum_{m=n_1}^{n-1} \frac{1}{a_1(m)}}\right] \le 0 \qquad (5)$$

Now , from $\sum_{m=n_2}^{n-1} \Delta z(m) = z(n) - z(n_2)$, we get

IV.

$$z(n) = z(n_2) + \sum_{m=n_2}^{n-1} \Delta z(s) = z(n_2) + \sum_{m=n_2}^{n-1} \frac{a_2(m)\Delta z(m)\sum_{i=n_1}^{m-1}\frac{1}{a_1(i)}}{a_2(m)\sum_{i=n_1}^{m-1}\frac{1}{a_1(i)}}$$

$$\ge \frac{a_2(n)\Delta z(n)}{\sum_{i=n_1}^{n-1}\frac{1}{a_1(i)}} \sum_{m=n_2}^{n-1}\left[\frac{1}{a_2(m)}\sum_{i=n_1}^{m-1}\frac{1}{a_1(i)}\right]$$

$$\frac{z(n)}{a_2(n)\Delta z(n)} \ge \frac{\sum_{m=n_2}^{n-1}\left[\frac{1}{a_2(m)}\sum_{i=n_1}^{m-1}\frac{1}{a_1(i)}\right]}{\sum_{i=n_1}^{n-1}\frac{1}{a_1(i)}} \qquad (6)$$

Let us define $\omega(n)$ as follows

$$\omega(n) = \phi(n)\frac{a_1(n)\Delta[a_2(n)\Delta z(n)]}{a_2(n)\Delta z(n)}, n \ge n_1. \qquad (7)$$

Notice that $\omega(n) > 0$ for $n \ge n_1$. Now

$$\Delta(\omega(n)) = \phi(n)\frac{\Delta[a_1(n)\Delta[a_2(n)\Delta z(n)]]}{a_2(n)\Delta z(n)} + \Delta\phi(n)\frac{a_1(n+1)\Delta[a_2(n+1)\Delta z(n+1)]}{a_2(n+1)\Delta z(n+1)}$$

$$-\phi(n)\frac{a_1(n+1)\Delta[a_2(n+1)\Delta z(n+1)]\Delta[a_2(n)\Delta z(n)]}{a_2(n)\Delta z(n)a_2(n+1)\Delta z(n+1)}$$

It follows from (1) and (7)

$$\Delta\omega(n)=\frac{-q(n)\phi(n)f(x(\sigma(n)))}{a_2(n)\Delta z(n)}+\Delta\phi(n)\frac{\omega(n+1)}{\phi(n+1)}$$
$$-\frac{\phi(n)}{a_1(n)}\frac{\omega(n+1)}{\phi(n+1)}\frac{a_1(n)\Delta[a_2(n)\Delta z(n)]}{a_2(n)\Delta z(n)}$$

$a_1(n)\Delta[a_2(n)\Delta z(n)]$ is a non increasing sequence and $a_2(n)\Delta z(n)$ is an increasing sequence .From (3)

$$\Delta(\omega(n))\le\Delta\phi(n)\frac{\omega(n+1)}{\phi(n+1)}-\frac{\phi(n)}{a_1(n)}\frac{\omega^2(n+1)}{\phi^2(n+1)}-kq(n)\phi(n)\frac{x(\sigma(n))}{a_2(n)\Delta z(n)}$$
$$\le-\left[\frac{\phi(n)}{a_1(n)}\frac{\omega^2(n+1)}{\phi^2(n+1)}-\Delta\phi(n)\frac{\omega(n+1)}{\phi(n+1)}\right]-kq(n)\phi(n)p_*(n)\frac{z(\sigma(n))}{a_2(n)\Delta z(n)}$$
$$\le-\left[\frac{\phi(n)}{a_1(n)}\frac{\omega^2(n+1)}{\phi^2(n+1)}-\Delta\phi(n)\frac{\omega(n+1)}{\phi(n+1)}\right]-kq(n)\phi(n)p_*(n)\Theta(n)$$
$$\le-\left[\sqrt{\frac{\phi(n)}{a_1(n)}}\frac{\omega(n+1)}{\phi(n+1)}-\frac{\Delta\phi(n)}{2}\sqrt{\frac{a_1(n)}{\phi(n)}}\right]^2+\frac{\Delta^2\phi(n)a_1(n)}{4\phi(n)}-kq(n)\phi(n)p_*(n)\Theta(n)$$

Which implies ,

$$\Delta(\omega(n))\le-kq(n)\phi(n)p_*(n)\Theta(n)+\frac{\Delta^2\phi(n)a(n)}{4\phi(n)}$$

Summing from $n_3(>n_2)$ to $n-1$,we get

$$\omega(n)-\omega(n_3)\le\sum_{m=n_3}^{n-1}\left[\frac{\Delta^2\phi(m)a_1(m)}{4\phi(m)}-kq(m)\phi(m)p_*(m)\Theta(m)\right]$$
$$\sum_{m=n_3}^{n-1}\left[kq(m)\phi(m)p_*(m)\Theta(m)-\frac{\Delta^2\phi(m)a_1(m)}{4\phi(m)}\right]\le\omega(n_3)$$

Letting $n\to\infty$,

$$\lim_{n\to\infty}Sup\sum_{m=n_3}^{n-1}\left[kq(m)\phi(m)p_*(m)\Theta(m)-\frac{\Delta^2\phi(m)a_1(m)}{4\phi(m)}\right]\le\omega(n_3)$$

Which contradicts (2) which completes the proof.

Theorem 2.2. Assume $(H_1)-(H_4)$ and (A) holds and we have

$$\sum_{j=n_0}^{\infty}\frac{1}{a_2(j)}\sum_{i=j}^{\infty}\frac{1}{a_1(i)}\sum_{m=i}^{\infty}q(m)=\infty \qquad (8)$$

Then any solution $x(n)$ of (1) is either oscillatory or $\lim_{n\to\infty}x(n)=0$

Proof . Suppose $x(n)$ is the positive solution of (1). By condition (A), there exists a possible case such that $z(n)>0,\Delta z(n)<0,\Delta[a_2(n)\Delta z(n)]>0,\Delta[a_1(n)\Delta[a_2(n)\Delta z(n)]]<0$

Since $z(n)>0$ and $\Delta z(n)<0$,there exists a finite limit , $\lim_{n\to\infty}z(n)=l.$ We shall prove that $l=0$;

assume that $l>0$, then for any $\varepsilon>0$, we have $l+\varepsilon>z(n)>l$; choose $0<\varepsilon<\frac{l(1-p)}{p}$,

$$z(n)=x(n)+p(n)x^\alpha(\tau(n))$$
$$x(n)=z(n)-p(n)x^\alpha(\tau(n))>l-pz(\sigma(n))>1-p(l+\varepsilon)=k(l+\varepsilon)$$
$$>kz(n)\ \text{ where , } k=\frac{l-p(l+\varepsilon)}{l+\varepsilon}>0$$

From (1),

$$\Delta[a_1(n)\Delta[a_2(n)\Delta z(n)]]=-q(n)f(x(\sigma(n)))$$
$$\le-kq(n)z(\sigma(n))$$

Summing from n to ∞, we get

$$\sum_{m=n}^{\infty}\Delta[a_1(m)\Delta[a_2(m)\Delta z(m)]]\le -k\sum_{m=n}^{\infty}q(m)z(\sigma(m))$$

$$\frac{kl}{a_1(n)}\sum_{m=n}^{\infty}q(m)\le \Delta[a_2(n)\Delta z(n)]$$

Summing again from n_1 to ∞, we get

$$\sum_{i=n_1}^{\infty}\left[\frac{kl}{a_1(i)}\sum_{m=i}^{\infty}q(m)\right]\le \sum_{i=n_1}^{\infty}\Delta[a_2(i)\Delta z(i)]$$

$$\frac{kl}{a_2(n_1)}\sum_{i=n_1}^{\infty}\frac{1}{a_1(i)}\sum_{m=i}^{\infty}q(m)\le -\Delta z(n_1)$$

Again taking summing from n_0 to ∞

$$kl\left[\sum_{j=n_0}^{\infty}\frac{1}{a_2(j)}\sum_{i=j}^{\infty}\frac{1}{a_1(i)}\sum_{m=i}^{\infty}q(m)\right]\le -\sum_{j=n_0}^{\infty}\Delta z(j)$$

$$\sum_{j=n_0}^{\infty}\frac{1}{a_2(j)}\sum_{i=j}^{\infty}\frac{1}{a_1(i)}\sum_{m=i}^{\infty}q(m)\le \frac{z(n_0)}{kl}$$

Which contradicts (8). Hence due to condition (8), we get the required result.

Theorem 2.3 . Let $0\le p(n)\le p_1\le 1$. Assume $(H_1)-(H_4)$ and (B) holds , If there exists a positive function δ, such that for all sufficiently large $n_2>n_1\ge n_0$, we have

$$\lim_{n\to\infty} Sup\sum_{m=n_2}^{n-1}\left[kq(m)\delta(m)p_*(m)\sum_{i=n_1}^{\sigma(m)-1}\frac{1}{a_2(i)}-\frac{1}{4a_1(m)\delta(m)}\right]=\infty \qquad (9)$$

Where $p_*(n)=1-\frac{p(n)}{M^{1-\alpha}}$ and $\delta(n)=\sum_{m=n}^{\infty}\frac{1}{a_1(m)}$ hold for all constants $M>0$. Then any solution $x(n)$ of(1) is oscillatory.

Proof. Suppose $x(n)$ is positive solution of (1).By condition (B), there exists a possible case such that

$$z(n)>0,\Delta z(n)>0,\Delta[a_2(n)\Delta z(n)]<0,\Delta[a_1(n)\Delta[a_2(n)\Delta z(n)]]<0.$$

Hence $a_1(n)\Delta[a_2(n)\Delta z(n)]$ is non increasing ,

$$a_1(m)\Delta[a_2(m)\Delta z(m)]\le a_1(n)\Delta[a_2(n)\Delta z(n)],\quad m>n\ge n_1$$

Dividing by $a_1(s)$ and summing n to $l-1$, we get

$$\sum_{m=n}^{l-1}\Delta[a_2(m)\Delta z(m)]\le a_1(n)\Delta[a_2(n)\Delta z(n)]\sum_{m=n}^{l-1}\frac{1}{a_1(m)}$$

$$a_2(l)\Delta z(l)\le a_2(n)\Delta z(n)+a_1(n)\Delta[a_2(n)\Delta z(n)]\sum_{m=n}^{l-1}\frac{1}{a_1(m)}$$

Letting $l\to\infty$

$$\lim_{l\to\infty}a_2(l)\Delta z(l)\le a_2(n)\Delta z(n)+a_1(n)\Delta[a_2(n)\Delta z(n)]\lim_{l\to\infty}\sum_{m=n}^{l-1}\frac{1}{a_1(m)}$$

V.

$$0\le a_2(n)\Delta z(n)+a_1(n)\Delta[a_2(n)\Delta z(n)]\sum_{m=n}^{\infty}\frac{1}{a_1(m)}$$

$$-\frac{a_1(n)\Delta[a_2(n)\Delta z(n)]\sum_{m=n}^{\infty}\frac{1}{a_1(m)}}{a_2(n)\Delta z(n)}\le 1$$

Now define $\phi(n)$ as follows

$$\phi(n)=\frac{a_1(n)\Delta[a_2(n)\Delta z(n)]}{a_2(n)\Delta z(n)}, n\ge n_1. \tag{10}$$

Notice that $\phi(n)<0$ for $n\ge n_1$. Thus

$$-\phi(n)\sum_{m=n}^{\infty}\frac{1}{a_1(m)}\le 1$$

$$-\phi(n)\delta(n)\le 1 \text{ where } \delta(n)=\sum_{m=n}^{\infty}\frac{1}{a_1(m)}$$

We get

$$\Delta\phi(n)=\Delta\left[\frac{a_1(n)\Delta[a_2(n)\Delta z(n)]}{a_2(n)\Delta z(n)}\right]$$

$$=\frac{[a_2(n)\Delta z(n)]\Delta[a_1(n)\Delta[a_2(n)\Delta z(n)]]}{[a_2(n)\Delta z(n)][a_2(n+1)\Delta z(n+1)]}-\frac{a_1(n)\Delta[a_2(n)\Delta z(n)]\Delta[a_2(n)\Delta z(n)]}{[a_2(n)\Delta z(n)][a_2(n+1)\Delta z(n+1)]}$$

$$=\frac{\Delta[a_1(n)\Delta[a_2(n)\Delta z(n)]]}{a_2(n+1)\Delta z(n+1)}-\frac{a_1(n)\Delta^2[a_2(n)\Delta z(n)]}{[a_2(n)\Delta z(n)][a_2(n+1)\Delta z(n+1)]}$$

$$=\frac{-kq(n)x(\sigma(n))}{a_2(n)\Delta z(n)}-\frac{a_1(n)\Delta^2[a_2(n)\Delta z(n)]}{[a_2(n)\Delta z(n)]^2} \text{ since } \Delta[a_2(n)\Delta z(n)]<0$$

$$\Delta\phi(n)=\frac{-kq(n)x(\sigma(n))}{a_2(n)\Delta z(n)}-\frac{\phi^2(n)}{a_1(n)}$$

In view of case ;

$$z(n)\ge a_2(n)\sum_{m=n_1}^{n-1}\frac{1}{a_2(m)}\Delta z(n),$$

$$\frac{z(n)}{a_2(n)\Delta z(n)}\ge\sum_{m=n_1}^{n-1}\frac{1}{a_2(m)} \tag{11}$$

Hence , $\Delta\left[\frac{z(n)}{\sum_{m=n_1}^{n-1}\frac{1}{a_2(m)}}\right]\le 0$, which implies that $\frac{z(\sigma(n))}{z(n)}\ge\frac{\sum_{m=n_1}^{\sigma(n)-1}\frac{1}{a_2(m)}}{\sum_{m=n_1}^{n-1}\frac{1}{a_2(m)}}$.

Now use (3) and (11) in $\Delta\phi(n)$, we have

$$\Delta\phi(n)\le\frac{-kq(n)p_*(n)z(\sigma(n))}{a_2(n)\Delta z(n)}-\frac{\phi^2(n)}{a_1(n)}$$

$$\le -kq(n)p_*(n)\sum_{m=n_1}^{\sigma(n)-1}\frac{1}{a_2(m)}-\frac{\phi^2(n)}{a_1(n)}$$

Thus multiply by $\delta(n)$ and summing from $n_2(>n_1)$ to $n-1$

$$\sum_{m=n_2}^{n-1}\Delta\phi(m)\delta(m)\le\sum_{m=n_2}^{n-1}-kq(m)p_*(m)\delta(m)\sum_{i=n_1}^{\sigma(m)-1}\frac{1}{a_2(i)}$$

$$-\sum_{m=n_2}^{n-1}\left[\frac{\phi^2(m)}{a_1(m)}\delta(m)+\frac{\phi(m)}{a_1(m)}\right]$$

$$\sum_{m=n_2}^{n-1} kq(m)p_*(m)\delta(s)\sum_{i=n_1}^{\sigma(m)-1}\frac{1}{a_2(i)}+\sum_{m=n_2}^{n-1}\left[\frac{\phi^2(m)}{a_1(m)}\delta(m)+\frac{\phi(m)}{a_1(m)}\right]$$

$$\leq -\phi(n)\delta(n)+\phi(n_2)\delta(n_2)$$

Since , $-\phi(n)\delta(n)\leq 1$

$$\sum_{m=n_2}^{n-1} kq(m)p_*(m)\delta(m)\sum_{i=n_1}^{\sigma(m)-1}\frac{1}{a_2(i)}+\sum_{m=n_2}^{n-1}\left[\phi(m)\sqrt{\frac{\delta(m)}{a_1(m)}}+\frac{1}{\sqrt[2]{a_1(m)\delta(m)}}\right]^2$$

$$-\sum_{m=n_2}^{n-1}\frac{1}{4a_1(m)\delta(m)}\leq 1+\phi(n_2)\delta(n_2)$$

$$\sum_{m=n_2}^{n-1}\left[kq(m)p_*(m)\delta(m)\sum_{i=n_1}^{\sigma(m)-1}\frac{1}{a_2(i)}-\frac{1}{4a_1(m)\delta(m)}\right]\leq 1+\phi(n_2)\delta(n_2)$$

Letting $n\to\infty$

$$\lim_{n\to\infty}\sup\sum_{m=n_2}^{n-1}\left[kq(m)p_*(m)\delta(m)\sum_{i=n_1}^{\sigma(m)-1}\frac{1}{a_2(i)}-\frac{1}{4a_1(m)\delta(m)}\right]\leq 1+\phi(n_2)\delta(n_2)$$

Which contradicts (9). This completes the proof.

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