APPROXIMATION PROPERTIES OF CHLODOWSKY TYPE MODIFICATION OF (p,q) −BERNSTEIN DURRMEYER OPERATORS

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Abstract: In this paper, we introduce Chlodowsky type modification of $(\boldsymbol{p},\boldsymbol{q})$ −Bernstein Durrmeyer operators and analyze some of its approximation properties. Here, we apply the concept of $(\boldsymbol{p},\boldsymbol{q})$ −Beta and Gamma functions, to estimate the moments of these operators. Then, we establish the convergence of the operators via well known Korovkin's type theorem. In the end, we estimate the rate of convergence of these operators by means of modulus of continuity and Peetre's $\boldsymbol{K}$ −functional.

Index Terms: $\boldsymbol{q}$ −Calculus, (p,q) −Calculus, (p,q) −Bernstein Durrmeyer operator, Modulus of continuity, Peetre K −functional.

Mathematical subject classification: 41A25, 41A35.

I. Introduction

Approximation theory has an important role in mathematical research. It has a great potential and scope of applications. In 1950, With the famous Korovkin's theorem came into the study, the positive linear operators has been the center of research in approximation theory. Some of the well known operators such as Bernstein, Szasz, Chlodowsky, Baskakov and their generalized forms were introduced. Then the study of q −calculus came into the existence. In 1987, Lupas [2] introduced the q −analogue of Bernstein polynomials which has better approximation properties than the classical one, for a convenient choice of q. For more details on q −generalizations of Bernstein operators, one may refer [1].

In recent years, the concept of (p,q) −calculus was introduced and has become an active area of research. Some of the recent developments in (p,q) −calculus are presented in articles [4, 5, 6, 8, 10]. Before describing the present article , we recall some basic definitions and results.

The (p,q)-number is defined as

$$[n]_{p,q}=\frac{p^n-q^n}{p-q}, n=0,1,2,3\ldots..,$$

which can also be written as

$$[n]_{p,q}=p^{n-1}[n]_{q/p}, n=0,1,2,3\ldots..$$

By putting $p=1$ in the definition of (p,q) −integers, we get q −integers as a particular case of (p,q) −integers.Thus (p,q) −integers are the extension of q-integers.

The (p,q) −factorial is defined as

$$[n]_{p,q}!=\prod_{k=1}^{n}[k]_{p,q}, n\geq 1, [0]_{p,q}!=1.$$

The (p,q) −binomial coefficient is defined as

$$\binom{n}{k}_{p,q}=\frac{[n]_{(p,q)}!}{[n-k]_{(p,q)}!\,[k]_{(p,q)}!}, 0\leq k\leq n.$$

The (p,q) −power basis is given by

$$(x-a)_{p,q}^{n}=(x-a)(px-qa)(p^2x-q^2a)\ldots\ldots\ldots(p^{n-1}x-q^{n-1}a).$$

Let f be an arbitrary function and c be a real number, then (p,q) −integral of $f(x)$ on $[0,c]$ is defined as

$$\int_0^c f(x)d_{p,q}x=(q-p)c\Sigma_{k=0}^{\infty}\frac{p^k}{q^{k+1}}f\left(\frac{p^k}{q^{k+1}}c\right), if\left|\frac{p}{q}\right|<1$$

and

$$\int_0^c f(x)d_{p,q}x=(p-q)c\Sigma_{k=0}^{\infty}\frac{q^k}{p^{k+1}}f\left(\frac{q^k}{p^{k+1}}c\right), if\left|\frac{p}{q}\right|>1.$$

The (p,q) −Gamma function is defined as

$$\Gamma_{p,q}(n+1)=\frac{(p-q)_(p,q)^n}{(p-q)_n}=[n]_(p,q)!\,,0<q<p\leq 1$$

and n is a non negative integer.

The (p,q) −Beta function of first kind is defined as follows:
let $m,n\in\mathbb{N}$ and $0<q<p\leq 1$ then

$$B_{m,n}=\int_0^1 (px)^{m-1}(p-pqx)_{p,q}^{n-1}d_{p,q}x.$$

The (p,q) −generalization of the relation between beta and gamma function is given in [8] as follows:

$$B_{p,q}(m,n)=p^{\frac{[n(2m+n-2)+n-2]}{2}}\frac{\Gamma_{p,q}(m)\Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}, \text{where } m,n\in\mathbb{N}.$$

From the above relation it can be easily seen that the (p,q) −analogue of beta function is not commutative.
The (p,q) −analogue of Bernstein operators for $x\in[0,1]$ and $0<q\le 1$ is defined in [6] as

$$B_{n,p,q}(f;x)=\sum_{k=0}^{n} b_{n,k}^{p,q}(1,x)f\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right)$$

where

$$b_{n,k}^{p_n,q_n}(1,x)=\binom{n}{k}_{p_n,q_n}p_n^{[k(k-1)-n(n-1)]/2}x^k(1-x)_{p_n,q_n}^{n-k}$$

and the moments of the (p,q) −Bernstein polynomials,estimated in[5], are as follows:

$(i)B_{n,p,q}(1;x)=1,$

$(ii)B_{n,p,q}(t;x)=x,$

$(iii)B_{n,p,q}(t^2;x)=x^2+\dfrac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}.$

In [3], Buyukyazici et al. have defined q −Bernstein Chlodowsky durrmeyer operators and investigated some approximation properties for these operators.

Recently, Gupta et al.[8] have introduced the (p,q) −Bernstein Durrmeyer operators and studied many approximation properties of these operators.

Inspired by these operators, here in the present article, we introduce (p,q) −Bernstein Chlodowsky Durrmeyer operators and discuss some of its approximation properties.

II. CONSTRUCTION OF THE OPERATOR

Let $0<q_n<p_n<1$ such that $\lim_{n\to\infty}p_n=\lim_{n\to\infty}q_n=1$ and $0\le x<\alpha_n$ where α_n is a sequence of positive real numbers such that $\lim_{n\to\infty}\alpha_n=\infty$ and $\lim_{n\to\infty}\frac{\alpha_n}{[n]_{p_n,q_n}}=0.$

The (p,q) −Bernstein basis function is defined as

$$b_{n,k}^{p_n,q_n}(1,\tfrac{x}{\alpha_n})=\binom{n}{k}_{p_n,q_n}p_n^{[k(k-1)-n(n-1)]/2}(x/\alpha_n)^k(1-x/\alpha_n)_{p_n,q_n}^{n-k}.$$

Now, we define (p,q) −Bernstein-Chlodowsky-Durrmeyer operators as follows:

$$D_n^{p_n,q_n}(f,x)=\frac{[n+1]_{p_n,q_n}}{\alpha_n}\sum_{k=0}^{n} p_n^{-(n^2+3n-k^2-k)/2}b_{n,k}^{p_n,q_n}(1,\tfrac{x}{\alpha_n})\int_{t=0}^{\alpha_n} b_{n,k}^{p_n,q_n}(p_n,p_nq_n\tfrac{t}{\alpha_n})f(t)d_{p_n,q_n}t \qquad (3.1)$$

where

$$b_{n,k}^{p_n,q_n}(p_n,p_nq_n\tfrac{t}{\alpha_n})=\binom{n}{k}_{p_n,q_n}(\tfrac{p_nt}{\alpha_n})(p_n-p_nq_n\tfrac{t}{\alpha_n})^{n-k}.$$

Lemma 1 *For the above defined sequence of operators $\{D_n^{p_n,q_n}\}$, We have*

$$(i)D_n^{p_n,q_n}(1,x)=1 \qquad (3.2)$$

$$(ii)D_n^{p_n,q_n}(t,x)=\frac{p_n^n\alpha_n}{[n+2]_{p_n,q_n}}+\frac{q_n[n]_{p_n,q_n}x}{[n+2]_{p_n,q_n}} \qquad (3.3)$$

$$(iii)D_n^{p_n,q_n}(t^2,x)=\frac{p_n^{2n}[2]_{p_n,q_n}\alpha_n^2}{[n+2]_{p_n,q_n}[n+3]_{p_n,q_n}}+\frac{(2q_n^2+q_np_n)[n]_{p_n,q_n}x\alpha_n}{[n+2]_{p_n,q_n}[n+3]_{p_n,q_n}}$$
$$+\frac{q_n^3[n]_{p_n,q_n}(x^2[n]_{p_n,q_n}+p_n^{n-1}x(\alpha_n-x))}{[n+2]_{p_n,q_n}[n+3]_{p_n,q_n}}. \qquad (3.4)$$

Proof. We shall apply the concepts of (p,q)-Beta and Gamma function to obtain the proof as follows:

$$(i)D_n^{p_n,q_n}(1,x)=\frac{[n+1]_{p_n,q_n}}{\alpha_n}\sum_{k=0}^{n} p_n^{-(n^2+3n-k^2-k)/2}b_{n,k}^{p_n,q_n}(1,\tfrac{x}{\alpha_n})$$
$$\int_{t=0}^{\alpha_n} b_{n,k}^{p_n,q_n}(p_n,p_nq_n\tfrac{t}{\alpha_n})d_{p_n,q_n}t$$
$$=\frac{[n+1]_{p_n,q_n}}{\alpha_n}\sum_{k=0}^{n} p_n^{-(n^2+3n-k^2-k)/2}b_{n,k}^{p_n,q_n}(1,\tfrac{x}{\alpha_n})$$
$$\alpha_n\int_{t=0}^{1} b_{n,k}^{p_n,q_n}(p_n,p_nq_nu)d_{p_n,q_n}u$$
$$=[n+1]_{p_n,q_n}\sum_{k=0}^{n} p_n^{-(n^2+3n-k^2-k)/2}b_{n,k}^{p_n,q_n}(1,\tfrac{x}{\alpha_n})*$$
$$\int_{t=0}^{1}\binom{n}{k}_{p_n,q_n}(p_nu)^k(p_n-p_nq_nu)^{n-k}d_{p_n,q_n}u$$
$$=[n+1]_{p_n,q_n}\sum_{k=0}^{n} p_n^{-(n^2+3n-k^2-k)/2}b_{n,k}^{p_n,q_n}(1,\tfrac{x}{\alpha_n})\binom{n}{k}_{p_n,q_n}$$
$$B_{p_n,q_n}(k+1,n-k+1)$$
$$=[n+1]_{p_n,q_n}\sum_{k=0}^{n} p_n^{-(n^2+3n-k^2-k)/2}b_{n,k}^{p_n,q_n}(1,\tfrac{x}{\alpha_n})$$
$$\frac{[n]!_{p_n,q_n}}{[k]!_{p_n,q_n}\cdot[n-k]!_{p_n,q\_n}}p_n^{(n^2+3n-k^2-k)/2}\frac{[k]!_{p_n,q_n}\cdot[n-k]!_{p_n,q_n}}{[n+1]!_{p_n,q_n}}$$

$= \sum_{k=0}^{n} p_n^{-(n^2+3n-k^2-k)/2} b_{n,k}^{p_n,q_n}(1, \frac{x}{\alpha_n})$

$= 1$ (3.5)

Again, We have

$(ii) D_n^{p_n,q_n}(t,x) = \frac{[n+1]_{p_n,q_n}}{\alpha_n} \sum_{k=0}^{n} p_n^{-(n^2+3n-k^2-k)/2} b_{n,k}^{p_n,q_n}(1, \frac{x}{\alpha_n})$

$\int_{t=0}^{\alpha_n} b_{n,k}^{p_n,q_n}(p_n, p_n q_n \frac{t}{\alpha_n}) t d_{p_n,q_n} t$

$= \frac{[n+1]_{p_n,q_n}}{\alpha_n} \sum_{k=0}^{n} p_n^{-(n^2+3n-k^2-k)/2} b_{n,k}^{p_n,q_n}(1, \frac{x}{\alpha_n})$

$\alpha_n^2 \int_{t=0}^{1} b_{n,k}^{p_n,q_n}(p_n, p_n q_n u) u d_{p_n,q_n} u$

$= [n+1]_{p_n,q_n} \alpha_n \sum_{k=0}^{n} p_n^{-(n^2+3n-k^2-k)/2} b_{n,k}^{p_n,q_n}(1, \frac{x}{\alpha_n})$

$p_n^{-1} \int_{t=0}^{1} \binom{n}{k}_{p_n,q_n} (p_n u)^{k+1} (p_n - p_n q_n u)^{n-k} d_{p_n,q_n} u$

$= [n+1]_{p_n,q_n} \alpha_n \sum_{k=0}^{n} p_n^{-(n^2+3n-k^2-k)/2} b_{n,k}^{p_n,q_n}(1, \frac{x}{\alpha_n}) \binom{n}{k}_{p_n,q_n}$

$p_n^{-1} B_{p_n,q_n}(k+2, n-k+1)$

$= [n+1]_{p_n,q_n} \alpha_n \sum_{k=0}^{n} p_n^{-(n^2+3n-k^2-k)/2} b_{n,k}^{p_n,q_n}(1, \frac{x}{\alpha_n})$

$\frac{[n]!_{p_n,q_n}}{[k]!_{p_n,q_n} \cdot [n-k]!_{p_n,q\_n}} p_n^{(n^2+5n-k^2-3k)/2} \frac{[k+1]!_{p_n,q_n} \cdot [n-k]!_{p_n,q_n}}{[n+2]!_{p_n,q_n}}$

$= \alpha_n \sum_{k=0}^{n} b_{n,k}^{p_n,q_n}(1, \frac{x}{\alpha_n}) p_n^{n-k} \frac{[k+1]_{p_n,q_n}}{[n+2]_{p_n,q_n}}$

$= \alpha_n \sum_{k=0}^{n} b_{n,k}^{p_n,q_n}(1, \frac{x}{\alpha_n}) p_n^{n-k} \frac{(p_n^k + q_n [k]_{p_n,q_n})}{[n+2]_{p_n,q_n}}$

$= \frac{p_n^n \alpha_n}{[n+2]_{p_n,q_n}} (\sum_{k=0}^{n} b_{n,k}^{p_n,q_n}(1, \frac{x}{\alpha_n}))$

$+ \frac{q_n [n]_{p_n,q_n} \alpha_n}{[n+2]_{p_n,q_n}} (\sum_{k=0}^{n} b_{n,k}^{p_n,q_n}(1, \frac{x}{\alpha_n}) \frac{p_n^{n-k} [k]_{p_n,q_n}}{[n]_{p_n,q_n}})$

Using the definition of (p,q) −Bernstein operators, we obtain

$D_n^{p_n,q_n}(t,x) = \frac{p_n^n \alpha_n}{[n+2]_{p_n,q_n}} + \frac{q_n [n]_{p_n,q_n} \alpha_n}{[n+2]_{p_n,q_n}} \cdot \frac{x}{\alpha_n}$

$= \frac{p_n^n \alpha_n + q_n [n]_{p_n,q_n} x}{[n+2]_{p_n,q_n}}$ (3.6)

Now, following the proof $[i]$, $[ii]$ and using the identity

$$[k+2]_{p,q} = p^{k+1} + q p^k + q^2 [k]_{p,q}$$

we get

$(iii) D_n^{p_n,q_n}(t^2,x) = \frac{p_n^{2n} [2]_{p_n,q_n} \alpha_n^2}{[n+2]_{p_n,q_n} [n+3]_{p_n,q_n}} + \frac{(2q_n^2 + q_n p_n)[n]_{p_n,q_n} x \alpha_n}{[n+2]_{p_n,q_n} [n+3]_{p_n,q_n}}$

$+ \frac{q_n^3 [n]_{p_n,q_n} (x^2 [n]_{p_n,q_n} + p_n^{n-1} x(\alpha_n - x))}{[n+2]_{p_n,q_n} [n+3]_{p_n,q_n}}$ (3.7)

III. CONVERGENCE OF THE OPERATORS

In this section, we shall obtain the convergence of the proposed sequence operators using well known Korovkin's theorem.

Theorem 1 *If a be a sufficiently large fixed positive real number then for all $f \in [0,a]$,*

$$\lim_{n\to\infty} ||D_n^{p_n,q_n}(f;x) - f(x)||_{C[0,a]} = 0.$$

where $||.||$ is the superemum norm.

Proof. From (3.2)-(3.4) and following [3], we observe that

$||D_n^{p_n,q_n}(1;x) - 1||_{C[0,a]} = 0,$

$||D_n^{p_n,q_n}(t;x) - x||_{C[0,a]} \leq \frac{\alpha_n}{[n+2]_{p_n,q_n}}$

and

$$||D_n^{p_n,q_n}(t^2;x) - x^2||_{C[0,a]} \leq \frac{2\alpha_n^2}{[n]_{p_n,q_n}^2} + 4a \frac{\alpha_n}{[n]_{p_n,q_n}}.$$

Then, the proof is completed by applying a Korovkin-type theorem.

Lemma 2 *Let $n > 3$ be a given natural number and let $0 < q_n < p_n \leq 1, q_0 = q_0(n) \in (0, p_n)$ be the least number such that*

$$p_n^{2n+1} q_n - p_n^{n+1} q_n^{n+1} + p_n^{2n-1} q_n^3 - p_n^{n-1} q_n^{n+3} + p_n^{2n} q_n^2 - p_n^n q_n^{n+2} - 2p_n^{2n+3} + 2p_n^n q_n^{n+3} > 0 \quad (3.8)$$

Then, for every $q_n \in (q_0, 1)$.

$$D_n^{p_n,q_n}((t-x)^2, x) \leq \frac{2}{[n+2]_{p_n,q_n}} (\phi^2(x) + \frac{1}{[n+3]_{p_n,q_n}})$$

where $\phi^2(x) = x(\alpha_n - x), x \in [0, \alpha_n]$.

Proof. In view of Lemma-1, we obtain

$$D_n^{p_n,q_n}((t-x)^2,x) = x^2\frac{1}{[n+2]_{p_n,q_n}[n+3]_{p_n,q_n}}(q_n^3[n]_{p_n,q_n}([n]_{p_n,q_n}-p_n^{n-1})$$
$$-2q_n[n]_{p_n,q_n}[n+3]_{p_n,q_n}+[n+2]_{p_n,q_n}[n+3]_{p_n,q_n})$$
$$+\alpha_n x\frac{p_n^{n-1}q_n(p_n+q_n)^2[n]_{p_n,q_n}-2p_n^n[n+3]_{p_n,q_n}}{[n+2]_{p_n,q_n}[n+3]_{p_n,q_n}}$$
$$+\alpha_n^2\frac{p_n^{2n}(p_n+q_n)}{[n+2]_{p_n,q_n}[n+3]_{p_n,q_n}} \quad (3.9)$$

by using the definition of the (p,q) −numbers, we have

$$p_n^{n-1}q_n(p_n+q_n)^2[n]_{p_n,q_n}-2p_n^n[n+3]_{p_n,q_n} = p_n^{n-1}q_n(p_n+q_n)^2\frac{p_n^n-q_n^n}{p_n-q_n}$$
$$-2p_n^n\frac{p_n^{n+3}-q_n^{n+3}}{p_n-q_n} \quad (3.10)$$
$$=\frac{1}{p_n-q_n}(p_n^{2n+1}q_n-p_n^{n+1}q_n^{n+1}+p_n^{2n-1}q_n^3$$
$$-p_n^{n-1}q_n^{n+3}+p_n^{2n}q_n^2-p_n^nq_n^{n+2}-2p_n^{2n+3}$$
$$+2p_n^nq_n^{n+3})>0, \quad (3.11)$$

for every $q_n\in(q_0,1)$.

furthermore,

$$p_n^{n-1}q_n(p_n+q_n)^2[n]_{p_n,q_n}-2p_n^n[n+3]_{p_n,q_n}\le 2[n+3]_{p_n,q_n}$$

and following [8], we have

$$p_n^{n-1}q_n(p_n+q_n)^2[n]_{p_n,q_n}-2p_n^n[n+3]_{p_n,q_n}+q_n^3[n]_{p_n,q_n}([n]_{p_n,q_n}-p_n^{n-1})$$
$$-2q_n[n]_{p_n,q_n}[n+3]_{p_n,q_n}+[n+2]_{p_n,q_n}[n+3]_{p_n,q_n}$$
$$=p_n^{n-1}q_n(p_n+q_n)^2[n]_{p_n,q_n}-2p_n^n(p_n^{n+2}+q_np_n^{n+1}$$
$$+q_n^2p_n^n+q_n^3[n]_{p_n,q_n})+q_n^3[n]_{p_n,q_n}^2-q_n^3p_n^{n-1})[n]_{p_n,q_n}$$
$$-q_n[n]_{p_n,q_n}(p_n^{n+2}+q_np_n^{n+1}+q_n^2p_n^n+q_n^3[n]_{p_n,q_n})$$
$$+(p_n^{n+1}+q_np_n^n+q^2[n]_{p_n,q_n})$$
$$(p_n^{n+2}+q_np_n^{n+1}+q_n^2p_n^n+q_n^3[n]_{p_n,q_n})\le 0. \quad (3.12)$$

Therefore, for $x\in[0.\alpha_n]$, we have

$$D_n^{p_n,q_n}((t-x)^2,x)=\alpha_n^2\frac{p_n^{2n}(p_n+q_n)}{[n+2]_{p_n,q_n}[n+3]_{p_n,q_n}}$$
$$+\frac{p_n^{n-1}q_n(p_n+q_n)^2[n]_{p_n,q_n}-2p_n^n[n+3]_{p_n,q_n}}{[n+2]_{p_n,q_n}[n+3]_{p_n,q_n}}x(\alpha_n-x)$$
$$+\frac{1}{[n+2]_{p_n,q_n}[n+3]_{p_n,q_n}}(p_n^{n-1}q_n(p_n+q_n)^2[n]_{p_n,q_n}$$
$$-2p_n^n[n+3]_{p_n,q_n}+q_n^3[n]_{p_n,q_n}([n]_{p_n,q_n}-p_n^{n-1})$$
$$-2q_n[n]_{p_n,q_n}[n+3]_{p_n,q_n}+[n+2]_{p_n,q_n}[n+3]_{p_n,q_n})x^2$$
$$\le\frac{2[n+3]_{p_n,q_n}}{[n+2]_{p_n,q_n}[n+3]_{p_n,q_n}}\phi^2(x)+\frac{2\alpha_n^2}{[n+2]_{p_n,q_n}[n+3]_{p_n,q_n}}$$
$$\le\frac{2}{[n+2]_{p_n,q_n}}(\phi^2(x)+\frac{1}{[n+3]_{p_n,q_n}}) \quad (3.13)$$

which was required.

IV. RATE OF CONVERGENCE

We denote $W^2=\{g\in C[0,\infty): g',g''\in C[0,\infty)\}$, for $\delta>0$, K-functional is defined as

$$K_2(f,\delta)=inf\{||f-g||+\eta||g''||: g\in W^2\}$$

Where norm $||.||$ denotes the uniform norm on $C[0,\infty)$. Following the well-known inequality given in DeVore and Lorentz[7], there exists an absolute constant $C>0$ such that

$$K_2(f,\delta)\le C\omega_2(f,\sqrt{\delta})$$

Where, the second order modulus of continuity for $f\in C[0,\infty)$ is defined as

$$\omega_2(f,\sqrt{\delta})=\sup_{0<h\le\sqrt{\delta}x}\ \sup_{x+h\in[0,\alpha_n]}|f(x+h)-f(x)|$$

The usual modulus of continuity for $f\in C[0,\infty)$ is defined as

$$\omega(f,\delta)=\sup_{0<h\le\sqrt{\delta}x}\ \sup_{x+h\in[0,\alpha_n]}|f(x+h)-f(x)|$$

Now, we have the following theorem:

Theorem 2 *Let* $n>3$ *be a given natural number and let* $0<q_n<p_n\le 1, q_0=q_0(n)\in(0,p_n)$*be defined as in Lemma-2. Then there exists an absolute constant* $C>0$ *such that*

$$\left|D_n^{p_n,q_n}(f;x)-f(x)\right|\le C\omega_2(f,[n+2]_{p_n,q_n}^{-\frac{1}{2}}\delta_n(x))\ +\omega\left(f,\frac{\alpha_n-x}{[n+2]_{p_n,q_n}}\right),$$

where $f\in C[0,\alpha_n],\delta_n^2(x)=\phi^2(x)+\frac{\alpha_n^2}{[n+3]_{p_n,q_n}}, x\in[0,\alpha_n]$ and $q\in(q0,1)$.

Proof. For $f\in C[0,\alpha_n]$, we define

$$\widetilde{D}_n^{p_n,q_n}(f;x) = D_n^{p_n,q_n}(f;x) + f(x) - f\left(\frac{p_n^n\alpha_n + q_n[n]_{p_n,q_n}x}{[n+2]_{p_n,q_n}}\right). \quad (4.1)$$

Then, by Lemma-2, we immediately get

$$\widetilde{D}_n^{p_n,q_n}(1;x) = D_n^{p_n,q_n}(1;x) = 1 \quad (4.2)$$

and

$$\widetilde{D}_n^{p_n,q_n}(t;x) = D_n^{p_n,q_n}(t;x) + x - \left(\frac{p_n^n\alpha_n + q_n[n]_{p_n,q_n}x}{[n+2]_{p_n,q_n}}\right) = x. \quad (4.3)$$

By Taylor's formula

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du,$$

we have

$$\widetilde{D}_n^{p_n,q_n}(g;x) = g(x) + \widetilde{D}_n^{p_n,q_n}\left(\int_x^t (t-u)g''(u)du;x\right) \quad (4.4)$$
$$= g(x) + D_n^{p_n,q_n}\left(\int_x^t (t-u)g''(u)du;x\right)$$
$$-\int_x^{\frac{p_n^n\alpha_n+q_n[n]_{p_n,q_n}x}{[n+2]_{p_n,q_n}}}\left(\frac{p_n^n\alpha_n+q_n[n]_{p_n,q_n}x}{[n+2]_{p_n,q_n}} - u\right)g''(u)du$$

Thus

$$|\widetilde{D}_n^{p_n,q_n}(g;x) - g(x)| \le D_n^{p_n,q_n}\left[\left|\int_x^t |t-u||g''(u)|du\right|;x\right]$$
$$+\left|\int_x^{\frac{p_n^n\alpha_n+q_n[n]_{p_n,q_n}x}{[n+2]_{p_n,q_n}}}\left|\frac{p_n^n\alpha_n+q_n[n]_{p_n,q_n}x}{[n+2]_{p_n,q_n}} - u\right||g''(u)|du\right|$$
$$\le D_n^{p_n,q_n}((t-x)^2;x)||g''||$$
$$+\left(\frac{p_n^n\alpha_n+q_n[n]_{p_n,q_n}x}{[n+2]_{p_n,q_n}} - x\right)^2||g''||. \quad (4.5)$$

Also, we have

$$D_n^{p_n,q_n}((t-x)^2;x) + \left(\frac{p_n^n\alpha_n+q_n[n]_{p_n,q_n}x}{[n+2]_{p_n,q_n}} - x\right)^2$$
$$\le \frac{2}{[n+2]_{p_n,q_n}}\left(\phi^2(x) + \frac{1}{[n+3]_{p_n,q_n}}\right)$$
$$+\left(\frac{p_n^n\alpha_n - ([n+2]_{p_n,q_n} - q_n[n]_{p_n,q_n})}{[n+2]_{p_n,q_n}}\right)^2. \quad (4.6)$$

further, it can be easily seen that

$$1 \le [n+2]_{p_n,q_n} - q_n[n]_{p_n,q_n} \le 2. \quad (4.7)$$

Now, using (4.7), we have

$$\left(\frac{p_n^n\alpha_n - ([n+2]_{p_n,q_n} - q_n[n]_{p_n,q_n})}{[n+2]_{p_n,q_n}}\right)^2 \delta_n^{-2}(x)$$
$$= \frac{1}{[n+2]_{p_n,q_n}^2}\{p_n^{2n}\alpha_n^2 - 2p_n^n([n+2]_{p_n,q_n} - q_n[n]_{p_n,q_n})x\alpha_n$$
$$+([n+2]_{p_n,q_n} - q_n[n]_{p_n,q_n})^2x^2\}\ \frac{[n]_{p_n,q_n}}{[n]_{p_n,q_n}x(\alpha_n - x) + \alpha_n^2},$$
$$\le \frac{p^{2n}\alpha_n^2 - 2p_n^n x\alpha_n + 4x^2}{[n+2]_{p_n,q_n}}\cdot\frac{[n]_{p_n,q_n}}{[n+2]_{p_n,q_n}}\cdot\frac{1}{[n]_{p_n,q_n}x(\alpha_n - x) + \alpha_n^2},$$
$$\le \frac{p^{2n} - 2p_n^n(\frac{x}{\alpha_n}) + 4(\frac{x}{\alpha_n})^2}{[n+2]_{p_n,q_n}}\cdot\frac{1}{[n]_{p_n,q_n}(\frac{x}{\alpha_n})(1 - (\frac{x}{\alpha_n})) + 1},$$
$$\le \frac{3}{[n+2]_{p_n,q_n}}. \quad (4.8)$$

for $n \in \mathbb{N}$ and $0 < q_n < p_n \le 1$. Now, using (4.6) and (4.8), for $x \in [0, \alpha_n)$ we have

$$D_n^{p_n,q_n}((t-x)^2;x) + \left(\frac{p_n^n\alpha_n+q_n[n]_{p_n,q_n}x}{[n+2]_{p_n,q_n}} - x\right)^2$$
$$\le \frac{5}{[n+2]_{p_n,q_n}}\delta_n^2(x). \quad (4.9)$$

Using (4.5) with the condition $n > 3$ and $x \in [0, \alpha_n)$, we obtain

$$|\widetilde{D}_n^{p_n,q_n}(g;x) - g(x)| \le \frac{5}{[n+2]_{p_n,q_n}}\delta_n^2(x)||g''(x)||. \quad (4.10)$$

Furthermore, for $f \in C[0,\infty)$ we have $||D_n^{p_n,q_n}(f,x)|| \le ||f||$,

Therefore,

$$|\widetilde{D}_n^{p_n,q_n}(f;x)| \le |D_n^{p_n,q_n}(f;x)| + |f(x)| + \left|f\left(\frac{p_n^n\alpha_n+q_n[n]_{p_n,q_n}x}{[n+2]_{p_n,q_n}}\right)\right|$$
$$\le 3||f||,\ for\ all\ f \in C[0,\infty). \quad (4.11)$$

Again, for $f \in C[0,\infty)$ and $g \in W^2$, we have

$$|D_n^{p_n,q_n}(f;x) - f(x)|$$
$$= \left|\widetilde{D}_n^{p_n,q_n}(f;x) - f(x) + f\left(\frac{p_n^n\alpha_n+q_n[n]_{p_n,q_n}x}{[n+2]_{p_n,q_n}}\right) - f(x)\right|$$
$$\le |\widetilde{D}_n^{p_n,q_n}(f-g;x)| + |\widetilde{D}_n^{p_n,q_n}(g;x) - g(x)| + |g(x) - f(x)|$$
$$+\left|f\left(\frac{p_n^n\alpha_n+q_n[n]_{p_n,q_n}x}{[n+2]_{p_n,q_n}}\right) - f(x)\right|$$

$$\leq 4||f-g||+\frac{5}{[n+2]_{p_n,q_n}}.\delta_n^2(x).||g''||$$
$$+\omega\left(f,\left|\frac{p_n^n\alpha_n-([n+2]_{p_n,q_n}-q_n[n]_{p_n,q_n})x}{[n+2]_{p_n,q_n}}\right|\right)$$
$$\leq (||f-g||+\frac{1}{[n+2]_{p_n,q_n}}.\delta_n^2(x).||g''||)+\omega\left(f,\frac{\alpha_n-x}{[n+2]_{p_n,q_n}}\right),$$
$(using(4.10)\ \ and\ (4.11))$.

Now, taking the infimum on the right hand side over all $g\in W^2$, we obtain

$$\left|D_n^{p_n,q_n}(f;x)-f(x)\right|\leq 5K_2\left(f,\frac{1}{[n+2]_{p_n,q_n}}.\delta_n^2(x)\right)$$
$$+\omega\left(f,\frac{\alpha_n-x}{[n+2]_{p_n,q_n}}\right)$$

Finally, we have

$$|D_n^{p_n,q_n}(f;x)-f(x)|\leq C\omega_2(f,[n+2]_{p_n,q_n}^{-\frac{1}{2}}.\delta_n(x))\ +\omega\left(f,\frac{\alpha_n-x}{[n+2]_{p_n,q_n}}\right)$$

which is the required result.

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