Smarandache curves in Euclidean space of parallel transport frame

Dr. Shankar Lal, Assistant Professor, Department of Mathematics, H.N.B. Garhwal University, S.R.T. Campus, Badshahithaul, Tehri Garhwal, Uttarakhand (**India**)

Abstract: In the present paper, we introduce Smarandache curves in the Euclidean space of parallel transport frame. In the section **one**, we give basic tools of parallel transport frame of a curve in 4-dimensional Euclidean space. In section **two**, we study the parallel transport frame of Euclidean space, besides we solve few theorems, corollary and illustrate examples. Again section **three,** we define parallel transport frame to the Smarandache curve and obtain some definitions, theorems and their apparatus. Further section **four**, we have also explained to Frenet frame of principal normal, binomial and derivatives in the curvature of the curve. In the **end,** we discussed about the Smarandache curve in the Euclidean space of all apparatus Ferret-Serret.

2010 MSC: 14H45 14H50 53A04

Key Words: Smarandache Curves, Frenet-Serret, Euclidean Space, parallel transport frame etc.

1. Introduction

Let $\beta : I \subset R \to E^4$ be arbitrary curve in the 4-dimensional Euclidean space in E^4. $X = (x_1, x_2, x_3, x_4)$, $Y = (y_1, y_2, y_3, y_4)$, and $Z = (z_1, z_2, z_3, z_4)$, where X, Y, Z be any three vectors in E^4. The curve β is parameterized by arc length of the function s if $< \beta'(s), \beta'(s) >= 1$, together with the inner product of E^4 given by

(1.1) $\quad < X, Y >= x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4$

In particular, the norm of a vector $X \in E^4$ is given by

$$\|X\| = \sqrt{< X, X >} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$$

$$\|X\| = \left(\sum_{i=1}^{4} |x_4|^p \right)^{\frac{1}{p}}, \qquad p = 1 \qquad \Rightarrow \qquad \|x_4\|_\infty = \max |x_4|$$

$$\Rightarrow \qquad \frac{\partial}{\partial x_k} \|x\|_p = \frac{x_k |x_k|^{p-2}}{\|x\|_p^{\,p-1}} \qquad \Rightarrow \qquad \frac{\partial}{\partial x} \|x\|_p = \frac{x \bullet |x_k|^{p-2}}{\|x\|_p^{\,p-1}}$$

If $p = 2$, then $\qquad \frac{\partial}{\partial x} \|x\|_2 = \frac{x}{\|x\|_2}$

The vector product X, Y and Y is defined by the determinant

(1.2) $$X \times Y \times Z = \begin{vmatrix} a & b & c & d \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

Let (t, m_1, m_2, m_3) be the moving Frenet frame along the unit speed curve β. Then t, n, b_1 and b_2 are the tangent, the principal normal, first and second binomial vectors of the curve β, respectively. Then Frenet-Serret frame is given by

$$(1.3)\quad \begin{bmatrix} t' \\ n' \\ b_1' \\ b_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b_1 \\ b_1 \end{bmatrix}$$

Where $< t,t >=< n,n >=< b_1,b_1 >=< b_2,b_2 >= 1$

$< t,n >=< t,b_1 >=< n,b_1 >=< t,b_2 >=< n,b_2 >= 0$

Here κ,τ,σ denote principal curvature to the Serret-Frenet frame of the curve β. While t', n', b_1' and b_2' are called tangent, principal normal and first and second binormal.

Let $\beta = \beta(t)$ be an arbitrary curve in E^4. The Serret-Frenet apparatus of the curve β can be solved by the following equations as given below

$$t = \frac{\beta'}{\|\beta'\|}, \qquad n = \|\beta'\|^2 \beta'' - < \beta', \beta'' > \beta', \qquad b_1 = \rho(b_2 \times t \times n)$$

$$b_2 = \rho\left\{\frac{t \times n \times \beta'''}{\|t \times n \times \beta'''\|}\right\}, \qquad \kappa = \frac{\|\|\beta'\|\beta'' - < \beta', \beta'' >\|}{\|\beta'\|^4}$$

$$\tau = \frac{\|t \times n \times \beta'''\|\|\beta'\|}{\|\|\beta'\|^2 \beta'' - < \beta', \beta'' > \beta'\|}, \qquad \sigma = \frac{< \beta'''', b_2 >}{\|t \times n \times \beta'''\|\|\beta'\|}.$$

Where ρ is taken ± 1 such that determinant of matrix $[t,n,b_1,b_2] = 1$.

We use the tangent vector $T(s)$ and three relatively parallel vector fields m_1, m_2 and m_3 to construct an alternative frame. We call this frame a parallel transport frame along the curve. The reason for the name parallel transport frame is because the normal component of the derivatives of the normal vector field is zero.

If the set (t, m_1, m_2, m_3) as parallel transport frame and

$$k_1 =< t', m_1 >, \qquad k_2 =< t', m_2 >, \qquad k_3 =< t', m_3 >$$

as parallel transport curvatures.

Using Euler angles an arbitrary rotation matrix is given by

$$\begin{bmatrix} \cos\theta\cos\psi & -\cos\phi\sin\phi + \sin\phi\sin\theta\sin\psi & \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi \\ \cos\theta\cos\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix}$$

, Where angels θ,ϕ,ψ are Euler angles.

2. Parallel Transport Frame of Euclidean Space

In this section, we give parallel transport frame of a Smarandache curve and we introduce the relations between the frame and Frenet frame of the Smarandache curve in Euclidean space E^4 using the Euler angles. The relation which is well known in Euclidean space E^4 is generalized for the first time in 4-dimensional Euclidean space E^4.

Theorem 1: Let (t, m_1, m_2, m_3) be a Frenet frame along a unit speed curve $\beta : I \subset R \to E^4$ and (t, m_1, m_2, m_3) denotes the parallel transport frame of the curve β. The relation may be expressed as the arc is given by

$$t = t(s),$$

$$n = m_1(\cos\theta\cos\psi) + m_2(-\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi) + m_3(\sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi)$$

$$b_1 = m_1(\cos\theta\sin\psi) + m_2(\cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi)$$

$$+m_3(-\sin\phi\cos\psi+\sin\theta\cos\phi\sin\psi)$$

$$b_2=m_1(-\sin\theta)+m_2(\sin\phi\cos\theta)+m_3(\cos\phi\cos\theta)$$

The alternative parallel transport frame equations E^4 are given by

$$(2.1)\quad \begin{bmatrix} t' \\ m_1' \\ m_2' \\ m_3' \end{bmatrix}=\begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & 0 & 0 \\ -k_2 & 0 & 0 & 0 \\ -k_3 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} t \\ m_1 \\ m_2 \\ m_3 \end{bmatrix}$$

where k_1, k_2 and k_3 are principal curvature functions according to parallel transport frame of the curve β and their expression as follows

$$k_1=\kappa\cos\theta\cos\phi,$$

$$k_2=\kappa(-\cos\phi\sin\psi+\sin\theta\sin\phi\cos\psi),$$

$$k_3=\kappa(\sin\phi\sin\psi+\sin\theta\cos\phi\cos\psi)$$

and $\kappa=\sqrt{k_1^2+k_2^2+k_3^2}$,

$$\tau=-\psi'+\phi'\sin\theta,$$

$$\sigma=\frac{\theta'}{\sin\psi},$$

$$\phi'\cos\theta+\theta'\cot\psi=0.$$

Where $\theta'=\dfrac{\sigma}{\sqrt{\kappa^2+\tau^2}}$, $\quad\psi'=-\tau-\sigma\dfrac{\sqrt{\sigma^2-\theta'^2}}{\sqrt{\kappa^2+\tau^2}}$, $\quad\phi'=-\dfrac{\sqrt{\sigma^2-\theta'^2}}{\cos\theta}$.

Proof: Given that the above theorem, differentiating the m_1, m_2, m_3 with respect to s, we get

$$m_1'=(-\kappa\cos\theta\cos\psi)t+(\theta'\sin\theta\cos\psi-\psi'\cos\theta\sin\psi-\tau\cos\theta\sin\psi)n$$
$$+(\theta'\sin\theta\cos\psi-\psi'\cos\theta\sin\psi-\tau\cos\theta\sin\psi)n$$
$$+(-\theta'\sin\theta\sin\psi-\psi'\cos\theta\cos\psi-\sigma\sin\theta)b_1$$
$$+(-\theta'\cos\theta-\sigma\cos\theta\sin\psi)b_2,$$

$$m_2'=-\kappa[-\cos\phi\sin\psi+\sin\theta\sin\phi\sin\psi]t+[\phi'\sin\phi\sin\psi-\psi'\cos\phi\cos\psi$$
$$+\phi'\sin\theta\cos\phi\cos\psi+\theta'\sin\phi\cos\theta\cos\psi-\psi'\sin\theta\sin\phi\sin\psi-k_2(\cos\phi\cos\psi$$
$$+\sin\theta\sin\phi\sin\psi)]n+[-\phi'\sin\phi\cos\psi-\psi'\cos\phi\sin\psi+\phi'\sin\theta\cos\phi\sin\psi$$
$$+\theta'\cos\theta\sin\phi\sin\psi+\psi'\sin\theta\sin\phi\cos\psi+\tau(-\cos\phi\sin\psi+\sin\theta\sin\phi\cos\psi)$$
$$-\sigma\cos\theta\sin\phi]b_1+[\phi'\cos\theta\cos\phi-\theta'\sin\theta\sin\phi$$
$$+\sigma(\cos\phi\sin\psi+\sin\theta\sin\phi\sin\psi]b_2$$

$$m_3'=[-\kappa(\sin\phi\sin\psi+\sin\theta\cos\phi\cos\psi)]t+[(\phi'\cos\phi\sin\psi+\psi'\sin\phi\cos\psi$$
$$-\phi'\sin\theta\sin\phi\cos\psi+\theta'\cos\theta\cos\phi\cos\psi-\psi'\sin\theta\cos\phi\sin\psi-\tau(\sin\phi\cos\psi$$
$$+\sin\theta\cos\phi\sin\psi)]n+[(-\phi'\cos\phi\cos\psi+\psi'\sin\phi\sin\psi-\phi'\sin\theta\sin\phi\sin\psi$$
$$+\theta'\cos\theta\cos\phi\sin\psi+\psi'\sin\theta\cos\phi\cos\psi+\tau(\sin\phi\sin\psi+\sin\theta\cos\phi\cos\psi)$$
$$+\sigma(\cos\theta\cos\phi)]b_1+(-\phi'\cos\theta\sin\phi-\theta'\sin\theta\cos\phi)$$
$$+\sigma(-\sin\phi\cos\psi+\sin\theta\cos\phi\sin\psi)]b_2.$$

Since m_1, m_2 and m_3 are relatively parallel vector field, normal component of the m_1', m_2' and m_3' must be zero and the equalities are satisfy

$$<m_1',m_2>=<m_1',m_3>=<m_2',m_1>=<m_2',m_3>=<m_3',m_1>=<m_3',m_2>=0$$

Also, if we consider that the parallel transport frame of the curve β we can easily complete proof that the above theorem.

Corollary 1: If we consider $\theta = \phi = \psi = 0$ that then we get the parallel transport frame in E^4. Now, we give an example for the curve which has not a Frenet frame at some points but, it has parallel transport frame on these points.

Example 1: Let $\beta(s) = (\sin s, 2s+1, 2s-1, s)$ be a curve in Euclidean space. Since $\beta''(0) = (0,0,0,0)$ we cannot calculate the Frenet frame vectors at the point $s = 0$. However, we can calculate using the parallel transport frame vectors as follows

$$m_1 = b_1 \sin\psi, \qquad m_2 = b_1 \cos\phi \cos\psi, \qquad m_3 = -b_1 \sin\phi \cos\psi,$$

Where ϕ and ψ are constant angles.

Theorem 2: Let $\beta : I \subset R \to E^4$ be a curve with nonzero curvatures $k_i\,(i = 1,2,3)$ according to parallel transport frame in Euclidean space E^4. Then lies β on a sphere if and only if $ak_1 + bk_2 + ck_3 + 1 = 0$ where $a, b,$ and c are non-zero constants.

Proof: Let β lies on a sphere with center P and radius R, then

$$<\beta - P, \beta - P> = R^2$$

Differentiating this equation with respect to s, it gives us

$$<t, \beta - P> = 0, \qquad \Rightarrow \qquad \beta - P = am_1 + bm_2 + cm_3$$

For some function $a, b,$ and c.

$$a' = <\beta - P, m_1> = <t, m_1> + <k_1 t, \beta - P> = 0$$

So a is a constant. Similarly, we can easily say that b and c are constants. Then differentiating the equation $<t, \beta - P>$ with respect to s we get

$$<k_1 m_1 + k_2 m_2 + k_3 m_3, \beta - P> + <t, t> = ak_1 + bk_2 + ck_3 + 1 = 0.$$

That is, between k_1, k_2 and k_3 has the linear relation such as

$$ak_1 + bk_2 + ck_3 + 1 = 0$$

Moreover, $\qquad R^2 = <\beta - P, \beta - P> = a^2 + b^2 + c^2 = \frac{1}{d^2} d$

, where d is the distance of the plane $axby + cz + 1 = 0$ from the origin.

Conversely, suppose that the equation holds

$$ak_1 + bk_2 + ck_3 + 1 = 0$$

If P is denoted by

$$P = \beta - am_1 - bm_2 - cm_3,$$

then differentiating the last equation we have

$$P' = t + (ak_1 + bk_2 + ck_3)t = 0$$

so P is constant. Similarly shows that

$$R^2 = <\beta - P, \beta - P> \quad \text{is constant.}$$

So, β lies on a sphere with center P and radius R.

Example 2: Let $\beta(s) = \left(\sin\frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}\cos s\right)$ be a curve in Euclidean space E^4. According the Frenet frame there are lots of formulas for showing that this curve is a spherical curve. But the formulas have some disadvantages which were define the above chapters. Then we calculate curvature functions of the curve β according to parallel transport frame.

$$k_1 = 0, \; k_2 = -\cos\phi \; k_3 = \sin\phi$$

, where ϕ is constant. The curve β satisfy the fallowing equation

$$ak_1 + bk_2 + ck_3 + 1 = 0.$$

Consequently, the curve β is a spherical curve. But, using the Frenet curvatures we cannot show that β is a spherical curve. Because has a zero torsion.

3. Parallel Transport frame in tb_1 and tm_1 to the Smarandache Curve in E^4

In this section, we define tb_1 and tm_1 Smarandache Curve according to parallel transport frames in E^4 and obtains some characterizations for such curves.

3.1 tb_1 Smarandache Curves according to the Parallel Transport Frame

In this subsection we define tb_1 Smarandache curves to the parallel transport frame and obtain their Frenet apparatus.

Definition 1: A regular curve in the 4-dimensional Euclidean space, whose position vector is obtained by Frenet frame vectors on another regular curve, is called Smarandache curve.

Definition 2: Let $\beta = \beta(s)$ be a unit-speed curve with constant and nonzero curvatures k_1, k_2, k_3 and (t, n, b_1, b_1) be moving frame on it tb_1 Smarandache curves are defined by $\alpha(S_\alpha) = \frac{1}{\sqrt{2}}[t(s) + b_1(s)]$.

Theorem 1: Let $\beta(s)$ be a unit speed curve with constant non zero curvatures k_1, k_2, k_3 and $\alpha(s_\alpha)$ be tb_1 Smarandache curves in the parallel transport frame defined by the frame vectors of $\beta(s)$. Then the Frenet apparatus of $\alpha(t_\alpha, n_\alpha, b_{1\alpha}, b_{2\alpha}, k_{1\alpha}, k_{2\alpha}, k_{3\alpha})$ could be formed by Frenet apparatus of $\beta(t, n, b_1, b_2, k_1, k_2, k_3)$.

Proof: Let $\alpha = \alpha(s_\alpha)$ be tb_1 Smarandache curve of the curve β. Then

By using the definition (2), we get

(3.1) $$\alpha(S_\alpha) = \frac{1}{\sqrt{2}}[t(s) + b_1(s)]$$

By differentiating (4.1) with respect to s, we get

(3.2) $$\frac{d\alpha(s_\alpha)}{ds} = \frac{d\alpha(s_\alpha)}{ds_\alpha}.\frac{ds_\alpha}{ds} = \frac{1}{\sqrt{2}}[(k_1 - k_2)n + k_3 b_2]$$

The tangent vector of the curve α is given by

(3.3) $$t_\alpha = A_1 n + A_2 b_2$$

Where $\frac{ds_\alpha}{ds} = \sqrt{\frac{(k_1 - k_2)^2 + k_3^2}{2}}$, $A_1 = \frac{k_1 - k_2}{\sqrt{(k_1 - k_2)^2 + k_3^2}}$ and $A_2 = \frac{k_3}{\sqrt{(k_1 - k_2)^2 + k_3^2}}$

Again differentiating the tangent vector of the curve α with respect to s_α, we can get α'' as follows

(3.4) $$\alpha'' = \frac{\sqrt{2}[-k_1(k_1 - k_2)t + (k_1 k_2 - k_2^2 - k_3^2)b_1]}{(k_1 - k_2)^2 + k_3^2}$$

The principal normal of the curve α is

(3.5) $$n_\alpha = A_3 t + A_4 b_1$$

Where $A_3 = \frac{-k_1(k_1 - k_2)}{\sqrt{k_1^2(k_1 - k_2)^2 + (k_1 k_2 - k_2^2 - k_3^2)^2}}$ and $A_4 = \frac{k_1 k_2 - k_2^2 - k_3^2}{\sqrt{k_1^2(k_1 - k_2)^2 + (k_1 k_2 - k_2^2 - k_3^2)^2}}$

(3.6) $$\alpha''' = A_5 n + A_6 b_2$$

Where $A_5 = \frac{2[-k_1^2(k_1 - k_2) - k_2(k_1 k_2 - k_2^2 - k_3^2)]}{\left[(k_1 - k_2)^2 + k_3^2\right]^{\frac{3}{2}}}$ and $A_6 = \frac{k_3(k_1 k_2 - k_2^2 - k_3^2)}{\left[(k_1 - k_2)^2 + k_3^2\right]^{\frac{3}{2}}}$

The second and first binomial vector of the curve α is given as follows

(3.7) $$b_{2\alpha} = \frac{(k_1k_2 - k_2^2 - k_3^2)t + k_1(k_1 - k_2)}{\sqrt{(k_1k_2 - k_2^2 - k_3^2)^2 + k_1^2(k_1 - k_2)^2}}$$

(3.8) $$b_{1\alpha} = \frac{-k_3n + (k_1 - k_2)b_2}{\sqrt{k_3^2 + (k_1 - k_2)^2}}$$

The first, second and third curvature of the curve α are

(3.9) $$k_{1\alpha} = \frac{2\left[(k_1k_2 - k_2^2 - k_3^2)^2 + k_1^2(k_1 - k_2)^2\right]}{\left[k_3^2 + (k_1 - k_2)^2\right]}$$

(3.10) $$k_{2\alpha} = \frac{\sqrt{2}k_3\left[k_1(k_1k_2 - k_2^2 - k_3^2)^2 + k_1^2(k_1 - k_2)\right]}{\left[(k_3^2 + (k_1 - k_2)^2\right] + \sqrt{(k_1k_2 - k_2^2 - k_3^2)^2 + k_1^2(k_1 - k_2)^2}}$$

(3.11) $$k_{3\alpha} = \frac{\sqrt{2}(-k_1A_4A_5 - k_2A_3A_5 - k_3A_3A_6)}{\sqrt{(k_3^2 + (k_1 - k_2)^2} + \sqrt{(k_1k_2 - k_2^2 - k_3^2)^2 + k_1^2(k_1 - k_2)^2}}$$

This is the required proof of the above equations.

4. TM_1 Smarandache curves in E^4 according to the parallel transport frame

We have studied to TM_1 Smarandache curves and we get their parallel transport frame and principal curvatures.

Definition 3: Let $\beta = \beta(s)$ be a unit speed curve in E^4 and $\{T_\beta, M_{1\beta}, M_{2\beta}, M_{3\beta}\}$ be its moving parallel transport frame. TM_1 Smarandache curves is defined by

(4.1) $$\alpha(s_\alpha) = \frac{1}{\sqrt{2}}(T_\beta + M_{1\beta}).$$

Theorem 2: Let $\beta = \beta(s)$ be the unit speed curve with constant principal curvatures $K_{1\beta}, K_{2\beta}, K_{3\beta}$ and $\alpha(s_\alpha)$ be TM_1 Smarandache curves in E^4 defined by the parallel transport frame vectors of $\beta = \beta(s)$. Then the parallel transport frame of α can be formed by the parallel transport frame of β and the principal curvatures of $\alpha(K_{1\alpha}, K_{2\alpha}, K_{3\alpha})$ can be obtained by the principal curvatures of β.

Proof: To calculate the parallel transport frame of TM_1 Smarandache curve base to $\beta(s)$ differentiating equation (5.1) with respect to s then

(4.2) $$T_\alpha \frac{ds_\alpha}{ds} = \frac{1}{\sqrt{2}}\left(-K_{1\beta}T_\beta + K_{1\beta}M_{1\beta} + K_{2\beta}M_{2\beta} + K_{3\beta}M_{3\beta}\right)$$

The tangent vector of the curve β can be written as

(4.3) $$T_\alpha = \left(\frac{-K_{1\beta}T_\beta + K_{1\beta}M_{1\beta} + K_{2\beta}M_{2\beta} + K_{3\beta}M_{3\beta}}{\sqrt{K_{1\beta}^2 + K_{2\beta}^2 + K_{3\beta}^2}}\right)$$

Where $\dfrac{ds_\alpha}{ds} = \dfrac{1}{\sqrt{2}}\sqrt{K_{1\beta}^2 + K_{2\beta}^2 + K_{3\beta}^2}$

Differentiating (5.3) with respect to s then

(4.4) $$T'_\alpha = \frac{dT_\alpha}{ds_\alpha} = \lambda_0 T_\beta + \lambda_1 M_{1\beta} + \lambda_2 M_{2\beta} + \lambda_3 M_{3\beta}$$

Where $\lambda_0 = \dfrac{-\sqrt{2}(K_{1\beta}^2 + K_{2\beta}^2 + K_{3\beta}^2)}{(2K_{1\beta}^2 + K_{2\beta}^2 + K_{3\beta}^2)}$, $\lambda_1 = \dfrac{-\sqrt{2}K_{1\beta}^2}{(2K_{1\beta}^2 + K_{2\beta}^2 + K_{3\beta}^2)}$, $\lambda_2 = \dfrac{-\sqrt{2}(K_{1\beta}^2 K_{2\beta}^2)}{(2K_{1\beta}^2 + K_{2\beta}^2 + K_{3\beta}^2)}$,

$\lambda_3 = \dfrac{-\sqrt{2}(K_{1\beta}^2 K_{3\beta}^2)}{(2K_{1\beta}^2 + K_{2\beta}^2 + K_{3\beta}^2)}$

The Frenet frame in the curvature of the curve α is

(4.5) $$k_{1\alpha} = \sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_2^3} = \frac{\sqrt{2}\sqrt{K_{1\beta}^2 + K_{2\beta}^2 + K_{3\beta}^2}}{\sqrt{2K_{1\beta}^2 + K_{2\beta}^2 + K_{3\beta}^2}}$$

The principal normal of the curve α is

(4.6) $$n_\alpha = \frac{\lambda_0 T_\beta + \lambda_1 M_{1\beta} + \lambda_2 M_{2\beta} + \lambda_3 M_{3\beta}}{\sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}}$$

The third derivative of the curve α is given by

(4.7) $$\alpha''' = (\lambda_0 T'_\beta + \lambda_1 M'_{1\beta} + \lambda_2 M'_{2\beta} + \lambda_3 M'_{3\beta})\frac{\sqrt{2}}{\sqrt{2K_{1\beta}^2 + K_{2\beta}^2 + K_{3\beta}^2}}$$

$$\Rightarrow T_\alpha \times n_\alpha \times \alpha''' = C_1 M_{1\beta} + C_2 M_{2\beta} + C_3 M_{3\beta}$$

Where $$C_1 = \frac{\sqrt{2}\left[\lambda_0\lambda_3 K_{1\beta}K_{2\beta} - \lambda_0\lambda_2 K_{1\beta}K_{3\beta} - (\lambda_1 K_{1\beta} + \lambda_2 K_{2\beta} + \lambda_3 K_{3\beta})(\lambda_3 K_{2\beta} - \lambda_2 K_{3\phi})\right]}{\sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}\left(2K_{1\beta}^2 + K_{2\beta}^2 + K_{3\phi}^2\right)}$$

$$C_2 = \frac{\sqrt{2}\left[\lambda_0\lambda_3 K_{1\beta}^2 - \lambda_0\lambda_1 K_{1\beta}K_{3\beta} - (\lambda_1 K_{1\beta} + \lambda_2 K_{2\beta} + \lambda_3 K_{3\beta})(\lambda_3 K_{1\beta} - \lambda_1 K_{3\beta})\right]}{\sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}\left(2K_1^2 + K_2^2 + K_3^2\right)}$$

$$C_3 = \frac{\sqrt{2}\left[\lambda_0\lambda_2 K_{1\beta}^2 - \lambda_0\lambda_1 K_{1\beta}K_{2\beta} - (\lambda_1 K_{1\beta} + \lambda_2 K_{2\beta} + \lambda_3 K_{3\beta})(\lambda_1 - \lambda_2)K_{2\beta}\right]}{\sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}\left(2K_1^2 + K_2^2 + K_3^2\right)}$$

We know the second and first binomial of the curve α is given by

(4.8) $$b_{2\alpha} = \frac{C_1 M_{1\beta} + C_2 M_{2\beta} + C_3 M_{3\beta}}{\sqrt{C_1^2 + C_2^2 + C_3^2}}$$

(4.9) $$b_{1\alpha} = b_{2\alpha} \times T_\alpha \times n_\alpha = \gamma_0 T_\beta + \gamma_1 M_{1\beta} + \gamma_2 M_{2\beta} + \gamma_3 M_{3\beta}$$

Where $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ are constants and $M_{1\alpha}$, $M_{2\alpha}$, and $M_{3\alpha}$ are the parallel transport frame of $M_{1\beta}$, $M_{2\beta}$, and $M_{3\beta}$ of the curve α is given.

Further the second and third curvature tensor of the curve α is given by

(4.10) $$k_{2\alpha} = \sqrt{\frac{C_1^2 + C_2^2 + C_3^2}{\sqrt{\lambda_0^1 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}}}$$

(4.11) $$k_{3\alpha} = \frac{-(C_1 K_{1\beta} + C_2 K_{2\beta} + C_3 K_{3\beta})(\lambda_1 K_{1\beta} + \lambda_2 K_{2\beta} + \lambda_3 K_{3\beta})}{\sqrt{C_1^2 + C_2^2 + C_3^2}}$$

The first, second and third curvature of the curve α is to parallel transport frame

(4.12) $$K_{1\alpha} = \sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}\cos\theta_\alpha \cos\psi_\alpha$$

(4.13) $$K_{2\alpha} = \sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}\left[-\cos\phi_\alpha \sin\psi_\alpha + \sin\phi_\alpha \sin\theta_\alpha \cos\psi_\alpha\right]$$

(4.14) $$K_{3\alpha} = \sqrt{\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2}\left[\sin\phi_\alpha \sin\psi_\alpha + \cos\phi_\alpha \sin\theta_\alpha \cos\psi_\alpha\right]$$

Discussion: In this paper we discussed an Einstein's theory opened a door to new geometries such as Smarandache curve of parallel transport frame in the Euclidean space. They adapted the geometrical models to relativistic motion of charged particles. As it stands, the Frenet-Serret formalism of a relativistic motion describes the dynamics of the charged particles. The position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache. In this work, we study special Smarandache Curve in the Euclidean space.

References

[1] M. Elzawy, Smarandache curves in Euclidean 4-space E^4, J. of Egyptian Mathematical society, vol. 25 (2017), pp 268-271.

[10] K. Llarslan, E. Nesovic, Some characterizations of osculating curves in the Euclidean space, Demonstratio Mathematica, vol. XLI (4) (2008), pp. 931-939.

[11] S. Yilmz, M. Turgut, A new version of Bishop frame and an application to spherical image, J. Math. Anal. Appl., vol. 371 (2010), pp. 764-776.

[12] L.R. Bishop, There is more than one way to frame a curve, Amer. Math. Vol. 82 (3) 1975, pp. 246-251.

[2] A.T. Ali, Special Smarandache curves in the Euclidean space, Int. J. Math. Combin vol. 2 (2010), pp 30-36.

[3] M. Cetin, H. Kocayigit, On the quaternionic Smarandache Curves in Euclidean 3-space, Int. J. Contem. Math. Sciences, vol. 8(no.3) (2013), pp. 139-150.

[4] T. Krpinar, E. Turhan, Biharmonic curves according to parallel transport frame in E^4, Bol. Soc. Paran. Mat. (3s) vol. 31(2) (2013). Pp. 213-217.

[5] F. celik, Z. Bozkurt, I. Gok, F.N. Ekmekci, Y. Yayl, Parallel transport frame in 4-dimensional Euclidean space E^4, Caspian J. Math. Sci. (CJMS), vol. 3(1) (2014), pp. 91-103.

[6] S. Senyurt, A. Calskan, An application according to spatial quaternionic Smarandache curve, appl. Math. Sci. vol. 9 (5) (2015), pp. 219-228.

[7] K. Bharathi, M. Nagaraj, Quaternion valued function of a real Serret-Ferret formulae, Indian J. pure appl. Math., vol. 16 (1985), pp. 741-756.

[8] M. Turgut, S. Yilmaz, Smarandache curve in Minkowaki space-time, Int. J. math. Comb. Vol. 3 (2008), pp. 51-55.

[9] M. Elzawy, S. Mosa, Smarandachecurves in the Galilean 4-space G_4, J. Egypt. Math. Soc., vol. 25 (2017) pp. 53-56.