Historical Development in Haar Wavelets and Their Application - An Overview

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Abstract: - In this paper, basic idea of wavelets, its connection with Fourier expansion and advantages of wavelet expansion over Fourier expansion is reviewed. Further the idea of removing Gibb's phenomena over Fourier phenomena is also discussed. The discrete wavelet transform is very much useful in solving the various problems in the field of science and technology. Therefore, these transformations have attained very good prestige of being very effective individually and by getting hybrid with other established techniques in solving the many of the real time problems. One of the oldest and mathematically most simple as compared with other wavelets which owes its origin to 1910 is the Haar wavelet. This review aims to provide the fantastic application of Haar wavelets in solving the various problem of science and technology. In current article some computational and mathematical capabilities of Haar wavelets and diverse applications in various field have been reviewed. Some future scope in the direction of developing the new hybrid method for solving the various advance problems have also been discussed.

Introduction

The decomposition, synthesis and analysis of functions, in various function spaces, is one of the most useful activities of mathematics, mathematical physics and lastly of Data/signal processing in engineering. It is one of the most intensively studied topics in mathematics since the beginning of 19th century. The bold declaration of Fourier (Pinsky, (2002)), about expressing any periodic function in the form of trigonometric series kept the entire scientific community on alert for nearly two centuries-especially in establishing convergence, uniform

convergence etc of Fourier Series. In this process one finds blossoming of rich topics like Lebesgue integral, functions of striking theories in Banach space, Hilbert space etc.

A function has power series representation if it is smooth and gives its local structure accurately. Fourier series of functions with classical orthogonal functions is useful for the global analysis of functions. These can be used for representing functions with no smoothness properties but they are inefficient for analysing the detail behaviour of a function near a point. A pure Fourier basis diagonalizes translation invariant linear operators. We look for a basis (of function spaces) that is well localized in frequency and nearly diagonalizes the operator i.e. their matrix entries decay rapidly away from the diagonal. Also, it is desirable a basis to be well localized in space for effective local analysis. What is missing is a method for analysing the local irregular behaviour of functions that aren't smooth and this is where wavelets come into play. Wavelet is a small impulsive function which can be operated in two ways; one way is translation w.r.t the initial condition and other is dilatation which means extension or compression of original wavelet. Mathematically wavelet can be represented as

$$\psi_{a,b} = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right)$$

where a and b respectively representing the dilation and translation factor. Wavelet transformation any function $\xi(t)$ is defined as

$$w(a,b) = \int_t \xi(t)\frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right)dt$$

It gives us information that at $t = \frac{b}{a}$, how much the scaled function is like the given solution function $\xi(t)$.

Wavelet has oscillatory behaviour in some region and decay rapidly to zero outside this region. A remarkable discovery of recent years is that the translations and dilations of certain wavelets can be used to form set of basic (basis) functions for representing general function into infinite series which have been playing dominant role in mathematics and applications. In fact, wavelets were implicit in several scientific fields but it is the greatest tribute to the efforts of many since last 10-20 years to know, for instance Littlewood-Paley theory in mathematics and pyramidal algorithm of DSP have the same roles. The synthesis of these theories leading to Wavelet analysis and the impetus to great interest was the result of research by Grossman and Morlet (1984) who called Littlewood-Paley theory as Wavelet theory. Meyer (1993) and his co-workers noticed that Calderon - Zygmund theory, in particular the Littlewood - Paley representation having discrete analog gave unified view of research in harmonic analysis.

First time the wavelets appeared in the thesis of Hungarian mathematician Alfred Haar [1] in 1909. The Haar wavelet is piecewise constant function and main property of this wavelets is compact support of wavelet. Unfortunately, it is not continuously differentiable therefore it was not studied much at that time because of this limitation. In 1980s multiresolution analysis invented by S. Mallat [2] and Y. Meyer [3] has given most prosperity to the field of wavelet analysis. Main advantage of multiresolution analysis was that it has provided a scope for other researchers to develop mathematically their own family of wavelets. Using work on multiresolution analysis Y. Meyer developed his wavelets and these wavelets were continuously differentiable but they do not have a compact support. After few years, Ingrid Daubechies [4], [5] took the idea from Mallat and Meyer 's work to create a new set of wavelet basis, which were orthonormal and have a compact support. These wavelets became the foundation of wavelet application. Right from the beginning, the wavelets were considered as scientific curiosity but because of huge research in the development of wavelets, it has turned to influential scientific mathematical tool, which can be used for many applications. Within the Daubechies family of wavelets, wavelets were generally classified by the number of vanishing moments. After applying the condition of vanishing movements, a set of linear and non-linear simultaneous algebraic equations on coefficients were obtained and solving these equations, numerical values for the coefficients were obtained. This straight forward approach has fulfilled the need of researchers for understanding the construction of wavelets and becomes very popular in the construction of wavelets. As a result, a new set of wavelets families came into existence like coiflet wavelets, symlets wavelets etc. Now these wavelets were continuous, differentiable and had compact support. These were extensively used in signal processing, image processing and there was some application in numerical analysis also. With the attractable properties of these wavelets, there was a big limitation of these wavelets that they do not possess any explicit form of expression and cannot be used comfortably for discretization. One has to construct the wavelets with the help of filter coefficients and because of this analytical differentiation or integration for these wavelets becomes impossible. This makes the process very complicated when the integrals of some nonlinear functions are required in the application. Then a new concept of connection coefficients developed to calculate these kinds of integrals but the process of calculation of these connection coefficients was very complicated and has to perform separately for each such integral. Other than this, method was applicable only for some simple type of nonlinearities in the equations (quadratic). Mishra and Sabina[6]also solved the differential equations using the connection coefficients in Galerkin method. Now because of this complexity in obtaining the solutions by wavelets has induced some pessimistic estimates. It was considered that solving the mathematical problems by wavelet method have no advantage over the conventional methods. Strang and Nguyen [7](1996, p. 394) wrote "The competition with other methods is severe. We do not necessarily predict that wavelets will win". It gave a new impulse to look for other possibilities to come out of this deadlock. Again researchers start thinking about all the wavelets family developed till that time .In 1997 Chen and Hsiao [8] has overcome the disadvantage of Haar wavelet of not being differentiable at the point of discontinuity. He approximated the highest order derivative present in the problem with Haar wavelet series instead of approximating the solution function by the Haar wavelet series. The rest all derivatives and solution function itself were founded by integrating the highest order derivative. This technique has been proved very faithful and researchers are using this technique in solving the mathematical models governed by differential, integral and intro-differential equation. In 2001 Ülo Lepik [9] took the idea of Chen and Hsiao [8] and used the wavelet transformations to analyse the linear vibration of single and two degree of freedom .He used the three type of wavelets transforms and found that, in the case of single degree of freedom vibrations, all three wavelet has given the same qualitative results but for second degree of freedom of motion two wavelets i.e. Mexican hat and Haar wavelets have given qualitatively different results are from third wavelet i.e. Morlet wavelet. In 2004 Ülo Lepik and Enn Tamme[10] used Haar wavelet to investigate the behaviour of solution for linear integral equations. Different kinds of integral equations (Fredholm and Volterra equations, integrodifferential equations etc) were considered .He found that the solution obtained by using Haar wavelets was more effective than conventional solutions with the same step size and in the case of Fredholm and Volterra equations the convergence rate was O(M^{-2}). In 2005 U¨.Lepik [11] developed a new technique based upon Haar wavelet to investigate the solution of with different types ODE and PDE and compared Chen and Hsiao method (CHM) with method of segmentation and approximation by piecewise constant approximation. He found that Chen and Hsiao wavelet method is mathematically very simple because the wavelet matrices H and the matrices of their integrals becomes more and more sparse which made the process very fast. But its instability for approximating the higher order derivatives becomes a disadvantage. In 2007 U. Lepik [12] applied Haar wavelet method to solve nonlinear evolution equation. The method was tested on Burgers and Sine Gordon equations and found that the method is in full competition with the other existing classical methods. The method was found very economical as far as the computational cost and simplicity is concerned. The method was found to be very suitable for boundary value problem as it automatically takes care of the boundary condition in the process of solution. In 2007 , a little survey on the use of the Haar wavelet transform was given by the Ü.Lepik [13] in which he has discussed some different types of integral and differential equations. In 2007 Ülo Lepik and Enn Tamme[14] has applied the Haar wavelet transform technique to test the applicability of the method on nonlinear Fredholm integral equations and the results obtained are very promising . In 2008 U. Lepik [15] developed a new technique of non-uniform Haar wavelets for solving the integral and differential equations and proved that Haar wavelet method with non-uniform mesh is suitable in the case of problems where abrupt or rapid changes in the solution take place. In 2008 Phang Chang, Phang Piau [16] developed the operational Matrices for a Haar Wavelets to solve the ODEs and performed all the calculations with the matrix representation of wavelets and all of its integrals which reduced the complexity of the process. In 2009 Ü. Lepik [17] developed the algorithm based upon Haar wavelets to solve the various fractional integral equations. It was found that the method is very simple and fast to solve these kind of fraction equations In 2009 E. Babolian and A.Shahsavaran[18] has proved the convergence of the Haar wavelets method by doing the error analysis which was a big question at that time and also handle the non- linearity in in his process to solve nonlinear Fredholm integral equations. In 2010 G. Hariharan, K. Kannan[19] has extended the utility of Haar wavelets technique to solved some nonlinear parabolic partial differential equations using Haar wavelet method where he has considered very well know nonlinear PDEs for testing the performance of the method. He proved that the proposed scheme is working better and has a good ability to handle the non-linearity in the process and it can be used to big group of nonlinear PDEs. In 2010 G.Hariharan [20] solved a physical model of deflection in a beam of finite length which is governed by a fourth order ODE with the associated boundary and initial conditions using Haar wavelets method. In the method a generalized operation matrix and the matrices of their integral was developed to solve this model and results were compared with the exact solution available in the literature. It was shown in the results that Haar wavelet method takes smaller time on CPU and is able to give better results for less degree of freedom as compare to the other methods. In 2011, G. Hariharan [21] extended the use of Haar wavelet Method to solve the Klein-Gordon and the Sine-Gordon Equations with modification in approximation. He found that the results are closer to the exact real values for a very small no of colocation points and the accuracy can further be improved by increasing the number of collocation points. In 2011 , Ülo Lepik[22] applied the Haar wavelet method on the physical model of buckling of elastic beams in which he has produced the solution for the different situation in buckling of elastic beams like crack simulation, beam vibrations on an elastic foundation, beams having flexible cross-section etc.in this application of Haar wavelets author revealed many advantageous features of the Haar wavelet method like high accuracy for small no of grid points, use of common subprograms for solving the different problems, treatment of singularities in intermediate boundary condition and simplicity for implementation etc. Author also proved that the method can also be applied to more complicated problems by taking the different examples. In 2011 V. Mishra, H. Kaur, and R.C. Mittal [23] has developed an algorithm by using Haar wavelet with collocation method for solving various types of ODEs ,IDEs and integral equations .Numerical experiment were performed to prove the reliability and efficiency of the algorithm developed and found that found that the wavelet-collocation algorithm is less time consuming less complicated as compare to the wavelet-Galerkin procedure for solving the similar types of problems. In 2011 Ü. Lepik[24] solved partial differential equations with the aid of two-dimensional Haar wavelets which can be used for solving the higher dimensional equation with less complexity. In 2012 Naresh Berwal , Dinesh Panchal and C. L. Parihar [25] solved the Wave-Like Equation by using Haar wavelet technique. In 2013 Hariharan [26] developed a Haar wavelet algorithm for Fractional Klein-Gordon Equations in which the author developed the wavelet operational matrices and the operational matrices of their fractional integrals. A group of algebraic equations have been obtained by using these operational matrices on fractional Klein–Gordon equations. Further the group of algebraic equations is transformed into a matrix system and solved for getting the unknown coefficients of the Haar wavelet series approximation of solution. Author claim is that method is very effective, simple, fast and flexible for other differential and integral systems. Also, complexity of calculating the correction coefficients can also we avoided in comparison with Daubechies wavelets. In 2013 S. Sekar [27] has solved the Integro-Differential Equations using single term Haar Wavelet and compare the results with Local Polynomial Regression (LPR) method and proved the efficiency of the Haar wavelet method. In 2013 Naresh Berwal and Dinesh Panchal[28] has proved the efficiency of Haar wavelet method by applying it onto L−C−R equation and comparing the results with exact solution. In 2013 Harpreet Kaur, et.al [29] Author have applied a quasi-linearization technique along with the Haar wavelet bases to solve the nonlinear Blasius equation at uniform collocation points. Blasius equation is a very important of fluid mechanics and needed some promising solution. Author claims that in applying the quasi-linearization technique along with Haar wavelet approximation does not require any iteration on the selected collocation points which makes the quasilinearization process of handling non linearity very easy. In 2014 Umer Saeed and Mujeeb Ur Rehman[30] extended the utility of method of approximation of solution using Haar wavelet series for the fractional order nonlinear oscillation equations and found that solutions on large intervals were in agreement with fourth order Runge-Kutta method . In 2014 Umer Saeed and Mujeeb ur Rehman[31] applied the Haar wavelet-Quasilinearization technique on Heat Convection-Radiation Equations for getting the approximate solution . In which author has first linearized the nonlinear heat transfer equation using quasilinearization technique. Then the linear system is solved by approximating the dependent variable and their derivatives by the truncated convergent series of Haar wavelet bases which results into a matrix system after using the collocation points in the resultant algebraic equations. Two special case of nonlinear heat transfer equation i.e distribution of temperature in lumped system of slab made of variable thermal conductivity material and lumper system cooling profile are studied using the proposed scheme. Author in the manuscript claim that Haar wavelet-quasilinearization technique are roughly coincide with exact solution and has given better results as compare to the other methods. In 2014 R.C. Mittal, Harpreet Kaur and Vinod Mishra [32] developed Haar wavelet-based algorithm to investigate the phenomena governed by the nonlinear coupled Burgers' equation. After applying the Haar wavelets with Collocation method the system of nonlinear coupled Burgers' partial differential equations transformed in the new system of ODEs. Resultant system of

ODEs then solved by the Runge Kutta technique. Author has also established the stability analysis of this hybrid scheme. The method was tested on some test problem and author claimed that it is giving the quite satisfactory results and the method can be extended to solve other higher order differential system of equation. In 2014 Asmita C. Patel & V. H. Pradhan [33] applied the Wavelet Galerkin scheme for solving the nonlinear partial differential Equations. They used Daubecheis wavelet with Galerkin method to solve the nonlinear partial differential equations. In 2014 Sangeeta Arora , Yadwinder Singh Brar and Sheo Kumar[34] implemented the Haar Wavelet Matrices techniques for finding the Numerical Solutions of Differential Equations . In 2014 Osama H. M., Fadhel S. F and Zaid A. M [35] applied Haar wavelet method to solve fractional Variational problems. In 2014 Santanu Saha Ray[36] has done a comparison of two most promising schemes for solving the Fractional differential equations .In his work he compared the performance of Haar wavelet method with optimal Homotopy Asymptotic method on Fractional Fisher type equations. Author claimed that both the methods are appropriate and reliable for solving these kinds of equation. But optimal Homotopy Asymptotic method provided better results as compared to Haar wavelet method for certain number of grid points. On the other hand, accuracy of the Haar wavelet method can be improved by increasing the no of grid points. In 2015 Ö. Oruç , F. Bulut, A. Esen [37] developed a new hybrid technique for the investigation of solution of modified Burgers Equation. In the algorithm temporal part was discretized and handled by finite differencing, spatial part was discretized by Haar wavelets whereas the non-linearities in the equation were handled by quasi-linearization technique. Author tested the method developed on three test problems and the claimed that method is fully consistent, fast, and very much economical in terms of computational cost. In 2015 Manoj Kumar and sapna pandit[38]solved the Fokker Plank Equations with constant and variable coefficients by using an algorithm based on Haar wavelet method. In 2015 S. C. Shiralashetti and A. B. Deshi[39] have addressed the multi-term FDEs by using collocation method with Haar wavelet bases and found that HWCM equally competent and easy to implement in comparison with other existing methods is very effective, is easy to implement and is able to approximate the solution accurately compared to existing methods. In 2015 M. Fallahpour, M. Khodabin and K. Maleknejad [40] have developed the Haar wavelet based method to tackle with more variables in two-dimensional linear Stochastic Volterra integral equation and difficulty of finding solution because of the randomness. After testing the method on the test examples author claims that method is reliable efficient and fast but can be improved to be more accurate by using other numerical methods. In 2015 S.C. Shiralashetti et.al [41] extended the applicability of Haar wavelet collocation method for the investigation of models governed by singular initial value problems . Authors of the manuscript have shown that HWCM is a powerful numerical method for the solution of the linear and non-linear singular initial value Problems as compare to other methods like Adomian decomposition Method (ADM) &Variational iteration method (VIM) etc. In 2015 Inderdeep Singh, Sangeeta Arora, Sheo Kumar [42] solved the wave equation by using Haar wavelet and proved that this method is better than other method . In 2016 S. C. Shiralashetti, M. H. Kantli and A. B. Deshi[43] developed Haar wavelet based collocation method to address the nonlinear ODEs emerging in the field of fluid dynamics with different boundary conditions. After testing the method on different problems of fluid dynamics with different boundary conditions author claimed that HWCM established a solid foundation for its use in solving these kinds of problem because of their simplicity and fast convergence. In 2016 S. C. Shiralashetti et.al [44] has solved the Klein–Gordon equations by using Haar wavelet method and shown that it works better than classical numerical methods like finite difference method. In 2016 Firdous A. Shah et.al [45] developed an explicit form of Operational matrix of Haar wavelets for solving the various linear and non-linear fractional Differential Equations .Many standard benchmark problems were tested and claimed the superiority of method by giving the numerical evidence in terms of fast convergence and better accuracy. In 2016 A. C. Patel and V. H. Pradhan[46] implemented Haar wavelet method on advection-dispersion equation representing one dimensional contaminant transport through a porous medium. In 2016 O. Oruc, F. Bulut and A. Esen [47] has developed the Haar wavelet based hybrid technique for the investigation of the phenomena governed by Regularized Long Wave Equation. Time derivatives were discretized by using finite differencing and space derivatives were approximated by truncated Haar wavelet series. Various test problem related to solitary wave motion has been analysed and claimed that method working well to analyze these kinds of problems. In 2016 Harpreet Kaur, Shin Min Kang [48] developed the time discretization of Haar wavelet Series approximations with Quasilinearization technique for solving well known nonlinear PDEs .Quasilinearization was used to tackle the nonlinearity in nonlinear PDEs .Haar wavelet method with collocation method is used to convert the given PDEs into a linear system of equations which were further solved by Thomas algorithm In 2017 S. C. Shiralashetti, et.al [49] used adaptive grid by adding the more grid points which were actually the midpoints of the regular uniform grid points in the regular uniform grid of Haar Wavelets. The new adaptive grid Haar wavelet technique was applied to solve the parabolic type of PDEs along with collocation method and has shown that the new technique gives better accuracy in comparison to regular HWCM and FDM. In 2017 , Somayeh Arbabi et.al [50] applied the two-dimensional Haar wavelets method for solving the systems of partial differential equations. Convergence and stability of the method was proved. Method was tested on the test problems and claimed that result obtained were in a very good coincidence with the exact solution.

Conclusion:

We have reviewed the Haar wavelet based different techniques available in the literature from 1903-2017 to solve the various problems of science and technology and following observations has been made for Haar wavelet based different techniques

i. Matrices of Haar wavelets and their integrals sparse in nature which are helpful in making the computation process fast and economic in terms of computational cost.
ii. Haar wavelet methods are simple and easy to implement through the computer programming language as common subprograms fits all the problems.
iii. Accuracy of the solution can be improved by increasing the level of resolution which supports the convergence of method for solution.
iv. Small grids are able to give very good results in Haar wavelet methods.
v. Haar wavelet methods are supportive for both linear and non linear problems of science and technology.

vi. The method is equally competitive and suitable for a various type of fractional, partial and ordinary differential equation. It is also suitable for solving the integral equations.

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