Homotopy Perturbation Method (HPM) and it's applications in different non-linear partial differntial equations: A Review

Mamta Kapoor*

*Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara 144411, Punjab, India.

Abstract

Homotopy perturbation method is an approach, which is grounded upon the concept of homotopy, which is basically taken from the area of topology, it is a general analytical method. In present paper, a brief review regarding homotopy perturbation method (HPM) is given as why homotopy perturbation method came into existence and which advancement, this method contains in comparison with previous work. in this paper, some applications of homotopy perturbation method along with non-linear partial differential equations are also discussed in terms of finding the analytical solution.

1. Introduction

The concept of HPM is independent of notion of inclusion of a trivial parameter given in the equation. By implementing homotopy concept in topology, a homotopy can be formed including the parameter, which belongs to [0, 1], taken as a small parameter. HPM was given by he [1-7] in 1999. By going through He's [4] paper given in 1999, the point came into notion that why Homotopy perturbation method came into existence. The theory presented in He's paper contains some interesting facts regarding HPM. Like, finding analytical technique for solving the non-linear problems is a matter of interest for scientist and researchers. The mostly well-known techniques for finding these analytical solutions are the perturbation methods but alike other techniques for finding analytical solutions, it got clear that perturbation techniques had their own limitations,

(a) As such perturbation techniques depend upon the hypothesis that a trivial parameter must be present in the equation and this minor parameter's existence becomes the reason for the restricted applicability of the perturbation techniques.

(b) To fetch the value of this small parameter is usually a difficult task which demands some special techniques. When the appropriate value of small parameter is taken into account in such case only correct results are possible otherwise any not suitable selection of small parameter can lead to the unacceptable result.

(c) The approximate solutions which are obtained by the perturbation practices are acceptable but generally for the small value of the parameter.

So by going through He' notion it is clear that all the limitations emerged from the assumption of minor parameter. He introduced a technique by implementing the homotopy approach.

According to He's notion, there is no requirement of the slight parameters in respective equation, that is why the limitations of the perturbation techniques can be easily overcome.

2. Basic notion of HPM

For any non-linear differential equation, the methodology of HPM is discussed as follows by He (1999) [4]. Non-linear differential equation:

$$B(u) - g(s) = 0 \text{, where } s \in D \quad \textbf{(1)}$$

Where B is general differential operator.

$$\text{Boundary condition: } C(u, \partial u / \partial n) = 0, \; r \in \partial D \quad \textbf{(2)}$$

C is the boundary operator, g(s) is analytical solution, D is domain of computation, ∂ D is the boundary of computational domain. The general differential operator can be splitted into two parts linear and non-linear. l is known as the linear part, n is known as the non-linear part.

Equation (1) can be updated as: $[l + n]\,(u) - g(s) = 0$

or

$$l(u) - n(u) - g(s) = 0 \quad \textbf{(3)}$$

By employing the homotopy technique [8-9], a homotopy was constructed as follows:

$$v(s, q) : D \times [0, 1] \rightarrow R$$

Which satisfied the following condition:

$$H(v, q) = (1 - q)\,[l(v) - l(u_0)] + q\,[B(v) - g(s)] = 0 \quad \textbf{(4)}$$

where q ∈ [0, 1], s ∈ D

$$\text{i.e. } H(v, q) = l(v) - q\,l(v) - l(u_0) + q\,l(u_0) + q\,B(v) - q\,g(s) = 0 \quad \textbf{(5)}$$

$$\text{i.e. } H(v, q) = l(v) - q\,l(v) - l(u_0) + q\,l(u_0) + q\,l(v) + q\,n(v) - q\,g(s) = 0 \quad \textbf{(6)}$$

$$\text{i.e. } H(v, q) = l(v) - l(u_0) + q\,l(u_0) + q\,[n(v) - g(s)] = 0 \quad \textbf{(7)}$$

Where q belongs to [0, 1] known as an embedding parameter, u_0 is known as initial approximation of the equation (1), which satisfies the boundary conditions.

From equation (7),

$$H(v, 0) = l(v) - l(u_0) = 0 \quad \textbf{(8)}$$

$$H(v, 1) = l(v) - l(u_0) + l(u_0) + n(v) - g(s) = 0$$

$$\text{i.e. } H(v, 1) = l(v) + n(v) - g(s) = 0$$

$$\text{i.e. } H(v, 1) = B(u) - g(s) = 0 \quad \textbf{(9)}$$

Where $l(v) - l(u_0)$ and $B(v) - g(s)$ are known as homotopic.

According to this technique the imbedding parameter q is taken as small parameter and the solution of equation (7) can be expressed as a power series of this parameter q.

$$v = v_0 + q\, v_1 + q^2\, v_2 + q^3\, v_3 + \cdots \ldots \ldots \ldots .. \tag{10}$$

By assuming that q = 1,

We will obtain that

$$v = v_0 + v_1 + v_2 + v_3 + \cdots \ldots \ldots \ldots \ldots .. \tag{11}$$

Approximate solution of equation (1) can be written as

$$u = \lim_{q \to 1} v \tag{12}$$

$$\text{i.e. } u = \lim_{q \to 1} v = v_0 + q\, v_1 + q^2\, v_2 + q^3\, v_3 + \cdots \ldots \ldots \ldots \ldots .. \tag{13}$$

The joining of the perturbation technique and homotopy method is known as the Homotopy Perturbation Method (HPM), which eliminates the restrictions caused due to the traditional perturbation techniques.

3. Homotopy perturbation method: applications to different non-linear equations

3.1 Application of Homotopy Perturbation method upon 1D Burgers' equation:

Concept of application of Homotopy Perturabation method upon 1D Burgers' equation is taken from the paper [10],

1D Burgers' equation is given as follows:

$$u_t + u\, u_x = \nu\, u_{xx} \tag{14}$$

Initial condition:

$$u(x, 0) = g_0(x) \quad \text{where } x \in [0, 1] \tag{15}$$

Boundary conditions:

$$u(0, t) = g_1(t) \quad , t > 0 \tag{16}$$

$$u_x(0, t) = g_2(t) \quad , t > 0 \tag{17}$$

Constructing homotopy,

$$(1\text{-}q)\, \frac{\partial U}{\partial t} - \frac{\partial U_0}{\partial t} + q\, \frac{\partial U}{\partial t} + U\, \frac{\partial U}{\partial x} - \nu\, \frac{\partial^2 U}{\partial x^2} = 0 \tag{18}$$

On solving above equation,

$$\frac{\partial U}{\partial t} - \frac{\partial U_0}{\partial t} = q\, [- \frac{\partial U_0}{\partial t} - U\, \frac{\partial U}{\partial x} + \nu\, \frac{\partial^2 U}{\partial x^2}] \tag{19}$$

Suppose that the solution of equation (19) is,

$$U = U_0 + q\, U_1 + q^2\, U_2 + \ldots \ldots \ldots \ldots \ldots \ldots \tag{20}$$

By using equation (20) into equation (19) we will get the following recurrence relation,

$$U_j = \int_0^t [\nu \frac{\partial^2 U_{j-1}}{\partial x^2} - \sum_{k=0}^{j-1} U_k \frac{\partial U_{j-k-1}}{\partial x}]\, dt, \tag{21}$$

Where $j = 1, 2, 3, \dots\dots\dots\dots$

and approximate solution will be obtained as,

$$U = U_0 + U_1 + U_2 + \dots\dots\dots\dots\dots\dots \tag{22}$$

3.2 Application of Homotopy perturbation method in 2d coupled Burgers' equation:

taking the concept of HPM for non-liear coupled 2d Burgers' equation from [10] following discussion is given. 2D non-linear coupled Burgers' equation is given as follows,

$$u_t + u\, u_x + v\, u_y = \frac{1}{Re}(u_{xx} + u_{yy}) \tag{23}$$

$$v_t + v\, v_x + v\, v_y = \frac{1}{Re}(v_{xx} + v_{yy}) \tag{24}$$

Initial Conditions:

$$u(x, y, 0) = \phi_1(x, y)\,,\ (x, y) \in D \tag{25}$$

$$v(x, y, 0) = \phi_2(x, y),\ (x, y) \in\ D \tag{26}$$

Boundary conditions:

$$u(x, y, t) = \psi_1(x, y, t),\ (\mathrm{x, y}) \in \partial \mathrm{D},\ t\ >\ 0 \tag{27}$$

$$v(x, y, t) = \psi_2(x, y, t),\ (\mathrm{x, y}) \in \partial \mathrm{D},\ t\ >\ 0 \tag{28}$$

Where D is the computational domain given by D = {(x, y) : a ≤ x ≤ b, c ≤ y ≤ d }

u(x, y, t) and v(x, y, t) are the velocity components are to be determined. $\phi_1(x, y)$

, $\phi_2(x, y)$, $\psi_1(x, y, t)$ and $\psi_2(x, y, t)$ are the known functions and Re is the Reynolds number. In order to solve equation (23) and (24) along with the initial conditions (25) and (26) following homotopy will be formed.

$$\text{(1-q)}\,\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t} + q\ \left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \left(\frac{1}{Re}\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right] = 0 \tag{29}$$

$$\text{(1-q)}\,\frac{\partial v}{\partial t} - \frac{\partial v_0}{\partial t} + q\ \left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - \left(\frac{1}{Re}\right)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right] = 0 \tag{30}$$

Where approximated solution of equations (29) and (30) is given as following,

$$\mathrm{u} = u_0 + q\, u_1 + q^2 u_2 + \dots\dots\dots\dots\dots \tag{31}$$

$$\mathrm{v} = v_0 + q\, v_1 + q^2 v_2 + \dots\dots\dots\dots\dots \tag{32}$$

By implemented the values of u and v from equation (31) and (32) into equations(29) and (30), following recurrence relations will be obtained.

$$u_j = \left(\frac{1}{Re}\right) \int_0^t \left(\frac{\partial^2 u_{j-1}}{\partial x^2} + \frac{\partial^2 u_{j-1}}{\partial y^2}\right) dt - \int_0^1 (u_k \frac{\partial u_{j-k-1}}{\partial x} + v_k \frac{\partial u_{j-k-1}}{\partial y})\ dt \quad \textbf{(33)}$$

$$v_j = \left(\frac{1}{Re}\right) \int_0^t \left(\frac{\partial^2 v_{j-1}}{\partial x^2} + \frac{\partial^2 v_{j-1}}{\partial y^2}\right) dt - \int_0^1 (u_k \frac{\partial v_{j-k-1}}{\partial x} + v_k \frac{\partial v_{j-k-1}}{\partial y})\ dt \quad \textbf{(34)}$$

Where the approximated solution is given as follows,

$$\text{u} = \lim_{q\to 1}(u_0 + qu_1 + q^2\ u_2 + \cdots ...) \quad \textbf{(35)}$$

$$\text{v} = \lim_{q\to 1}(v_0 + q\ v_1 + q^2\ v_2 + \cdots ...) \quad \textbf{(36)}$$

4. Review of literature

In previous years a lot of research had seen done by scientists and researchers in order to develop the analytical solutions of different class of diffrential equations. Different novel techniques have been implemented to get the solution of non-linear partial differential equations. Liao [11-15] established a new method based on homotopy from the area of topology, known as the Homotopy perturbation method in 1992 afterwards He gave the series solution of this method for the non-linear partial differential equation. Many researcher have implemented He' HPM in order to solve different class of differential equations. Ganji [16] gave the application of He's homotopy perturbation method to different type of non-linear equations used in area of heat transfer. Odibat and Momani [17] employed the notion of modified HPM as the application to the Ricatti differential equation with fractional order. Cveticanin [18], implemented the concept of HPM in the are of pure non-liear differential equations. Ghori et al. [19] presented the application of HPM in the squeezing flow of a Newtonian fluid. Abbasbandy [20] implemented the notion of HPM for the concept of functional integral equations. Tari et al. [21] gave the solutions of different type of differential equations by implementing the notion of variational iteration method, homotopy perturbation method and homotopy analysis method. Ozis and Yildtim [22] employed the notion of He' HPM for the travelling wave solution of KdV equation.

5. Conclusion:

In present review paper, the technique of homotopy perturbation method is discussed after reviewing the literature and this point came into light that HPM is a very important technique to discuss the analytical solution of class on non-linear partial differential equations. Applications of this method are also discussed for the 1D Burgers' equation as well as for 2D coupled Burgers' equation as per literatrure. It can be said that this review paper will help in further research regarding homotopy perturbation method as it has the most useful and important points and discussion related with homotopy perturbation method.

References

1. He, J. H. (2004). The homotopy perturbation method for nonlinear oscillators with discontinuities. *Applied mathematics and computation*, *151*(1), 287-292.
2. He, J. H. (2005). Application of homotopy perturbation method to nonlinear wave equations. *Chaos, Solitons & Fractals*, *26*(3), 695-700.

3. He, J. H. (2006). Homotopy perturbation method for solving boundary value problems. *Physics letters A*, *350*(1-2), 87-88.
4. He, J. H. (1999). Homotopy perturbation technique. *Computer methods in applied mechanics and engineering*, *178*(3-4), 257-262.
5. He, J. H. (2000). A coupling method of a homotopy technique and a perturbation technique for non-linear problems. *International journal of non-linear mechanics*, *35*(1), 37-43.
6. He, J. H. (2004). Comparison of homotopy perturbation method and homotopy analysis method. *Applied Mathematics and Computation*, *156*(2), 527-539.
7. He, J. H. (2003). Homotopy perturbation method: a new nonlinear analytical technique. *Applied Mathematics and computation*, *135*(1), 73-79.
8. Liao, S. J. (1995). An approximate solution technique not depending on small parameters: a special example. *International Journal of Non-Linear Mechanics*, *30*(3), 371-380.
9. Liao, S. J. (1997). Boundary element method for general nonlinear differential operators. *Engineering Analysis with Boundary Elements*, *20*(2), 91-99.
10. Biazar, J., & Ghazvini, H. (2009). Exact solutions for nonlinear burgers' equation by homotopy perturbation method. *Numerical Methods for Partial Differential Equations*, *25*(4), 833-842.
11. Liao, S. J. (1992). *The proposed homotopy analysis technique for the solution of nonlinear problems* (Doctoral dissertation, Ph. D. Thesis, Shanghai Jiao Tong University).
12. Liao, S. J. (1999). An explicit, totally analytic approximate solution for Blasius' viscous flow problems. *International Journal of Non-Linear Mechanics*, *34*(4), 759-778.
13. Liao, S. (2004). On the homotopy analysis method for nonlinear problems. *Applied Mathematics and Computation*, *147*(2), 499-513.
14. Liao, S. (2006). Series solutions of unsteady boundary- layer flows over a stretching flat plate. *Studies in Applied Mathematics*, *117*(3), 239-263.
15. Liao, S. (2009). Notes on the homotopy analysis method: some definitions and theorems. *Communications in Nonlinear Science and Numerical Simulation*, *14*(4), 983-997.
16. Ganji, D. D. (2006). The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer. *Physics letters A*, *355*(4-5), 337-341.
17. Odibat, Z., & Momani, S. (2008). Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order. *Chaos, Solitons & Fractals*, *36*(1), 167-174.
18. Cveticanin, L. (2006). Homotopy–perturbation method for pure nonlinear differential equation. *Chaos, Solitons & Fractals*, *30*(5), 1221-1230.
19. Ghori, Q. K., Ahmed, M., & Siddiqui, A. M. (2007). Application of homotopy perturbation method to squeezing flow of a Newtonian fluid. *International Journal of Nonlinear Sciences and Numerical Simulation*, *8*(2), 179-184.
20. Abbasbandy, S. (2007). Application of He's homotopy perturbation method to functional integral equations. *Chaos, Solitons & Fractals*, *31*(5), 1243-1247.
21. Tari, H., Ganji, D. D., & Rostamian, M. Approximate Solutions of K (2, 2), KdV and Modified KdV Equations by Variational Iteration Method, Homotopy.
22. Öziş, T., & Yıldırım, A. (2007). Traveling wave solution of Korteweg-de Vries equation using He's homotopy perturbation method. *International Journal of Nonlinear Sciences and Numerical Simulation*, *8*(2), 239-242.