Rayleigh waves with impedance boundary conditions using consistent couple stress model

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Abstract

The paper aims to study about Rayleigh waves in an elastic solid half space using consistent couple stress theory under impedance boundary conditions. Impedance boundary conditions are used to imitate the impact of a thin layer over a half space. Consistent couple stress elasticity involves an additional size dependent elastic material parameter, known as internal characteristic length, which takes care of internal microstructures of the material. The influence of characteristic length becomes important when its dimensions are comparable with the order of internal microstructure of the material. Dispersion equation for propagation of Rayleigh waves under impedance boundary conditions is derived using consistent couple stress theory. Dispersion equation of Rayleigh waves for a traction free couple stress elastic half-space is obtained as a particular case. The effects of impedance and characteristic length parameters are studied on the phase velocity of Rayleigh waves. The dispersion curves are plotted for Rayleigh waves propagation under impedance conditions as well as under traction free conditions.

Keywords: Consistent couple stress theory, Characteristic length, Impedance conditions, Rayleigh wave

1. Introduction

Classical theory of elasticity is developed on assumptions that materials are homogeneous and are distributed continuously throughout the entire volume. The inner molecular structure of material was not taken into consideration in classical theory of elasticity. These deficiencies of classical theory of elasticity have led to the development of size dependent microcontinuum theories of elasticity [1-5]. These generalized microcontinuum theories involve certain additional material parameters which take care of internal microstructures of the material. Hadjesfandiari and Dargush [6] proposed consistent couple stress theory of elasticity, which involve three material parameters, out of which two are Lame's constants of classical mechanics and third parameter is couple stress coefficient ($\eta = \mu l^2$) and it further depends upon a length parameter known as internal characteristic length (l) of the material. This length parameter creates a difference between classical theory and consistent couple stress theory of elasticity. This characteristic length is negligible but it becomes important as the dimensions of body become comparable with this length parameter. In general, characteristic length is of the order of average internal cell size or internal microstructure of material. Moreover, stresses defined in constitutive relations of consistent couple stress theory are also non-symmetric in nature. Constitutive relations involve an additional expression defining couple stresses in terms of displacement vector and these couple stresses are skew-symmetric in nature. Researchers have applied consistent couple stress theory of elasticity to examine various problems [7-8].

Rayleigh surface waves have been extensively studied by the researchers in the fields of seismology, geophysics, acoustics and non-destructive testing techniques [9-11]. In its usual framework, Rayleigh waves exist in a traction free half space. However, to mathematically simulate the effects of thin layer of some other material over a half space, Tiersten [12] explained impedance boundary like conditions. In these impedance type boundary conditions, a linear combination of unknown function and their derivatives is prescribed on the boundary and stresses are assumed to be dependent upon displacement components and their derivatives. Researchers have explored Rayleigh waves using these types of boundary conditions [13-17].

Keeping in view, the capability of couple stress theory to examine materials at microstructural level, we intend to study the Rayleigh waves propagation in an elastic solid half space using impedance boundary conditions in the frame work of consistent couple stress theory of elasticity.

2. Governing Equations and Solution

Consider a homogeneous isotropic elastic solid is occupying the half space $z \geq 0$. We assumed the cartesian frame of reference having its origin on the surface $z = 0$ and wave is propagating along x-axis direction, whereas the z-axis is pointing downwards into the half space. It is supposed that all particles along any line parallel to y-axis are equally displaced, so there is no variation along the direction of y-axis, that is $\frac{\partial}{\partial y} \equiv 0$. The governing equation of consistent couple stress theory in cartesian tensor notation is given as [6]

$$(\lambda + \mu + \eta\nabla^2)u_{k,ki} + (\mu - \eta\nabla^2)\nabla^2 u_i = \rho\ddot{u}_i \tag{1}$$

The constitutive relations are gives as [6]

$$\sigma_{ji} = \lambda u_{k,k}\delta_{ij} + \mu\left(u_{i,j} + u_{j,i}\right) - \eta\nabla^2\left(u_{i,j} - u_{j,i}\right) \tag{2}$$

$$\mu_{ji} = 4\eta\left(\omega_{i,j} - \omega_{j,i}\right), \text{ where } \omega_i = \frac{1}{2}\epsilon_{ijk}u_{k,j} \tag{3}$$

where μ, λ are Lame's parameters, ρ is density, u_i are displacement components, $\eta = \mu l^2$ is couple stress coefficient, l is characteristic length, σ_{ji} is stress tensor, δ_{ij} is Kronecker's delta, μ_{ji} is couple stress tensor, ϵ_{ijk} is alternating tensor, $i, j, k = 1, 2, 3$.

To solve the equation (1), we constrain our argument to two-dimensional medium, so let us assume displacement vector as $\vec{u} = (u, 0, w)$ and by Helmholtz decomposition rule, we consider

$$\vec{u} = \nabla\phi + \nabla \times \vec{\psi}, \nabla.\vec{\psi} = 0 \tag{4}$$

where ϕ is a scalar potential functions and $\vec{\psi} = (0, \psi, 0)$ is a vector potential function. Hence, we get

$$u = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial z}, w = \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial x} \tag{5}$$

Substituting equation (5) in equation (1), we get

$$\nabla^2\phi = \frac{1}{C_P^2}\frac{\partial^2\phi}{\partial t^2} \tag{6}$$

$$\nabla^2\psi - l^2\nabla^4\psi = \frac{1}{C_S^2}\frac{\partial^2\psi}{\partial t^2} \tag{7}$$

where $C_p^2 = \frac{(\lambda+2\mu)}{\rho}$, $C_s^2 = \frac{\mu}{\rho}$. Now, consider the solutions of equations (6) and (7) as

$$\{\phi,\psi\} = \{h(z),k(z)\}\mathrm{e}^{i\xi(x-ct)}, \quad \text{where } \xi \text{ is wave number and } c \text{ is phase velocity} \tag{8}$$

Putting these solutions in equations (6) and (7), we obtain

$$\frac{d^2h(z)}{dz^2} - \left(\xi^2 - \frac{\xi^2c^2}{C_P^2}\right)h(z) = 0 \tag{9}$$

$$\frac{d^4k(z)}{dz^4} - \left(2\xi^2 + \frac{1}{l^2}\right)\frac{d^2k(z)}{dz^2} + \left(\frac{\xi^2}{l^2} + \xi^4 - \frac{\xi^2c^2}{l^2C_S^2}\right)k(z) = 0 \tag{10}$$

By solving differential equations in (9) and (10), we obtain

$$\phi = (A_1e^{-\alpha z} + A_2e^{\alpha z})e^{i\xi(x-ct)} \tag{11}$$

$$\psi = \left(B_1e^{-\alpha z} + B_2e^{-\beta z} + B_3e^{\alpha z} + B_4e^{\beta z}\right)e^{i\xi(x-ct)} \tag{12}$$

where $a^2 = \xi^2\left(1 - \frac{c^2}{C_P^2}\right), (\alpha^2+\beta^2) = 2\xi^2 + \frac{1}{l^2}, \alpha^2\beta^2 = \xi^4 + \frac{\xi^2}{l^2}\left(1 - \frac{c^2}{C_S^2}\right)$

As wave must die down with depth into the half space, so we consider the solutions as

$$\phi = A_1e^{-\alpha z}e^{i\xi(x-ct)} \tag{13}$$

$$\psi = \left(B_1e^{-\alpha z} + B_2e^{-\beta z}\right)e^{i\xi(x-ct)} \tag{14}$$

where A_1, B_1, B_2 are arbitrary constants.

The force stresses and couple stress components are given as

$$\sigma_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) - \eta\left(\frac{\partial^3 u}{\partial x^2\partial z} - \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 u}{\partial z^3} - \frac{\partial^3 w}{\partial z^2\partial x}\right)$$

$$\sigma_{zz} = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial w}{\partial z}$$

$$\mu_{zy} = 2\eta\left(\frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 w}{\partial z\partial x}\right)$$

3. Boundary Conditions

Following Godoy *et al.* [15], impedance boundary conditions at the surface $z = 0$ are given as

$$\sigma_{zx} + \varpi I_1 u = 0, \sigma_{zz} + \varpi I_2 w = 0, \mu_{zy} = 0 \tag{15}$$

where $\varpi = \xi c$ is the circular frequency, I_1, I_2 are impedance parameters. We assumed that the impedance parameters are real and $I_1 = 0 = I_2$ gives us stress free boundary conditions. Here, we are taking non-dimensional parameters as $I_1^* = \frac{I_1}{\sqrt{\mu\rho}}$, $I_2^* = \frac{I_2}{\sqrt{\mu\rho}}$

4. Derivation of Secular Equations

4.1 Rayleigh waves in consistent couple stress elastic half space with impedance parameters

Using boundary conditions mentioned in (15), we get

$$\left(-2a + \frac{\varpi I_1^*}{c_S}\right) i\xi A_1 + \left((l^2 k_\alpha^2 - S_\alpha) + \frac{\alpha\varpi I_1^*}{c_S}\right) B_1 + \left((l^2 k_\beta^2 - S_\beta) + \frac{\beta\varpi I_1^*}{c_S}\right) B_2 = 0 \tag{16}$$

$$\left(P - \frac{a\varpi I_2^*}{c_S}\right) A_1 + \left(-2\alpha + \frac{\varpi I_2^*}{c_S}\right) i\xi B_1 + \left(-2\beta + \frac{\varpi I_2^*}{c_S}\right) i\xi B_2 = 0 \tag{17}$$

$$\alpha k_\alpha B_1 + \beta k_\beta B_2 = 0 \tag{18}$$

where

$$P = 2\xi^2 - \frac{c_P^2}{c_S^2}(\xi^2 - a^2), K_\alpha = (\xi^2 - \alpha^2), K_\beta = (\xi^2 - \beta^2), S_\alpha = (\xi^2 + \alpha^2), S_\beta = (\xi^2 + \beta^2)$$

The secular equation is given as

$$\alpha k_\alpha \left[\left(-2a + \frac{\varpi I_1^*}{C_S}\right)\left(-2\beta + \frac{\varpi I_2^*}{C_S}\right)\xi^2 + \left(P - \frac{a\varpi I_2^*}{C_S}\right)\left((l^2 k_\beta^2 - S_\beta) + \frac{\beta\varpi I_1^*}{C_S}\right)\right] -$$

$$\beta k_\beta \left[\left(-2a + \frac{\varpi I_1^*}{C_S}\right)\left(-2\alpha + \frac{\varpi I_2^*}{C_S}\right)\xi^2 + \left(P - \frac{a\varpi I_2^*}{C_S}\right)\left((l^2 k_\alpha^2 - S_\alpha) + \frac{\alpha\varpi I_1^*}{C_S}\right)\right] = 0 \tag{19}$$

4.1.1 Rayleigh waves in consistent couple stress half space with traction free boundary conditions

By taking $I_1^* = I_2^* = 0$, the equation (19) reduces to dispersion equation for Rayleigh waves in couple stress traction free half space

$$\alpha k_\alpha \left(4a\beta\xi^2 + P(l^2 k_\beta^2 - S_\beta)\right) - \beta k_\beta\left(4\alpha a\xi^2 + P(l^2 k_\alpha^2 - S_\alpha)\right) = 0 \tag{20}$$

Equation (20) is same as obtained by Sharma and kumar [18].

4.2 Rayleigh waves in classical elastic half space with impedance parameters

By considering the characteristic length $l = 0$ in equations (1)-(2), the consistent couple stress model reduces to classical elastic model and further, following the same procedure as above, we will get dispersion equation for Rayleigh wave propagation in homogeneous isotropic elastic half space under impedance conditions. Using boundary conditions mentioned in (15), we obtain

$$\left(-2a + \frac{\varpi I_1^*}{C_S}\right) i\xi A + \left(-(b^2 + \xi^2) + \frac{\varpi b I_1^*}{C_S}\right) B = 0 \tag{21}$$

$$\left(\frac{\lambda}{\mu}(a^2 - \xi^2) + 2a^2 - \frac{a\varpi I_2^*}{C_S}\right) A + \left(-2b + \frac{\varpi I_2^*}{C_S}\right) i\xi B = 0 \tag{22}$$

where $a^2 = \xi^2 \left(1 - \frac{c^2}{C_P^2}\right), b^2 = \xi^2 \left(1 - \frac{c^2}{C_S^2}\right)$

The secular equation is given as

$$\left(-2a + \frac{\varpi I_1^*}{C_S}\right)\left(-2b + \frac{\varpi I_2^*}{C_S}\right)\xi^2 + \left(\frac{\lambda}{\mu}(a^2 - \xi^2) + 2a^2 - \frac{a\varpi I_2^*}{C_S}\right)\left(-(b^2 + \xi^2) + \frac{\varpi b I_1^*}{C_S}\right) = 0 \tag{23}$$

4.2.1 Rayleigh waves in an elastic solid half space with traction free boundary conditions

By taking $I_1^* = I_2^* = 0$, the equation (23) gives us dispersion equation for Rayleigh waves in a half space with traction free surface and is same as obtained by Graff [19]

$$4\mu ab\xi^2 - (b^2 - \xi^2)(\lambda(a^2 - \xi^2) + 2\mu a^2) = 0 \tag{24}$$

5. Numerical results and discussion

For numerical results and discussions, we are considering material parameters [20] as E =Youngs modulus=14 GPa, Poisson ratio= ν =0.37, Density= ρ =1500 kg/m^3,the value of longitudinal velocity is $C_P = 4063m/s$ and shear velocity is $C_S = 1846m/s$.

5.1 Rayleigh waves in context of classical theory of elasticity

Fig. 1 shows the variation of Rayleigh speed (c/C_s) with respect to wave number (ξ) in an elastic half-space without impedance parameter that is $I_1^* = I_2^* = 0$and with impedance parameter having values $I_1^* = 2, I_2^* = 0$ and $I_1^* = 0, I_2^* = 2$. It is observed that impedance conditions lead to decrease in phase velocity of Rayleigh waves.

5.2 Effects of impedance parameters on Rayleigh wave velocity in the context of consistent couple stress theory

Fig. 2 shows the profile of normalized Rayleigh wave speed (c/C_s) with normalized wave number (ξl) in couple stress elastic half-space exhibiting microstructure. Here, impedance parameters $I_1^* = 3, I_2^* = 5$ are kept fixed and characteristic lengths (l), is taken as $l = 0.00001m$. It can be observed that for small wave numbers the phase velocity increases sharply with increasing wave number before becoming almost constant with higher values of wave number. This means profiles of phase velocity are dispersive in nature for short wave number range. The effect of microstructure is clearly visible on the Rayleigh waves as predicted by [21] that the Rayleigh waves exhibit dispersion.

5.2.1 Effects of Impedance parameter I_1^*

To examine the impact of non-dimensional impedance parameter I_1^* explicitly on the phase velocity of Rayleigh wave, here in Fig. 3, we have consider three non-zero values of $I_1^* = 3, 4, 5$ and I_2^* is taken as zero. Here material characteristic length is assumed to be 0.00001 m. It is found that Rayleigh waves show dispersion on the lower wave number range and it almost becomes constant for large values of wave number. Moreover, it can also be noted that phase speed decreases with increase in impedance parameter I_1^*. The variation in phase velocity for a continuous range of impedance parameter I_1^* is shown in Fig. 4

5.2.2 Effects of Impedance parameter I_2^*

Fig. 5 shows the variation of normalized Rayleigh wave speed with dimensionless wave number for different values of impedance parameter, where impedance parameter I_2^*, having values 3, 4, 5 is varied and $I_1^* = 0$. Also, by taking characteristic length to be of the order $10^{-5}m$. It is found that Rayleigh waves show dispersion in the lower wave number range and it almost becomes constant for large values of wave number. It is noted that phase velocity decreases with increase in impedance parameter I_2^*. The variation in phase velocity for a continuous range of impedance parameter I_2^* is shown in Fig. 6.

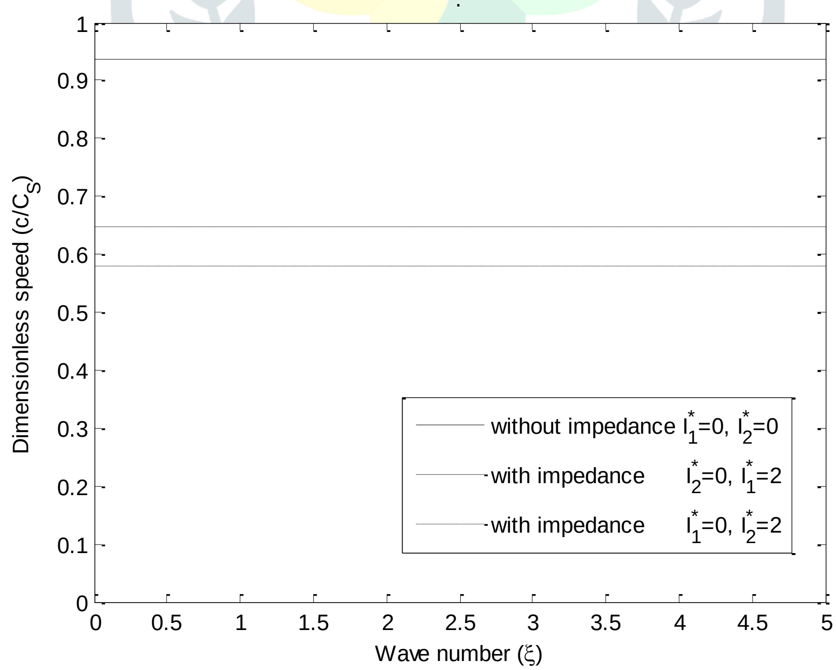

Figure 1. Variation of Rayleigh wave speed with wave number for different values of impedance parameter in classical theory

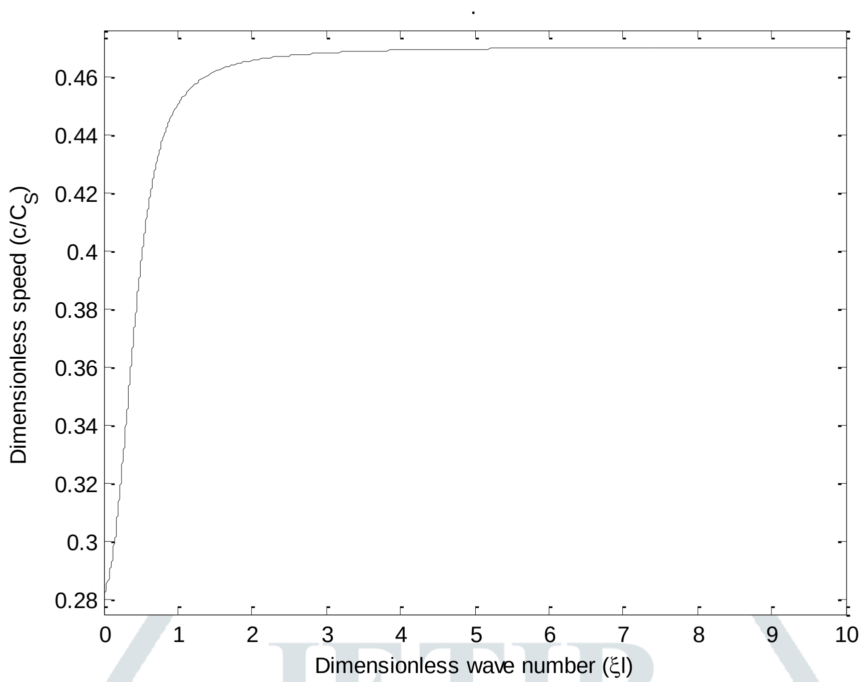

Figure 2. Variation of dimensionless Rayleigh wave speed with wave number, when impedance parameter $I_1^* = 3, I_2^* = 5$ are kept fixed with characteristic length $l = 0.00001m$

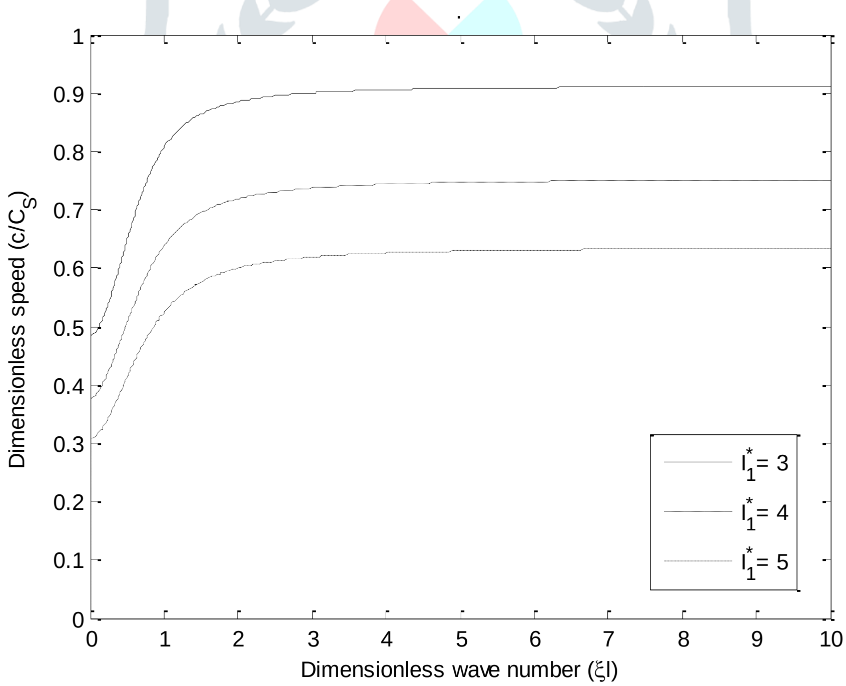

Figure 3. Variation of dimensionless Rayleigh wave speed with dimensionless wave number for three different values of impedance parameter $I_1^* = 3, 4, 5$

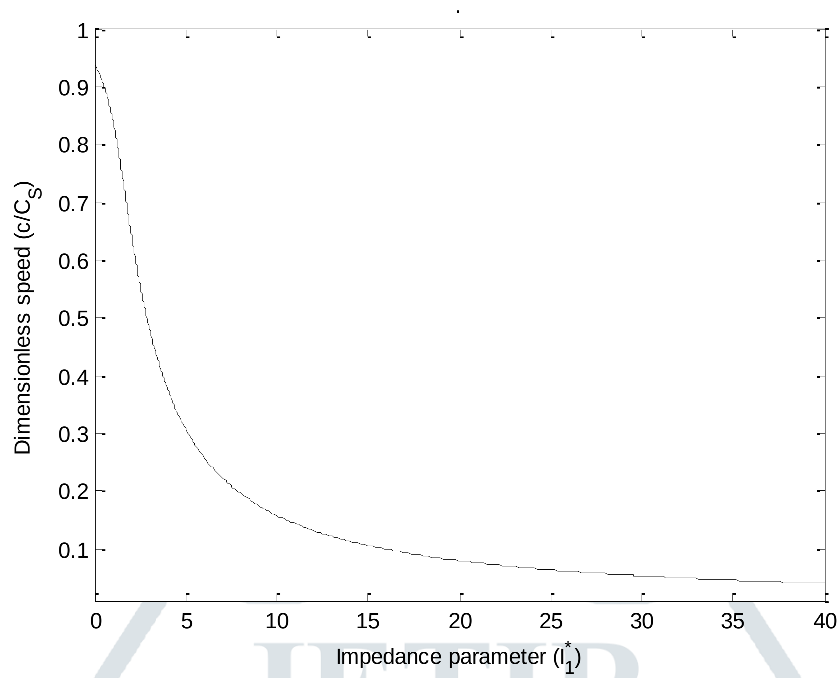

Figure 4. Variation of dimensionless Rayleigh wave speed with impedance parameter I_1^*

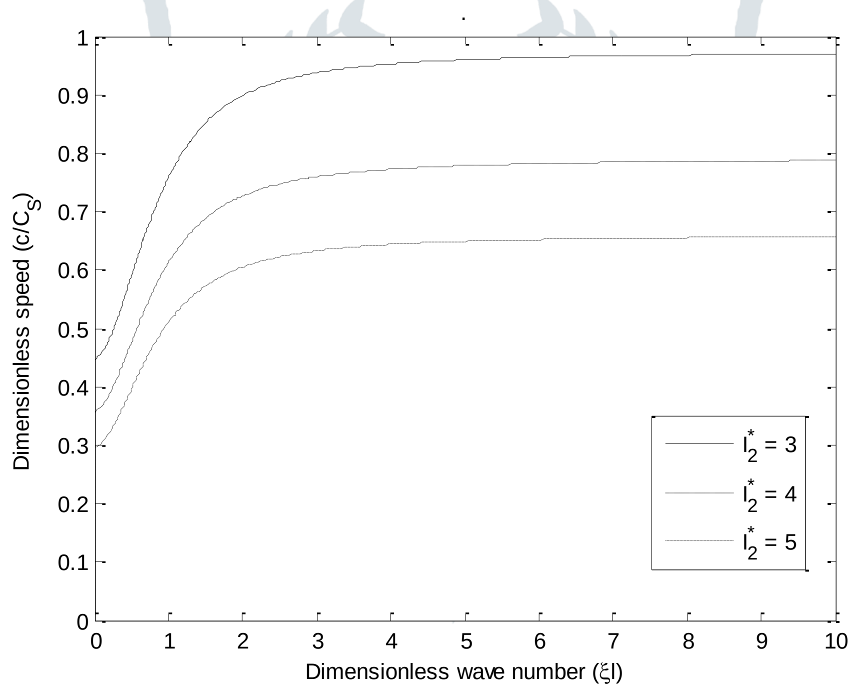

Figure 5. Variation of dimensionless Rayleigh wave speed with dimensionless wave number for three different values of impedance parameter $I_2^* = 3, 4, 5$

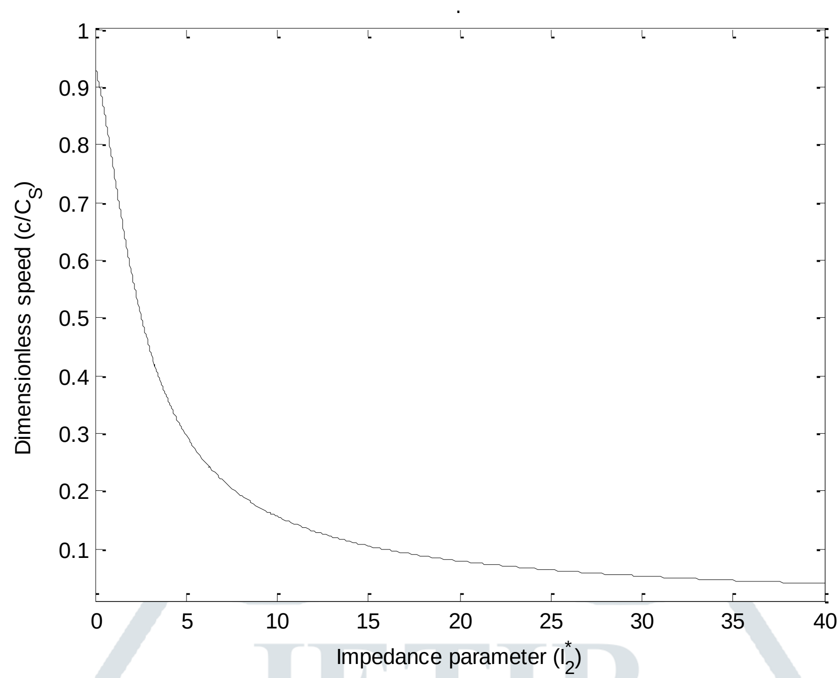

Figure 6. Variation of dimensionless Rayleigh wave speed with impedance parameter I_2^*

6. Conclusion

In the present study, it is found that Rayleigh wave does not exhibit any dispersion in context of classical mechanics. It is simply the validation of the results which are already established in [19]. Impedance parameter affects phase velocity profiles significantly and Rayleigh wave speed decreases considerably in the presence of impedance parameter. Secular equation for the propagation of Rayleigh waves with impedance conditions has been derived within the framework of consistent couple stress theory. One of the important observations of the study is dispersive nature of Rayleigh waves in context of consistent couple stress elasticity. Similar type of phenomenon has already been predicted by [21]. It is observed that impedance parameters go against phase velocity in the context of consistent couple stress elasticity also.

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