Response of fractional ordered microploar thermoelastic half under sinusoidal heating source

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Abstract

This article study the deformation in a fractional ordered micropolar thermoelastic elastic half space whose boundary is subjected to sinusoidal type heating source. The fractional order theory of thermoelasticity with one relaxation time has been employed to investigate the problem. The problem has been solved by using Integral transform technique. The inversion of transforms is obtained numerically. The numerically computed results for stress and temperature distribution in the half space are depicted graphically.

Keywords: Micropolar, Thermoelasticity, Fractional calculus

1. Introduction

Eringen's [1] theory of micropolar elasticity is one of well established theory to study the material properties with microstructure. Eringen [2] and Nowacki [3] introduced thermal effects and developed theory of micropolar thermoelasticity.

Recently fractional calculus has been employed to establish a number of useful models for studying physical phenomena, especially in the fields of heat transfer, solid mechanics, etc. Different models of fractional order thermoelasticity based upon fractional calculus are presented by various authors. In this paper we consider the theory of thermoelasticity based upon fractional calculus proposed by Sherief et al. [4], Ezzat [5] and Youssef [6].

Using Lord -Shulman [7] theory of thermoelasticity, Sherief et al [4] proposed heat conduction law given by

$$K^*\nabla^2 T = \rho C^*\left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^{\alpha'+1}}{\partial t^{\alpha'+1}}\right)T + \nu T_0\left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^{\alpha'+1}}{\partial t^{\alpha'+1}}\right)\nabla.\vec{u}$$

Where α' is the fractional order parameter with values lies in 0 and 1. T represents temperature distribution in the medium, T_0 represents reference temperature. K^* denotes coefficient of thermal conductivity, τ_0 is the thermal relaxation times, C^* symbolizes specific heat at constant strain, $\vec{u}$ is the displacement vector.

$\nu = (3\lambda + 2\mu + K)\alpha_t$, where α_t denotes coefficient of thermal linear expansion

Youssef [5] introduced another modal of thermoelasticity by using fractional calculus

$$K^* I^{\alpha'-1}\nabla^2 T = \rho C^*\left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right)T + \nu T_0\left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right)\nabla.\vec{u}$$

Ezzat [6] used Taylor series and proposed the equation of heat conduction given by

$$K^*\nabla^2 T = \rho C^*\left(\frac{\partial}{\partial t} + \frac{\tau_0^{\alpha'}}{\alpha'!}\frac{\partial^{\alpha'+1}}{\partial t^{\alpha'+1}}\right)T + \nu T_0\left(\frac{\partial}{\partial t} + \frac{\tau_0^{\alpha'}}{\alpha'!}\frac{\partial^{\alpha'+1}}{\partial t^{\alpha'+1}}\right)\nabla.\vec{u}$$

Recently Kumar et.al [8] studied the interaction of these fractional order theories in micropolar elastic solid subjected to a ramp type heating. Boundary conditions play a very important role to determine the particular solution of the problem and to analyze the behavior of the material under different types of conditions. In the present article, deformations in micropolar thermoelastic half space with fractional order heat transfer under sinusoidal heating source have been studied. Using integral transform on various variables analytical expressions for components of displacement, stress and temperature distribution are obtained in transformed domain.

2. Equations of motion

Equations of motion for homogeneous, isotropic micropolar thermoelastic [Eringen [2]] solid are

$$t_{ij} = \lambda u_{r,r}\delta_{ij} + \mu(u_{i,j} + u_{j,i}) + K(u_{j,i} - \epsilon_{ijr}\phi_r) - \nu T\delta_{ij} \quad (1)$$

$$m_{ij} = \alpha\phi_{r,r}\delta_{ij} + \beta\phi_{i,j} + \gamma\phi_{j,i} \quad (2)$$

$$(\mu + K)\nabla^2\vec{u} + (\lambda + \mu)\nabla(\nabla.\vec{u}) + K\nabla\times\vec{\phi} - \nu\nabla T = \rho\frac{\partial^2\vec{u}}{\partial t^2} \quad (3)$$

$$(\alpha + \beta + \gamma)\nabla(\nabla.\vec{\phi}) - \gamma\nabla\times(\nabla\times\vec{\phi}) + K\nabla\times\vec{u} - 2K\vec{\phi} = \rho j\frac{\partial^2\vec{\phi}}{\partial t^2} \quad (4)$$

Where $\lambda, \mu, K, \alpha, \beta, \gamma$ are material constants, $\vec{\phi}$ is the microrotation vector, ρ represents density, j is the microinertia.

Following [4] , [5] and [6], heat conduction equation can be written as

$$K^*\nabla^2 T = \rho C^*\left(\frac{\partial^{p_1}}{\partial t^{p_1}} + \frac{\tau_0^{p_2}}{p_2!}\frac{\partial^{\alpha'+1}}{\partial t^{\alpha'+1}}\right)T + \nu T_0\left(\frac{\partial^{p_1}}{\partial t^{p_1}} + \frac{\tau_0^{p_2}}{p_2!}\frac{\partial^{\alpha'+1}}{\partial t^{\alpha'+1}}\right)\nabla.\vec{u} \quad (5)$$

where

$p_1 = 1, p_2 = 1$	for Sherief theory
$p_1 = 1, p_2 = \alpha'$	for Ezzat theory
$p_1 = \alpha', p_2 = 1$	for Youssef theory
$p_1 = 1, p_2 = 1, \alpha' = 1$	For Lord and Shulman theory

3. Formulation of the Problem

Consider a fractional ordered homogeneous, isotropic, micropolar thermoelastic half space in an undisturbed state at uniform temperature T_0 .The Cartesian coordinate system (x_1, x_2, x_3) is taken at any point on the plane surface and x_3-axis points vertically downwards into the medium. For the two dimensional problem we consider

$$\vec{u} = (u_1, 0, u_3)\ ,\ \vec{\phi} = (0, \phi_2, 0) \tag{6}$$

Also we used following dimensionless quantities to simplify the solution

$$x_i' = \frac{\omega^*}{c_1}x_i,\ u_i' = \frac{\rho\omega^* c_1}{\nu T_0}u_i,\ \phi_2' = \frac{\rho c_1^2}{\nu T_0}\phi_2, m_{ij}' = \frac{\omega^*}{c_1 \nu T_0}m_{ij}, t' = \omega^* t,\ T' = \frac{T}{T_0}$$

$$\tau_0' = \omega^*\tau_0, t_{ij}' = \frac{t_{ij}}{\nu T_0},\ F_1' = \frac{F_1}{\nu T_0}, T_1' = \frac{T_1}{T_0}$$

where $\omega^* = \frac{\rho c^* c_1^2}{K^*}$, $c_1^2 = \frac{\lambda+2\mu+K}{\rho}$ (7)

Consider the potential functions Φ & ψ as

$$u_1 = \frac{\partial\Phi}{\partial x_1} - \frac{\partial\psi}{\partial x_3}\ ,\ u_3 = \frac{\partial\Phi}{\partial x_3} + \frac{\partial\psi}{\partial x_1} \tag{8}$$

Substituting (8) in the equations (3)-(5), using (7) and after leaving the primes we get

$$\nabla^2\Phi - T - \frac{\partial^2\Phi}{\partial t^2} = 0 \tag{9}$$

$$a_1\nabla^2\psi + a_2\phi_2 - \frac{\partial^2\psi}{\partial t^2} = 0 \tag{10}$$

$$\left(\nabla^2 - 2a_3 - a_4\frac{\partial^2}{\partial t^2}\right)\phi_2 - a_3\nabla^2\psi = 0 \tag{11}$$

$$\nabla^2 T-(\omega^*)^{p_1-1}\left(\frac{\partial^{p_1}}{\partial t^{p_1}}+\frac{\tau_0^{p_2}}{p_2!}(\omega^*)^{\alpha'+1-(p_1+p_2)}\frac{\partial^{\alpha'+1}}{\partial t^{\alpha'+1}}\right)T \\ -a_5\left(\frac{\partial^{p_1}}{\partial t^{p_1}}+\frac{\tau_0^{p_2}}{p_2!}(\omega^*)^{\alpha'+1-(p_1+p_2)}\frac{\partial^{\alpha'+1}}{\partial t^{\alpha'+1}}\right)\nabla^2\Phi=0 \tag{12}$$

Where $a_1=\frac{\mu+K}{\rho c_1^2}$, $a_2=\frac{K}{\rho c_1^2}$, $a_3=\frac{Kc_1^2}{\gamma\omega^{*2}}$, $a_4=\frac{\rho jc_1^2}{\gamma}$, $a_5=\frac{\nu^2 T_0(\omega^*)^{(p_1-2)}}{\rho K^*}$

Consider the following definition for Laplace and Fourier transform

$$\overline{f}(x_1,x_3,s)=\int_0^\infty f(x_1,x_3,t)e^{-st}dt \tag{13}$$

$$\tilde{\overline{f}}(\xi,x_3,s)=\int_{-\infty}^{\infty}\overline{f}(x_1,x_3,s)e^{i\xi x_1}dx_1 \tag{14}$$

Applying the Laplace transform and Fourier transform as given in (13)-(14) on (9)-(12), we obtained

$$(D^4+b_1D^2+b_2)\tilde{\overline{\Phi}}=0 \tag{15}$$

$$(D^4+b_3D^2+b_4)\tilde{\overline{\psi}}=0 \tag{16}$$

where

$b_1=-(a_5l_1+l_2+l_3)$,$b_2=l_2\,l_3+a_5l_1\xi^2$,$b_3=-(l_5+\left(\frac{l_4}{a_1}\right)-\frac{a_2a_3}{a_1})$,$b_4=\frac{l_4\,l_5-a_2\,a_3\,\xi^2}{a_1}$

$l_1=s^{p_1}+\frac{\tau_0^{p_2}}{p_2!}(\omega^*)^{\alpha'+1-(p_1+p_2)}s^{\alpha'+1}$, $l_2=\xi^2+(\omega^*)^{(p_1-1)}l_1$, $l_3=\xi^2+s^2$, $l_4=a_1\xi^2+s^2$,$l_5=\xi^2+2a_3+a_4s^2$, $D=\frac{d}{dx_3}$ (17)

Using the conditions $\tilde{\overline{\Phi}},\tilde{\overline{\psi}},\tilde{\overline{\phi}}_2$ *and* $\tilde{\overline{T}}\to 0$ *as* $x_3\to\infty$, the solution of equations (15) and (16) is taken as

$$\left\{\tilde{\overline{\Phi}},\tilde{\overline{T}}\right\}=\sum_{i=1}^{2}\{1,r_i\}A_ie^{-m_ix_3} \tag{18}$$

$$\left\{\tilde{\overline{\psi}},\tilde{\overline{\phi}}_2\right\}=\sum_{j=3}^{4}\{1,s_j\}A_je^{-m_jx_3} \tag{19}$$

where

$r_i=m_i-l_3$, $i=1,2$,$s_j=\frac{1}{a_2}(l_4-a_1m_j^2)$ $j=3,4$

where m_1,m_2 are the roots of the equation (15) and m_3,m_4 are the roots of the equation (16)

We obtained the displacement component $\tilde{\overline{u}}_1$ and $\tilde{\overline{u}}_3$ from equation (8) as

$$\tilde{\overline{u}}_1=-\mathrm{i}\,\xi A_1e^{-m_1x_3}-i\xi A_2e^{-m_2x_3}+A_3m_3e^{-m_3x_3}+A_4m_4e^{-m_4x_3} \tag{20}$$

$$\tilde{\bar{u}}_3 = -m_1 A_1 e^{-m_1 x_3} - m_2 A_2 e^{-m_2 x_3} - i\,\xi A_3 e^{-m_3 x_3} - i\,\xi A_4 e^{-m_4 x_3} \qquad (21)$$

4. Boundary Conditions

The surface of the half space is assumed to be stress-free and subjected to a sinusoidal type heating source i.e

$$t_{33}(x_1, 0, t) = t_{31}(x_1, 0, t) = m_{32} = 0\,,\ T(x_1, 0, t) = G(t)F(x_1) \qquad (22)$$

where G(t) as defined by $G(t) = \begin{cases} 0 & t < 0 \\ T_1 + A\sin\left(\frac{2\pi t}{t_0}\right) & 0 \le t \le t_0 \\ T_1 & t > t_0 \end{cases}$ (23)

where T_1 is the constant average temperature and A is amplitude of the sinusoidal temperature wave. t_0 is a fixed moment of time during which the surface has been exposed to sinusoidal heating source. $F(x_1)$ is an arbitrary function of x_1 and is consider as

$$F(x_1) = \delta(x_1) \qquad (24)$$

Here $\delta(\ \ )$ is Dirac delta function.

Applying integral transforms (13)-(14) on (22) we get

$$\tilde{\bar{t}}_{33}(\xi, 0, s) = 0,$$
$$\tilde{\bar{t}}_{31}(\xi, 0, s) = 0\,,$$
$$\tilde{\bar{m}}_{32} = 0\,, \qquad (25)$$
$$\tilde{\bar{T}}(\xi, 0, s) = \bar{G}(s)$$

where $\bar{G}(s) = \frac{T_1}{s} + \frac{\omega' A(1 - e^{-s t_0})}{s^2 + \omega'^2}$

Applying integral transforms (13)-(14) on (1)-(2) and using (7), we get

$$\tilde{\bar{t}}_{33} = -a_6 i\xi \tilde{\bar{u}}_1 + D\tilde{\bar{u}}_3 - \tilde{\bar{T}} \qquad (26)$$

$$\tilde{\bar{t}}_{31} = -a_7 i\xi \tilde{\bar{u}}_3 + a_1 D\tilde{\bar{u}}_3 - a_2 \tilde{\bar{\phi}}_2 \qquad (27)$$

$$\tilde{\bar{m}}_{32} = a_8 D\tilde{\bar{\phi}}_2 \qquad (28)$$

and $a_6 = \frac{\lambda}{\rho c_1^2}$, $a_7 = \frac{\mu}{\rho c_1^2}$, $a_8 = \frac{\gamma w^{*2}}{\rho c_1^4}$

Substitute the values of $\tilde{\bar{u}}_1, \tilde{\bar{u}}_3, \tilde{\bar{T}}, \tilde{\bar{\phi}}_2$ the boundary condition (25) and using (26)- (28), we obtained

$$\tilde{\bar{t}}_{33} = \frac{\bar{G}(s)}{\Delta}\left(d_{11}\Delta_1 e^{-m_1 x_3} + d_{12}\Delta_2 e^{-m_2 x_3} + d_{13}\Delta_3 e^{-m_3 x_3} + d_{14}\Delta_4 e^{-m_4 x_3}\right) \qquad (29)$$

$$\tilde{\bar{t}}_{31} = \frac{\bar{G}(s)}{\Delta}\left(d_{21}\Delta_1 e^{-m_1 x_3} + d_{22}\Delta_2 e^{-m_2 x_3} + d_{23}\Delta_3 e^{-m_3 x_3} + d_{24}\Delta_4 e^{-m_4 x_3}\right) \qquad (30)$$

$$\tilde{\tilde{m}}_{32} = \frac{\bar{G}(s)}{\Delta}(d_{33}\Delta_3 e^{-m_3x_3} + d_{34}\Delta_4 e^{-m_4x_3}) \quad (31)$$

$$\tilde{\tilde{T}} = \frac{\bar{G}(s)}{\Delta}(r_1\Delta_1 e^{-m_1x_3} + r_2\Delta_2 e^{-m_2x_3}) \quad (32)$$

where

$\Delta = \begin{vmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{vmatrix}$, where Δ_i , $i = 1,2,3,4$ areobtained from Δ by interchanging i^{th}column by the column $[0, 0,0,1]^T$

and

$d_{1i} = -a_6\xi^2 + m_i^2 - r_i i = 1,2$, $d_{1j} = i\xi m_j(-a_6 + 1)$, $j = 3,4$, $d_{2p} = i\xi m_p(a_7 + a_1)$, $p = 1,2$

$d_{2q} = -(a_7\xi^2 + a_1 m_q + a_2 s_q)$, $q = 3,4$, $d_{31} = 0, d_{32} = 0$, $d_{3l} = s_l m_l$, $l = 3,4$

$d_{4n} = r_n$, $d_{43} = 0$, $d_{44} = 0$, $n = 1,2$

By putting suitable values of p_1, p_2 and α', we can obtained temperature, stress and displacement components for Sherief , Youssef , Ezzat and Lord – Shulman theory .

5. Solution in physical domain

Analytically the inverse of integral transforms becomes difficult due to complicated expressions of different field variables. So numerical computations have been carried out to find the solution in physical domain, we invert the Laplace and Fourier transforms by means of method described by Kumar et.al [9]

6. Numerical results and analysis

In order to illustrate the contribution of fractional parameter, effect of sinusoidal heating on different field variables a numerical analysis is carried out, Following [8] [10], data for a magnesium crystal is given below

$\lambda = 9.4\ X10^{11}\text{kgm}^{-1}\text{s}^{-2}$, $\mu = 4.0X10^{11}\text{kgm}^{-1}\text{s}^{-2}$, $T_0 = 298K$, $K = 1.0X10^{11}\text{kgm}^{-1}\text{s}^{-2}$

$j = 0.2X10^{-19}\text{m}^2$, $\gamma = 0.779X10^{-9}\text{kgms}^{-2}$, $\rho = 1.74X10^3\text{kgm}^{-3}$, $\alpha_t = 2.36X10^{-5}\text{K}^{-1}$

$C^* = 9.623X10^2\text{Jkg}^{-1}\text{K}^{-1}$, $K^* = 2.510\text{Wm}^{-1}\text{K}^{-1}$, $\tau_0 = 0.02\text{s}$

The computations are carried out on the surface of the plane $x_3 = 1$ in the range $0 \leq x_1 \leq 2.5$.The numerically computed results for normal stress, tangential stress , tangential couple stress and temperature distribution are depicted with respect to distance x_1 shown in Figs.1-4. We have investigated how the stresses and temperature vary with distance x_1 for different values of parameter t_0 for Lord-Shulman theory. Variations are shown for three different values of sinusoidal parameter $t_0 =$ 0.2,0.3 0.5. As can be seen from figures (1-3) all the stress components shows oscillatory behavior and all components tend to vanish with increase in distance. Significant variations are noticed for different values sinusoidal parameter t_0. Figure 4 describe the variations in temperature distribution T with distance x_1 and as shown this field decreases with distance from the source location. Small variations are also observed in temperature distribution for considered values of sinusoidal parameter t_0.

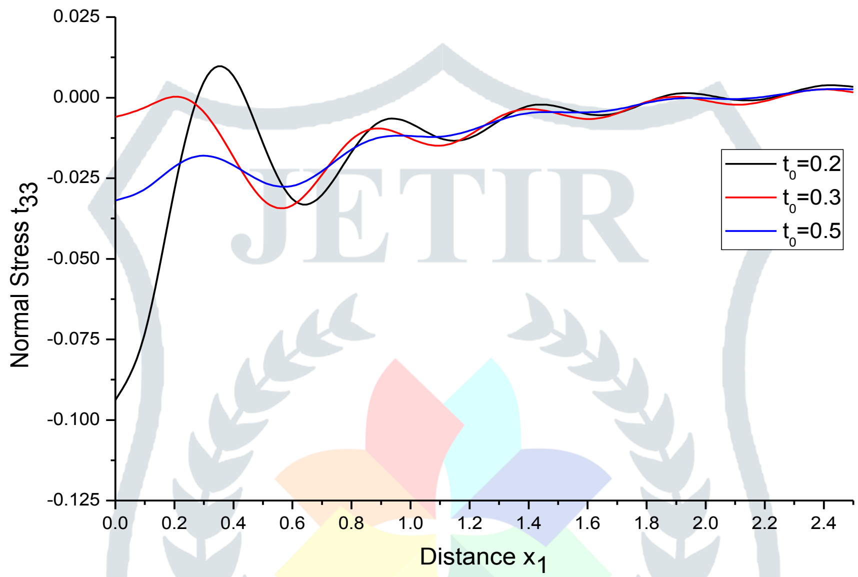

Figure 1. Normal stress w.r.t. distance for different values of parameter t_0

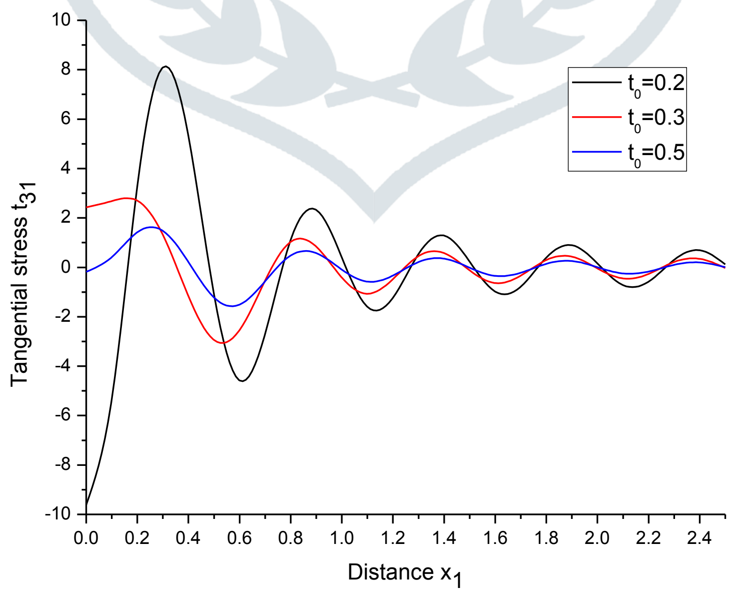

Figure 2. Tangential stress w.r.t. distance for different values of parameter t_0

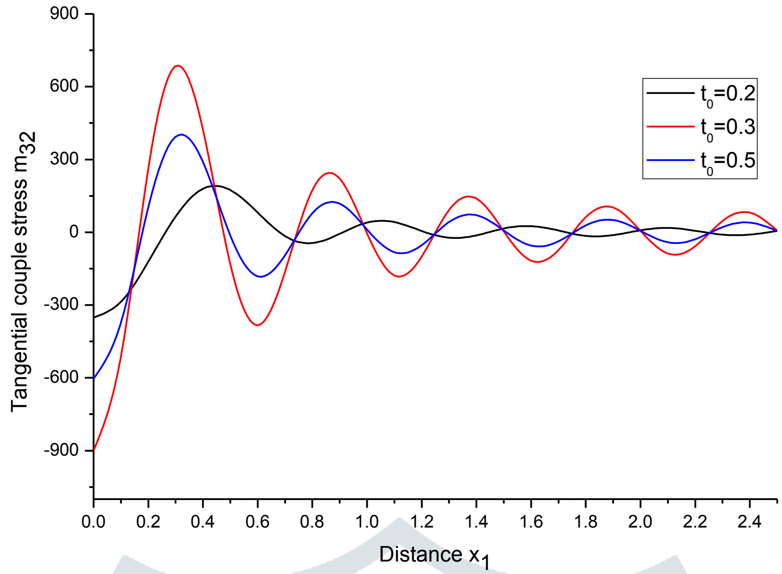

Figure 3. Tangential couple stress w.r.t. distance for different values of parameter t_0

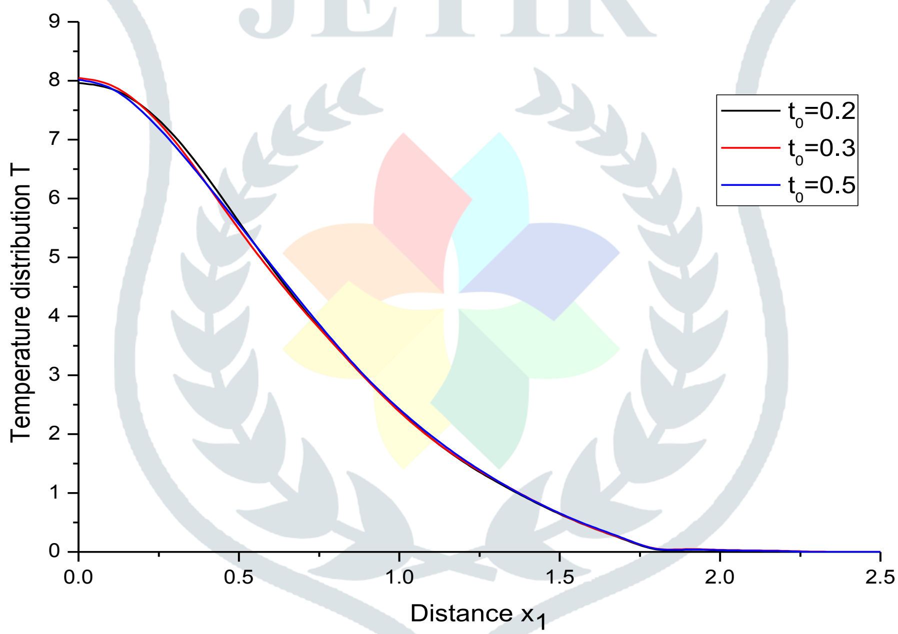

Figure 4. Temperature distribution w.r.t. distance for different values of parameter t_0

7. CONCLUSION

In the present manuscript, the deformation in a fractional ordered micropolar thermoelastic half space due to sinusoidal heating source has been studied. Using integral transform techniques the analytical expressions for displacement and stress components together with temperature distribution are obtained in the transformed domain. The results obtained in transformed domain are inverted numerically to physical domain. Sinusoidal heating source has significant effects on all the field variables and effect decreases with increase in distance from the source.

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