

Effect of Delayed Carbohydrates Uptake on Foliage Growth

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Abstract: The growth of plants depends on photosynthesis and carbon assimilation, When the carbohydrates uptake gets delayed, the foliage growth gets adversely affected. This delay effect is presented here with the help of delay parameter τ . In this paper, carbohydrates uptake and foliage growth mechanism with delay is analysed. Two state variables considered are: carbon store and foliage density. Boundedness, positivity, equilibrium points and stability analysis of equilibrium point is performed. Time delay is taken as the bifurcating parameter which disturbs the stability of feasible interior equilibrium and leads to Hopf bifurcation. Numerical simulation supports the theoretical results and system shows quite a complex dynamic.

Introduction

Plants and trees act as a huge sink of carbon dioxide. So, growth of plants and trees is very important for our climate. On the contrary, the adverse effect of climate changes on plant growth is equally important as well. The crop and forest production are affected by nutrient deficiency and virus and pest attack. This model proposed in this paper has its roots in the most important work done in the field of mathematical modelling in plant growth by Thornley [20]. Olofinboba [13] found while comparing the carbohydrates storage in African and European trees that most of the reserve carbohydrates are stored in parenchyma cells in the xylem of the trunk for birch species. Takahashi [19] measured the diameters and distributions of xylem vessels in 6 birch species that grow in Siberia. The cohesion-tension theory given by Dixon and Joly [4], showed that xylem is the main pathway for the transport of water and nutrients up the tree. Newell et al [12] performed experiments to measure how much carbon dioxide is absorbed in plants and then transformed into carbohydrates. Barbaroux et al [1] proved that there are three types of carbohydrates, namely monosaccharides, oligosaccharides and polysaccharides. Boros and Micle [2] showed the effect of copper on seed germination and growth of sunflower. Cu [3] showed that growth of the crop decreases if it is cultivated in a contaminated field. Geng et al [5] studied the arsenic sensitivity in relation to phosphorus. Kalra and Kumar [6,7,8,9] studied the impact of delay parameter in plant growth dynamics effected by toxicant using different models. Liao et al [10] studied root distribution and elemental accumulation of Chinese brake from arsenic contaminated soils where forty hectares of arsenic polluted land was irrigated by local farmers. The decrease in plant biomass and oscillatory behaviour for a value of delay under the effect of toxicant was shown by Naresh et al [11]. Pavel et al [14] used *Lepidium sativum* as a test plant to assess the phytotoxic effects of chromium and cadmium. It was found that root development and dry biomass were adversely affected by toxic stress. Pigna et al [15] reported that applying phosphorus fertilizers enhances the arsenic level in soil. Ruan [16] gave a detailed analysis of various kinds of stabilities like Absolute stability, conditional stability and bifurcation in predator-prey system with discrete delays. Ruan and Wei [17] analysed the nature of zeros of exponential characteristic equation. The stability analysis of equilibrium points involving a non-linear system of delay differential equations is carried out by Ruan [18]. The extent of plant injury by elevated zinc concentration was assessed by Tsonev and Lidon [21], considering its specific and strong dependence on the environmental conditions and availability of other heavy metals. Tu and Ma [22] showed that natural formation and anthropogenic activities are the main reasons of entrance of arsenic into terrestrial and aquatic environment.

In the view of above, therefore, in this paper the problem related role of delay in stability and bifurcation analysis of carbohydrates uptake and foliage growth mechanism is studied using the following mathematical model

Proposed Model

The Mechanism of nutrient and plant biomass is depicted by succeeding framework of equations including two variable, carbohydrates store C and foliage density F .

$$\frac{dF}{dt} = \alpha FC(t - \tau) - \beta F \quad (1)$$

$$\frac{dC}{dt} = I - \gamma FC - \delta C \quad (2)$$

Where: $F(0) > 0, C(0) > 0$ for all t and $C(t - \tau) = \text{constant}$ for $t \in [0, \tau]$.

The parameters considered in this model are interpreted in Table 1:

Table 1: Description of the parameters of the system (1)- (2)

Parameter	Description
I	Carbohydrates input flowing in the system due to photosynthesis
α	Carbohydrates uptake rate
β	Mortality of foliage
γ	Carbohydrates utilization
δ	Mortality rate of carbohydrates
τ	Delay parameter

Analysis of the Model

Boundedness:

Let $\theta = F + C$

$$\Rightarrow \frac{d\theta}{dt} = \frac{dF}{dt} + \frac{dC}{dt}$$

Let $\psi = \min(\alpha, \beta, \gamma, \delta)$ and $C(t - \tau) \cong C(t)$

$$\Rightarrow 0 \leq \theta \leq \frac{I}{\psi} \text{ means } 0 \leq (F + C) \leq \frac{I}{\psi}$$

Hence, all the solutions of the system of equations (1)- (2) lie in the two-dimensional region $\mathcal{R} = \{(F, C) \in R^{2+} : 0 \leq (F + C) \leq \frac{I}{\psi}\}$, $t \rightarrow \infty$, where $\psi = \min(\alpha, \beta, \gamma, \delta)$ with initial conditions $F \geq 0, C \geq 0, \forall t > 0$ and $C(t - \tau) = \text{constant}$ for $t \in [0, \tau]$.

Positivity of Solutions:

From equation (2): $\frac{dC}{dt} \geq -(\delta + \gamma F)C$

$$\Rightarrow \frac{dC}{dt} \geq -\left(\delta + \gamma \frac{I}{\psi}\right)C \Rightarrow C \geq e^{-(\delta + \gamma \frac{I}{\psi})t}$$

From equation (1): $\frac{dF}{dt} \geq -\beta F \Rightarrow F \geq e^{-\beta t}$

So, $C \geq 0, F \geq 0 \forall t$.

Hence solution set of the system of equations stay positive for all t . It ensures that system persists.

Equilibrium point $E^*(F^* \neq 0, C^* \neq 0)$:

It is assumed that at the point of equilibriums: $C(t - \tau) \cong C(t)$.

At point of equilibrium $E^*(F^* \neq 0, C^* \neq 0)$: $\frac{dF^*}{dt} = 0$ and $\frac{dC^*}{dt} = 0$

$$\Rightarrow C^* = \frac{\beta}{\alpha} > 0 \text{ and } F^* = \frac{\alpha I}{\gamma\beta + \alpha\delta} > 0$$

Stability of Equilibrium E^* and Hopf bifurcation:

The system of equations governing the nutrient-plant biomass mechanism at the equilibrium E^* is given by:

$$\frac{dF^*}{dt} = \alpha F^* C^* (t - \tau) - \beta F^* \quad (3)$$

$$\frac{dC^*}{dt} = I - \gamma F^* C^* - \delta C^* \quad (4)$$

The characteristic equation associated with the system of equations (3)-(4) is given by:

$$\lambda^2 + a\lambda + \mathcal{b} + ce^{-\lambda\tau} = 0 \quad (5)$$

$$\text{Where } a = (\beta + \gamma F^* + \delta), \mathcal{b} = \beta(\delta + \gamma F^*), c = \alpha\gamma F^* C^*$$

Clearly a, \mathcal{b}, c are positive at all times.

When $\tau = 0$, the equation (5) becomes:

$$\lambda^2 + a\lambda + \mathcal{b} + c = 0 \quad (6)$$

By Routh-Hurwitz's criteria, roots of equation (6) will have negative real part i.e. the system is stable if:

$$(\mathcal{H}_1): a > 0;$$

$$(\mathcal{H}_2): (\mathcal{b} + c) > 0$$

which obviously is true.

Now, we would like to check the shifting of negative real part of the roots to positive real parts with variations in the values of τ .

Let $\lambda = i\omega$ be a root of equation (5), then equation (5) becomes:

$$(i\omega)^2 + a(i\omega) + \mathcal{b} + ce^{-(i\omega)\tau} = 0$$

$$\Rightarrow -\omega^2 + a(i\omega) + \mathcal{b} + c(\cos \omega\tau - i\sin \omega\tau) = 0$$

Separating real and imaginary parts:

$$\omega^2 - \mathcal{b} = c \cos \omega\tau \quad (7)$$

$$a\omega = c \sin \omega\tau \quad (8)$$

Squaring and adding:

$$\omega^4 + (a^2 - 2\mathcal{b})\omega^2 + (a^2 - c^2) = 0 \quad (9)$$

The two roots of equation (9) are:

$$\omega_{1,2}^2 = \frac{(2\mathcal{b} - a^2) \pm \sqrt{(a^2 - 2\mathcal{b})^2 - 4(a^2 - c^2)}}{2} \quad (10)$$

None of the two roots $\omega_{1,2}^2$ is positive if:

$$(\mathcal{H}_3): (2\mathcal{b} - a^2) < 0 \text{ and } (a^2 - c^2) > 0 \text{ or } (a^2 - 2\mathcal{b}) < 4(a^2 - c^2)$$

That means equation (10) does not have positive roots if condition (\mathcal{H}_3) holds.

We have the following Conjecture (Ruan [12])

Conjecture 1. If $(\mathcal{H}_1) - (\mathcal{H}_2)$ hold, then all the roots of equation (5) have negative real parts for all $\tau \geq 0$.

On the other hand, if:

$$(\mathcal{H}_4): (a^2 - c^2) < 0 \text{ or } (2b - a^2) > 0 \text{ and } (a^2 - 2b)^2 = 4(a^2 - c^2)$$

Then, +ve root of equation (7) is ω_1^2 .

On the same basis, if:

$$(\mathcal{H}_5): (a^2 - c^2) > 0 \text{ or } (2b - a^2) > 0 \text{ and } (a^2 - 2b)^2 > 4(a^2 - c^2)$$

Then, two +ve roots of equation (7) are $\omega_{1,2}^2$.

In both- (\mathcal{H}_4) and (\mathcal{H}_5) , the equation (5) has purely imaginary roots when τ takes certain values. The critical values τ_j^\pm of τ can be calculated from the system of equations (7)-(8), given by:

$$\tau_j^\pm = \frac{1}{\omega_{1,2}} \cos^{-1} \left(\frac{\omega_{1,2}^2 - b}{c} \right) + \frac{2j\pi}{\omega_{1,2}}, j = 0, 1, 2, \dots \quad (11)$$

The above discussion can be condensed in succeeding conjecture (Ruan [12])

Conjecture 2. (I) If $(\mathcal{H}_1) - (\mathcal{H}_2)$ and (\mathcal{H}_4) hold and $\tau = \tau_j^+$, then equation (5) has a pair of purely imaginary roots $\pm i\omega_1$.

(II) If $(\mathcal{H}_1) - (\mathcal{H}_2)$ and (\mathcal{H}_5) hold and $\tau = \tau_j^+$ ($\tau = \tau_j^-$ respectively), then equation (5) has a pair of purely imaginary roots $\pm i\omega_1$ ($\pm i\omega_2$ respectively).

Our expectation is the shifting of negative real part of some roots of equation (5) to positive real part when $\tau > \tau_j^+$ and $\tau < \tau_j^-$. To look into this possibility, let us denote:

$$\tau_j^\pm = \mu_j^\pm(\tau) + i\omega_j^\pm(\tau); j = 0, 1, 2, 3, \dots$$

The roots of equation (5) satisfy: $\mu_j^\pm(\tau_j^\pm) = 0, \omega_j^\pm(\tau_j^\pm) = \omega_{1,2}$

We can verify that the following transversality condition holds:

$$\frac{d}{d\tau} \left(\text{Re } \lambda_j^+(\tau_j^+) \right) > 0 \text{ and } \frac{d}{d\tau} \left(\text{Re } \lambda_j^-(\tau_j^-) \right) < 0$$

It concludes that τ_j^\pm are bifurcating values. The succeeding postulate gives the scattering of the zeros of the equation (5) (Ruan [12])

Postulate: Let $\tau_j^\pm (j = 0, 1, 2, 3, \dots)$ be defined by equation (11).

(I) If $(\mathcal{H}_1), (\mathcal{H}_2)$ hold, then all the roots of equation (5) have -ve real parts for all $\tau \geq 0$.

(II) If $(\mathcal{H}_1), (\mathcal{H}_2)$ and (\mathcal{H}_4) hold and when $\tau \in [0, \tau_0^+)$, then all the roots of equation (5) have -ve real parts. When $\tau = \tau_0^+$, then equation (5) has a pair of purely imaginary roots $\pm i\omega_1$. When $\tau > \tau_0^+$, equation (5) has at least one root with +ve real part.

(III) If $(\mathcal{H}_1), (\mathcal{H}_2)$ and (\mathcal{H}_5) hold, then there is a +ve integer m such that

$0 < \tau_0^+ < \tau_0^- < \tau_1^+ < \tau_1^- \dots < \tau_{m-1}^- < \tau_m^+$ and there are m switches from stability to instability. This means, when $\tau \in [0, \tau_0^+), (\tau_0^-, \tau_1^+), \dots, (\tau_{m-1}^-, \tau_m^+)$, all the roots of equation (5) have negative real parts. When $\tau \in (\tau_0^+, \tau_0^-), (\tau_1^+, \tau_1^-), \dots, (\tau_{m-1}^+, \tau_{m-1}^-)$ and $\tau > \tau_m^+$, equation (5) has at least one root with +ve real part.

Numerical Example

The following set of parametric values is taken to represent graphically the dynamics depicted by the system of equations (1)- (2).

$$I = 0.7, \alpha = 0.4, \beta = 0.3, \gamma = 0.7, \delta = 0.2$$

Where: $N(0) = 2, B(0) = 2.5$

The change of behaviour of the system of equations (1)-(2) from being stable to complex dynamics about the equilibrium $E^*(5.1058, 2.9820)$ for different values of delay parameter τ is shown below:

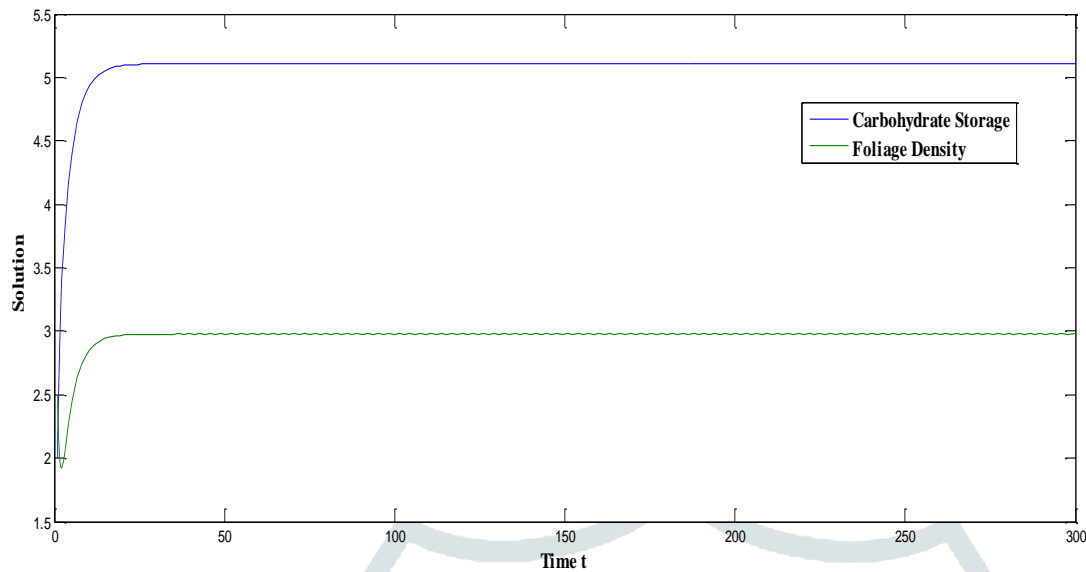


Figure 1. The equilibrium $E^*(5.1058, 2.9820)$ is stable in the absence of delay i.e. $\tau = 0$

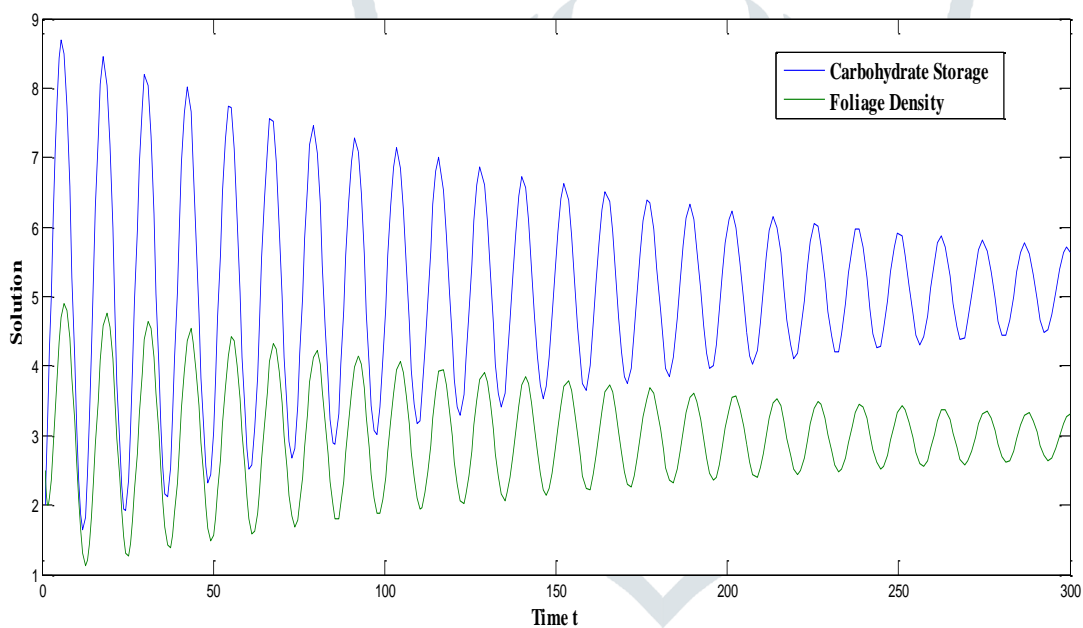


Figure 2. The equilibrium $E^*(5.1058, 2.9820)$ is asymptotically stable when delay is below the critical point i.e. $\tau < 2.41$

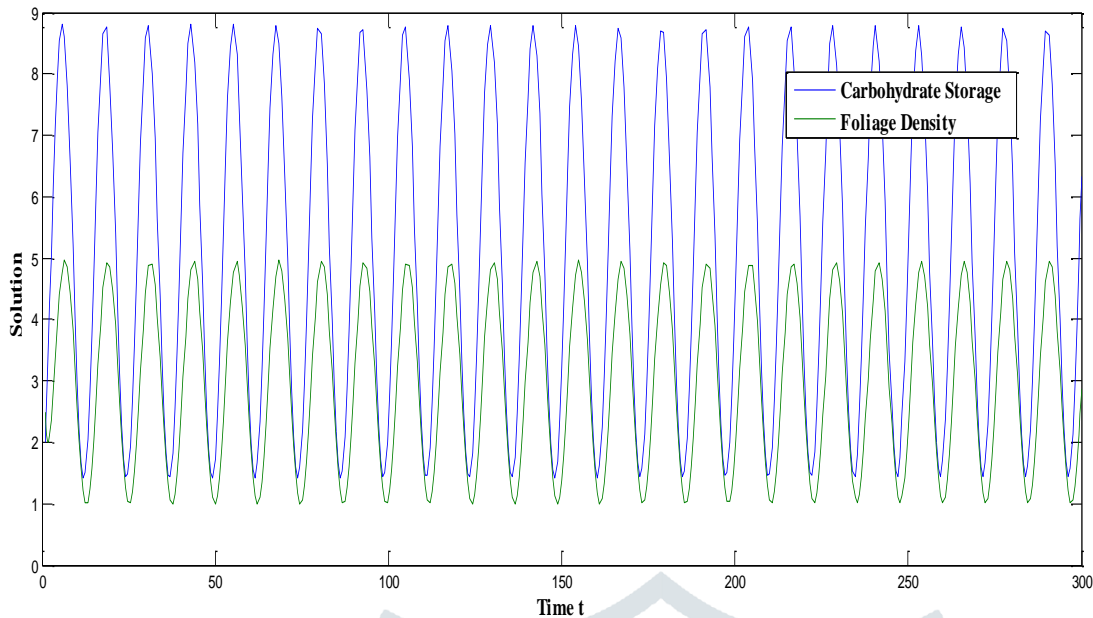


Figure 3. The equilibrium $E^*(5.1058, 2.9820)$ shows Hopf bifurcation when the delay passes through the critical point i.e. $\tau \geq 2.41$

Conclusion

The role of delay on carbohydrates uptake and foliage growth mechanism is studied with the help of proposed model. The state variables considered are: carbohydrates store and foliage density. The boundedness of the system is proved using usual comparison theorem. Positivity of the solutions shows that both the variables considered being real in natural phenomenon always remain positive at all the times. The stability analysis of the equilibrium E^* is carried out. In the absence of delay, the equilibrium E^* is absolutely stable as shown in figure 1. The same fact is also supplemented by $(\mathcal{H}_1) - (\mathcal{H}_2)$ as in conjecture 1. When the value of delay parameter τ is below the critical point i.e. $\tau < 2.41$, the equilibrium starts losing stability and leads to asymptotical stability as shown in figure 2. The moment, the delay parameter τ crosses the critical value i.e. $\tau \geq 2.41$, the equilibrium exhibits the complex dynamics in the form of Hopf bifurcation. This observation of complex behaviour shown by the system (1)-(2) is in agreement with $(\mathcal{H}_4) - (\mathcal{H}_5)$ as in conjecture 2.

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