A review of special summability methods and generation of some new sequence spaces

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Abstract-In this paper we present review of some special summability methods and we will also discuss results associated to these methods.

Weighted mean method

Definition. Let P_n represents the sequence of nonnegative numbers so that $P_0 > 0$,that is

$P_n \geq 0$, n = 1,2,... and $P_0 > 0$.The weighed mean technique $(\overline{N}, P_n)$is represented by the infinite matrix (a_{nk}),in which (a_{nk}) is defined by

$$a_{nk}=\begin{cases} \frac{p_k}{P_n}, & k \leq n \\ 0, & k > n. \end{cases}$$

Theorem: The $(\overline{N}, P_n)$ method is regular if and only if $\lim_{n\to\infty} P_n = \infty$.

Theorem:(Limitation theorem)If $P_n > 0$, for all n and $\{s_n\}$ is $(\overline{N}, P_n)$ summable to s, then

$$s_n - s = o\left(\frac{P_n}{p_n}\right), n \to \infty.$$

Theorem: If ${P_{n+1}}/{P_n} \geq 1 + \delta > 1$,then $\{S_n\}$ cannot be $(\overline{N}, P_n)$ summable unless it is convergent.[1],[2],[3]

The Abel's Method and (C,1) method

Abel's method is not possible to be defined by an infinite matrix method so that we have "non-matrix summability methods"[3]. The Abel's technique can also be called as a semi continuous technique.

Definition: A sequence $\{a_n\}$ is called Abel summable, written as (A) summable to L if

$\lim_{x\to 1-} (1 - x) \sum_k a_k x^k$ exists finitely and is equal to L,

Theorem: If $\{a_n\}$ converges to L , then $\{a_n\}$ is Abel summable to L.

Definition: A sequence $\{a_n\}$ is said to be $(C, 1)$ summable to L if $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} a_k$ exists finitely and equals L.

From the above definition, we see that the notion of $(C, 1)$ summability is to take the arithmetic mean of the terms of the given sequence and study the convergence of these means.

From the above definition we can see that $(C, 1)$ summability is represented through the infinite matrix

$$a_{nk} = \begin{cases} \frac{1}{n+1}, k \leq n; \\ 0, \quad k > n. \end{cases}$$

Theorem: The $(C, 1)$ method is regular.

Holder's Method

Definition: The $(H, 1)$ method is represented through matrix $\left(h_{nk}^{(1)}\right)$,where

$$h_{nk}^{(1)} = \begin{cases} \frac{1}{n+1}, k \leq n; \\ 0, k > n. \end{cases}$$

If m is a positive integer, the Holder method [4] of order m,denoted by(H, m),is represented though the infinite matrix $\left(h_{nk}^{(m)}\right)$,with

$$\left(h_{nk}^{(1)}\right) = \left(h_{nk}^{(1)}\right)\left(h_{nk}^{(1m-1)}\right),$$

here the product of two matrices denotes usual matrix multiplication.

The Hausdroff Method

Definition: Let $x = \{x_k\}$ is a real number sequence. Define

$$(\Delta^0 x)_n = x_n$$

$$(\Delta^1 x)_n = x_n - x_{n+1},$$

and

$$\left(\Delta^j x\right)_n = \left(\Delta^{j-1} x\right)_n - \left(\Delta^{j-1} x\right)_{n+1}, j = 2,3 \ldots$$

The sequence $x = \{x_k\}$ is said to be "totally monotone" if

$\left(\Delta^j x\right)_n \geq 0$ for all n, j.

Definition: Define the matrix $\delta = (\delta_{nk})$ by

$$\delta_{nk} = \begin{cases} (-1)^k n_{C_k}, if k \leq n; \\ 0, \quad if k > n. \end{cases}$$

Definition- If $\mu = (\mu_{nk})$ is a diagonal matrix, then the method defined by the infinite matrix $u = (u_{nk})$,where

$$u = \delta\mu\delta = (\delta_{nm})(\mu_{mj})(\delta_{jk})$$

is called a Hausdorff method, denoted by (H, μ).

The Natarajan Method

In a process to elaborate the Norlund method, Natarajan[5] introduced the (M, λ_n) method as follows:

Given a sequence $\{\lambda_n\}$ of numbers such that $\sum_n |\lambda_n| < \infty$,the (M, λ_n) process is represented thtough infinite matrix (a_{nk}),where

$$a_{nk} = \begin{cases} \lambda_{n-k}, k \leq n; \\ 0, k > n. \end{cases}$$

Natarajan method (M, λ_n) is a nontrivial summability method, that is, it is not equivalent to convergence.

Theorem: The (M, λ_n)-method is regular if and only if $\sum_n \lambda_n = 1$.

Theorem: Any two (M, λ_n) and (M, μ_n) methods which are regular are consistent.

Theorem: If $\{a_n\}$ is (M, λ_n) −summable to s,where (M, λ_n) is regular, then $\{a_n\}$ is said to be Abel's-summable to s.

The Euler method:

Definition: Let $r \in \boldsymbol{C} - \{1,0\}, \boldsymbol{C}$ is the complex number field. The Euler technique [6],[7],[8] of r order or the (E, r) technique is represented through infinite matrix $e_{nk}^{(r)} = \begin{cases} {n_{C_k}} r^k (1-r)^{n-k}, k \leq n \\ 0, \qquad k > n \end{cases}$

For $r \in \{1,0\}$, the (E, r) technique is defined respectively trough the infinite matrices $\left(e_{nk}^{(1)}\right)$ and $\left(e_{nk}^{(0)}\right)$, where

$$e_{nk}^{(0)} = \begin{cases} 1, k = n; \\ 0, k \neq n. \end{cases}$$

$$e_{nk}^{(0)} = 0, n = 0,1,2 \dots.; k = 1,2, \dots$$

$$e_{nk}^{(0)} = 1, n = 0,1,2, \dots$$

Theorem: The (E, r) is regular if and only if r is real and $0 < r \leq 1$.

The idea of a paranorm is intimately associated to linear metric spaces. This is a mere extension modulus of a complex numbers or absolute value of real numbers.[9]

A linear space along with a function $H: X \to R_+$ which fulfils the given axioms is called a Para normed space (X, H).

1) $H(\theta) = 0$;

2): $H(x) = H(-x)$ for all $x \in X$;

3):$H(x + y) \leq H(x) + H(y)$ for all $x, y \in X$;and

4): Suppose (α_n) is a scalars sequence such that $\alpha_n \to \alpha \; as \; n \to \infty$ and (x_n) is a sequence in X with $H(x_n - x) \to 0, as \; n \to \infty$, then $H(\alpha_n x_n - \alpha x) \to 0 \; as \; n \to \infty$

A Para normed space (X, H) is defined as complete given (X, d) is complete with metric $d(x, y) = H(x - y)$.

Nakano and Simons were the pioneers at the initial stage in the studies of paranormed sequence. Maddox [10] and many others delved into its nuances further. Multiple others ([11],[12])went on to study paranormed sequence spaces through the use of Orlicz function.

Orlicz Sequence Space l_ϕ

Orlicz sequence was developed to discuss Banach space related theory [13],[14],[15]. An Orlicz function is defined as a function $\phi: [0, \infty) \to [0, \infty)$ which holds the property of being non-decreasing, continuous and convex with

$\phi(0) = 0, \phi(u) > 0 \; for \; u > 0$, and $\phi(u) \to \infty \; as \; u \to \infty$.

Tzafriri and Lindenstrauss [8] applied Orlicz function to develop sequence space

$$l_\phi = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} \phi\left(\frac{|x_k|}{\rho}\right) < \infty for \; some \; \rho > 0 \right\}$$

of scalars, that turns into Banach space along with Luxemburg norm given by

$$\|x\|_\phi = inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} \phi\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

l_ϕ is known an Orlicz sequence space which is intimately associated to l_p space with $\phi(x) = x^p, (1 \leq p < \infty)$. They possess very rich geometrical and topological properties that re devoid in ordinary l_p spaces.

Other Sequence Spaces:

We will represent $e \; and \; e^{(n)} (n = 1,2, \dots)$ for the sequences so that $e_k = 1$ for $k = 1,2, \dots$

And

$$e_k^{(n)} = \begin{cases} 1, (for\ k = n) \\ 0, (for k \neq n). \end{cases}$$

Let m is a nonnegative integer, the m-section of a sequence $x = \{x_k\}$ by $x^{[m]}$, i.e

$$x^{[m]} = \sum_{k=1}^{\infty} x_k e^{(k)}.$$

The space of BS, bounded series represents sequences X with $\sup_n |\sum_{k=1}^{n} x_k| < \infty$.

The space BS is equipped with the following norm

$$\|x\|_{BS} = \sup_n |\sum_{k=1}^{n} x_k|,$$

Gives a Banach space which is isometrically isomorphic to l^{∞},through linear mapping

$$(x_n)_{n\in N} \to \left(\sum_{k=1}^{n} x_k\right)_{n\in N}$$

A sequence $\theta = (k_r)$ of positive integers with $k_0 = 0, 0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$ is known a Lacunary sequence. Intervals $I_r = (k_{r-1}, k_r]$ are determined by θ and the ratio $\frac{k_r}{k_{r-1}}$ are denoted as q_r.

References

1. Altay, Bilal, and Feyzi Başar. "Generalization of the sequence space ℓ (p) derived by weighted mean." Journal of Mathematical Analysis and Applications 330.1 (2007): 174-185.
2. Candan, Murat, and Asuman Güneş. "Paranormed sequence space of non-absolute type founded using generalized difference matrix." Proceedings of the National Academy of Sciences, India Section A: Physical Sciences 85.2 (2015): 269-276.
3. Natarajan, P. N. "Classical Summability Theory." (2017).
4. Canak, Ibrahim, Umit Totur, and Zerrin Onder. "A Tauberian theorem for (C, 1, 1) summable double sequences of fuzzy numbers." Iranian Journal of Fuzzy Systems 14.1 (2017): 61-75.
5. Braha, Naim L., and Mikail Et. "The sequence space $E_n^q(M, p, s)$ and N_k lacunary statistical convergence." Banach Journal of Mathematical Analysis 7.1 (2013): 88-96.
6. Polat, Harun, and Feyzi Basar. "Some Euler spaces of difference sequences of order m." Acta Math. Sci. B 27 (2007): 254-266.

7. Et, Mikail, and Rıfat Çolak. "On some generalized difference sequence spaces." *Soochow Journal of Mathematics* 21.4 (1995): 377-386.

8. Lascarides, Constantine G. "A study of certain sequence spaces of Maddox and a generalization of a theorem of Iyer." *Pacific Journal of Mathematics*38.2 (1971): 487-500.

9. Bhardwaj, Vinod K., and Indu Bala. "Banach space valued sequence space lM (X, p)." *Int. J. of Pure and Appl. Maths* 41.5 (2007): 617-626.

10. Khan, Vakeel A. "On a new sequence space defined by Orlicz functions." *Commun. Fac. Sci Univ. Ank. Series A* 1.57 (2008): 25-33.

11. Lindenstrauss, Joram, and Lior Tzafriri. "Orlicz sequence spaces." Classical Banach Spaces. Springer Berlin Heidelberg, 1973. 67-92.

12. Parashar, S. D., and B. Choudhary. "Sequence spaces defined by Orlicz functions." Indian Journal of Pure and Applied Mathematics 25 (1994): 419-419.

13. Kolk, Enno. "On generalized Orlicz sequence spaces defined by double sequences." Fasciculi Mathematici 55.1 (2015): 65-80.