Study of solitary waves

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Abstract

Differential equations have a striking capacity to foretell the world around us. They are accustomed in number of disciplines of sciences and engineering. Some differential equations are easy to sole while some leads to a complex form that is not easy to get the analytical solution of these complex problems which created the demand for advanced numerical methods. Numerical method is used to find the approximate solution of the given numerical problems or differential equations. There are two ways to find the solution of differential equation: Analytically and Numerically. The analytic approach gives the exact solution whereas numerical method gives the approximate solution. Then the question arises that why we need approximate solutions if we already have exact solutions? The answer to this basic question is that there are some complex differential equations whose exact solutions cannot be found, then in that cases we can find approximate solution which will tend to approach exact solution after removing errors. The aim of this work is to present collocation method with hybrid B-Spline basis functions to get the solution for a partial differential equation.

Introduction

Study of solitary waves play an important role to know the equivalent processes of particle physics. Generalized equal width (GEW) equation is one of the partial differential equations that results in the secondary solitary waves solutions. This non-linear wave equation was first expressed by Peregrine and Benjamin et.al [1, 2]. This equation has many applications related to generation of waves, for example it is used to model unidirectional transmitting of waves in water cannel and has application in study of long crested waves that generate near the seashore etc.

Generalized equal width (GEW) equation is of the form

$u_t + \varepsilon u^p u_x - \mu u_{xxt} = 0$

where ε, μ and *p* are the positive constants, to be solve with the boundary conditions u approaching zero as x tends to infinity

For p=1, the given equation reduced to an equal width equation (EW). Up to now the EW equation are solved by many numerical and exact solution methods. These methods includes Galerkins method with cubic B-spline finite elements[3], Petrov-Galerkin method using a quadratic B-spline finite elements [4], Collocation method based on cubic, quartic and septic B-spline [5,6], and a least squares technique with space-time linear finite elements [7].

For p=2, above equation results in a modified equal width equation (MEW). The MEW equation has also been solved numerically using various numerical techniques. For instance, the equation has been solved by the lumped-Galerkin method based on B-spline function [8], collocation method [9] and Petrov-Galerkin approach [10].

Thus, the GEW equation has a foundation upon the EW equation and is connected to the generalized regularized long wave (GRLW) equation and the generalized Korteweg-de-Vries (GKdV) equation. These all are wave equations that result in solitary solutions. Until now, the GEW wave equation has been solved by using many analytical and numerical solution techniques. Panahipour [11] solved GEW wave equation by using radial basis function (RBF) approach. Moving least squares collocation (MLSC) method was implemented by Kalpan and Derel [12] to get the numerical solution of GEW equation. Karakoc and Zeybek [13] applied septic B-Spline collocation method and used two distinct linearization techniques for solving the GEW wave equation. For the numerical simulations of the GEW wave equation and GRLW equation, Battal and Zeybek [14] used lumped Galerkin approach with cubic B-spline functions. Mohammadi [15] used exponential B-spline collocation method solve GRLW equation. Collocation method with quadratic B-spline was studied for the GEW wave equation by Evans and Raslan [16].

In this work the GEW equation is treated for the numerical simulation by a new approach. This approach is based upon the concept of hybrid basis functions which are the linear combinations of cubic B-spline and cubic trigonometric B-spline basis function with collocation method.

The collocation method is a well-known numerical technique to solve various differential equations with a good accuracy. This technique has been employed to solve a variety of problems in engineering and physical sciences. This method is popular due to its capability to obtain highly accurate solution with less computational work.

B-spline

B-spline is an important tool in computer graphics and is successfully implemented in finding the solution of differential equations arising in various areas of engineering and sciences. B-spline is a name given to basic type of spline. Most other spline functions can be written as the linear combinations of B-splines. The test functions include various other forms of B-splines such as standard B-spline, cubic B-spline, quartic B-spline, trigonometric cubic B-spline, exponential cubic B-spline and some other basis functions has also been used due to their disctinct properties such as sinc function, lagrange interpolating polynomials, legendre polynomial etc. Till now, a number of equations have been solved by various numerical methods namely B-spline basis function, collocation method, finite element method and differential quadrature method etc. But B-spline collocation method has proved to be a good technique to solve various differential equations. Some of the equations which are solved till today include solution with different order B-spline with different types- standard, exponential, trigonometric and quadrature.

The concept of B-spline is actively used to find the numerical solution of partial differential equations. In mathematics, a spline function is a smooth piecewise polynomial approximation. The points where spline intersect are called knots and where basis spline functions crosses the number of points are called control points. Each control point is associated with unique basis function. The core of basis function is that they definitely are continuous at knots but its derivatives may or may not be continuous depending whether the neighbouring knots are distant or not.

$$x(u) = \sum_{i=0}^{n} N_{i,k}(u)x \; ; \; 0 \le u \le n-k+2$$

here k is the order of curve, n+1 are the control points and curve is made up of n-k+2 segments. The basis function is given by

$$N_{i,k}(u) = \frac{(u-t_i)N_{i,k-1}(u)}{t_{i+k-1}-t_i} + \frac{(t_{i+k}-u)N_{i+1,k-1}(u)}{t_{i+k}-t_{i+1}}$$

Where t_i $(0 \le i \le n+k)$ are the knot values which are given as

$$\begin{cases} t_i = 0 & \text{if } i<k \\ t_i = i-k+1 & \text{if } k \le i \le n \\ t_i = n-k+2 & \text{if } i>n \end{cases}$$

where

$$N_{i,k}(u) = \begin{cases} 1 \text{ if } t_i \le u \le t_{i+1} \\ 0 \text{ otherwise} \end{cases}$$

There are various degrees of B-splines defined and available in this work the third-degree basis function is applied [17,18].

Third degree B-spline

The third degree B-spline which is also known as Cubic B-spline basis function is defined with the x_m as a centre knot and same number of knots on its each side. The third degree B-spline is given by the formula

$$B_{m,3} = \frac{1}{h^3} \begin{cases} (x-x_{m-2})^3 & x \in [x_{m-2}, x_{m-1}) \\ (x-x_{m-2})^3 - 4(x-x_{m-1})^3 & x \in [x_{m-1}, x_m) \\ (x_{m+2}-x)^3 - 4(x_{m+1}-x)^3 & x \in [x_m, x_{m+1}) \\ (x_{m+2}-x)^3 & x \in [x_{m+1}, x_{m+2}) \\ 0 & \text{otherwise} \end{cases}$$

From the above definition, the values upto two derivates of $B_m(x)$ by differentiating it with respect to x. The values of $B_m(x)$ and its first and second order derivatives are given in a tabular form as under:

	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}
$B_m(x)$	0	1	4	1	0
$B'_m(x)$	0	3/h	0	-3/h	0
$B''_m(x)$	0	$6/h^2$	$-12/h^2$	$6/h^2$	0

To solve a differential equation with collocation approach using B-Spline basis function, the approximate solution U(x,t) can be defined as follows:

$$U(x,t) = \sum_{j=m-k+2}^{m+k-2} c_j B_j(x)$$

Where k denotes the degree of B-spline with m number of nodes, while the unknown are given by c_m, which are the constants obtained from boundary conditions of the differential equation.

Now, by substituting k=3 we obtain a approximate solution for the cubic B-spline

By substituting the values of $B_m(x)$ at the knots as given in the mentioned table in the above equation, the approximate solution can be calculated. In the same way, we can calculate the approximate solution for its first as well as second derivative in terms of time parameters cm's which can be written as

$$U(x_m,t) = c_{m-1} + 4c_m + c_{m+1}$$
$$hU'(x_m,t) = 3(c_{m+1} - c_{m-1})$$
$$h^2U''^{(x_m,t)} = 6(c_{m-1} - 2c_m + c_{m+1})$$

Trigonometric B-Spline

The cubic trigonometric B-spline basis functions at knots is given by

$$CTB_j(x) = \frac{1}{\omega}$$

$$\begin{cases} p^3(x_m), & if\ x \in [x_m, x_{m+1}) \\ p(x_m)\big(p(x_m)q(x_{m+2}) + q(x_{m+3})p(x_{m+1})\big) + q(x_{m+4})p^2(x_{m+1}), & if\ x \in [x_{m+1}, x_{m+2}) \\ q(x_{m+4})\big(p(x_{m+1})q(x_{m+3}) + q(x_{m+4})p(x_{m+2})\big) + p(x_m)q^2, & if\ x \in [x_{m+2}, x_{m+3}) \\ q^3(x_{m+4}), & if\ x \in [x_{m+3}, x_{m+4}) \\ 0 & otherwise \end{cases}$$

where $p(x_m) = \sin(\frac{x-x_m}{2})$, $q(x_m) = \sin(\frac{x_m-x}{2})$, $\omega = \sin\left(\frac{h}{2}\right)\sin(h)\sin(\frac{3h}{2})$ and $h = \frac{b-a}{N}$

In this work, the modified form of cubic trigonometric B-spline basis functions is as mentioned below which are obtained from trigonometric B-spline basis in order to make the resulting martrix system of equations diagonally dominant

$CTB_1(X) = T_1(X) + 2T_0(X)$

$CTB_2(X) = T_2(X) - T_0(X)$

$CTB_m(X) = T_m(x)$ for m= 3,..., N-2

$CTB_{N-1}(X) = T_{N-1}(x) – T_{N+1}(x)$

$CTB_N(x) = T_N(X) + 2T_{N+1}(X)$

Mathematical Formulation

Let us consider GEW equation as

$$u_t + \in u^P u_x - \mu u_{xxt} = 0 \qquad x \in \Omega = (a, b) \subset R, \quad t > 0 \qquad (1)$$

with the initial condition

$$u(x, 0) = f(x), \qquad x \in \Omega = [a, b] \qquad (2)$$

and the boundary condition

$$u(a, t) = g_a(t), \quad u(b, t) = g_b(t), \quad t \geq 0 \tag{3}$$

where $\in$ and μ are positive parameters and $f(x)$, $g_a(t)$ and $g_b(t)$ are known functions. The time derivative in equation (1) is to be discretized in the usual finite difference way and space derivatives by the θ-weighted ($0 < \theta < 1$) scheme between two consecutive time levels n and $n+1$ as

$$\frac{u^{n+1} - u^n}{\partial t} + \theta\,(\in u^p u_x)^{n+1} + (1-\theta)\,(\in u^p u_x)^n - \frac{\mu}{\partial t}\,(u_{xx}^{n+1} - u_{xx}^n) = 0 \tag{4}$$

where $u^n = u(x, t^n)$, $t^n = t^{n-1} + \delta t$ and δt is a time step size.

The non-linear term $(u^p u_x)^{n+1}$ in (4) can be approximated by the formulas derived from the Taylor expansion.

$$(u^P)^{n+1} \approx (u^P)^n + \delta t\,((u^P)_t)^n$$

$$\approx (u^P)^n + \delta t p\,(u^{P-1})^n\left(\frac{u^{n+1} - u^n}{\partial t}\right) + O(\delta t^2)$$

$$(u_x)^{n+1} \approx (u_x)^n + \delta t \frac{u_x^{n+1} - u_x^n}{\partial t} + O(\delta t^2)$$

therefore

$$(u^P u_x)^{n+1} = (u^P u_x)^n + \delta t\,((u_t^P)^n u_x + (u^P)^n u^n{}_{xt}) + O(\delta t^2)$$

$$= (u^P u_x)^n + \partial t\left(\frac{(u^P)^{n+1} - (u^P)^n}{\partial t} u_x^n + (u^P)^n \frac{u_x^{n+1} - u_x^n}{\partial t}\right) +$$

$$= (u^P)^{n+1} u_x^n + (u^P)^n u_x{}^{n+1} - (u^P)^n u_x{}^n + O(\delta t^2)$$

$$= (u^P)^n u_x^{n+1} + P\,(u^{P-1})^n u_x^n u^{n+1} - P\,(u^P)^n u_x^n + O(\delta t^2) \tag{5}$$

Hence the equation (4) can be modified as

$$u^{n+1} + \delta t \in ((u^n)^P u_x{}^{n+1} + P\,(u^n)^{P-1} u_x{}^n u^{n+1}) - \mu\, u_{xx}{}^{n+1}$$

$$= u^n + \delta t \in (((P+1)\,\theta - 1)\,(u^n)^P u_x{}^n) - \mu u_{xx}{}^n \tag{6}$$

Now, let us choose the collocation points x_i, $i = 1, \ldots N$ such that x_i, $i = 2, \ldots, N-1$ are interior points and x_i, $i = 1, N$ are boundary points and then apply the following approximation

$$U^N(x, t) = \sum_{j=-1}^{N+1} c_j(t)\, L_j(x), \tag{7}$$

where $c_j(t)$ is time dependent unknowns to be calculated and $L_j(x)$ is linear combination between cubic B-Spline Collocation method which is explained as follows

To construct numerical solution, consider nodal points (x_j, t_n) defined in the region $[a,b]*[0,T]$, where

$$a = x_0 < x_1 < \ldots < x_N = b, \quad h = x_{j+1} - x_j = \frac{b-a}{N}, \quad j = 0, 1, \ldots, N,$$

$$0 = t_0 < t_1 < \ldots < t_n < \ldots < T, \quad t_n = n\Delta t, \quad n = 0, 1, \ldots .$$

Through this, linear combination between cubic B-Splines (LCCBS) with different basis functions is used to solve the considered GEW equation (1). The approximated solution is given in above mentioned eq. (7) where $L_j(x)$ is representing

$$L_j(x) = \gamma \, CTB_j(x) + (1-\gamma) \, B_j(x)$$

In the above defined expression, magnitude of gamma plays an imperative role. It reduces to standard cubic B-spline when its value is zero while it is come to express as only trigonometric cubic B-spline basis functions when it is one. Hence, the value of gamma can be in between 0 and 1.

Because of local support property of B-spline basis function, there exists only three nonzero basis functions, namely, Lj-1, Lj and Lj+1. Using the values of the two basis functions with the simplification required for the unknown functions C_{-1} nmad $C_{N+1}$from the initial and boundary conditions give

$$c_{-1} = \frac{g_1 - \alpha_2 c_0 + \alpha_1 c_1}{\alpha_1} \quad \text{and} \quad c_{N+1} = \frac{g_2 - \alpha_1 c_{N-1} + \alpha_2 c_N}{\alpha_1}$$

On substituting the values in the formulation form in (6) leads to (N+1) square matrix system AX=B given as follows:

$$A = \begin{bmatrix} \left(r - \frac{\alpha_2}{\alpha_1} s\right) & (v-s) & 0 & 0 & 0 & 0 & 0 \\ s & r & v & 0 & 0 & 0 & 0 \\ 0 & s & r & v & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \left(r - v\frac{\alpha_2}{\alpha_1}\right) \end{bmatrix}$$

$$B = \begin{bmatrix} R - \frac{s}{\alpha_1} g_1 \\ R \\ R \\ \vdots \\ \vdots \\ R \\ R - \frac{v}{\alpha_1} g_2 \end{bmatrix}$$

This is a tri-diagional system of equations required to be solved to obtain the solutions.

Conclusion

In this manuscript, a new technique of linear combination of two B-spline basis functions is implemented for the solution of famous GEW equation. The method is simple and straight forward. We can further calculate its numerical solution and can run this through test problems to check the accuracy of the solution we have derived.

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