New Identities on Fibonacci Triple Sequence

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Keywords: Multiplicative Triple Fibonacci sequence

Abstract: In this paper firstly, we have defined one of the schemes of Multiplicative triple Fibonacci sequence, have established and prove new generalised identities based on it.

1. Introduction

The contribution of Fibonacci and Lucas number are significant over the centuries. We have seen example of coupled differential equation in various application of mathematics. Many mathematicians have generalised fascinating properties on well-known Fibonacci sequence, but the concept of Fibonacci triple sequence is less known to us. It was first introduced by Jin-Zai Lee & Jia-Sheng Lee [1] in 1987. There are different schemes possible for Fibonacci triple sequence, in this paper we have established some new results of multiplicative triple Fibonacci Sequences of the one of the schemes.

2. Multiplicative Triple Fibonacci sequence

The Multiplicative Triple Fibonacci sequence is defined by the recurrence relation

$$\alpha_{n+2} = \gamma_{n+1}\gamma_n\,, \qquad \beta_{n+2} = \alpha_{n+1}\alpha_n, \qquad \gamma_{n+2} = \beta_{n+1}\beta_n \tag{2.1}$$

for all integer $n \geq 0$, with initial conditions

$$\alpha_0 = a,\ \ \alpha_1 = d,\ \ \beta_0 = b,\ \ \beta_1 = e,\ \ \gamma_0 = c,\ \ \gamma_1 = f$$

Where $a,\ d,\ b,\ e,\ c$ and f are real numbers

Theorem 2.1 If α_n, β_n and γ_n are define by equation (2.1) then (for $n > 1$)

$$\alpha_{n+9} = \left(\prod_{i=n+1}^{n+6} \beta_i\right)\left(\prod_{j=n+2}^{n+4} \gamma_j\right)\left(\prod_{k=n+3}^{n+4} \gamma_k\right)\gamma_{n+4} \tag{2.2}$$

Proof: Theorem can be proved by mathematical induction method on n

For $n = 1$ by equations (2.1) and (2.2)

$$\left(\prod_{i=2}^{7} \beta_i\right)\left(\prod_{j=3}^{5} \gamma_j\right)\left(\prod_{k=4}^{5} \gamma_k\right)\gamma_5 = (\beta_2\,\beta_3\beta_4\beta_5\beta_6\beta_7)(\gamma_3\gamma_4\gamma_5)(\gamma_4\gamma_5)\gamma_5$$

by using equation (2.1) repeatedly we have

$$\left(\prod_{i=2}^{7} \beta_i\right)\left(\prod_{j=3}^{5} \gamma_j\right)\left(\prod_{k=4}^{5} \gamma_k\right)\gamma_5 = \alpha_{10}$$

which proves for $n = 1$

Suppose the theorem is true for $n = m$, so by equation (2.2)

$$\alpha_{m+9} = \left(\prod_{i=m+1}^{m+6} \beta_i\right)\left(\prod_{j=m+2}^{m+4} \gamma_j\right)\left(\prod_{k=m+3}^{m+4} \gamma_k\right)\gamma_{m+4} \tag{2.3}$$

Now to prove for $n = m + 1$, by using equation (2.1) and (2.2)

$$\left(\prod_{i=(m+1)+1}^{(m+1)+6} \beta_i\right)\left(\prod_{j=(m+1)+2}^{(m+1)+4} \gamma_j\right)\left(\prod_{k=(m+1)+3}^{(m+1)+4} \gamma_k\right) \gamma_{(m+1)+4}$$
$$= (\beta_{m+2}\, \beta_{m+3}\beta_{m+4}\beta_{m+5}\beta_{m+6}\beta_{m+7})(\gamma_{m+3}\gamma_{m+4}\gamma_{m+5})(\gamma_{m+4}\gamma_{m+5})\gamma_{m+5}$$

by using equation (2.1) we have

$$\left(\prod_{i=(m+1)+1}^{(m+1)+6} \beta_i\right)\left(\prod_{j=(m+1)+2}^{(m+1)+4} \gamma_j\right)\left(\prod_{k=(m+1)+3}^{(m+1)+4} \gamma_k\right) \gamma_{(m+1)+4} = \alpha_{(m+1)+9}$$

which proves the theorem.

Theorem 2.2 If α_n, β_n and γ_n are define by equation (2.1) then (for $n > 1$)

$$\beta_{n+9} = \left(\prod_{i=n+1}^{n+6} \gamma_i\right)\left(\prod_{j=n+2}^{n+4} \alpha_j\right)\left(\prod_{k=n+3}^{n+4} \alpha_k\right) \alpha_{n+4} \qquad (2.4)$$

Proof: Theorem can be proved by mathematical induction method on n

For $n = 1$ by equations (2.1) and (2.4)

$$\left(\prod_{i=2}^{7} \gamma_i\right)\left(\prod_{j=3}^{5} \alpha_j\right)\left(\prod_{k=4}^{5} \alpha_k\right) \alpha_5 = (\gamma_2\, \gamma_3\gamma_4\gamma_5\gamma_6\gamma_7)(\alpha_3\alpha_4\alpha_5)(\alpha_4\alpha_5)\alpha_5$$

by using equation (2.1) we have

$$\left(\prod_{i=2}^{7} \gamma_i\right)\left(\prod_{j=3}^{5} \alpha_j\right)\left(\prod_{k=4}^{5} \alpha_k\right) \alpha_5 = \beta_{10}$$

which proves for $n = 1$

Suppose the theorem is true for $n = m$, so by equation (2.4)

$$\beta_{m+9} = \left(\prod_{i=m+1}^{m+6} \gamma_i\right)\left(\prod_{j=m+2}^{m+4} \alpha_j\right)\left(\prod_{k=m+3}^{m+4} \alpha_k\right) \alpha_{m+4} \qquad (2.5)$$

Now to prove for $n = m + 1$, by using equation (2.1) and (2.4)

$$\left(\prod_{i=(m+1)+1}^{(m+1)+6} \gamma_i\right)\left(\prod_{j=(m+1)+2}^{(m+1)+4} \alpha_j\right)\left(\prod_{k=(m+1)+3}^{(m+1)+4} \alpha_k\right) \alpha_{(m+1)+4}$$
$$= (\gamma_{m+2}\, \gamma_{m+3}\gamma_{m+4}\gamma_{m+5}\gamma_{m+6}\gamma_{m+7})(\alpha_{m+3}\alpha_{m+4}\alpha_{m+5})(\alpha_{m+4}\alpha_{m+5})\alpha_{m+5}$$

by using equation (2.1) we have

$$\left(\prod_{i=(m+1)+1}^{(m+1)+6} \gamma_i\right)\left(\prod_{j=(m+1)+2}^{(m+1)+4} \alpha_j\right)\left(\prod_{k=(m+1)+3}^{(m+1)+4} \alpha_k\right) \alpha_{(m+1)+4} = \beta_{(m+1)+9}$$

which proves the theorem.

Theorem 2.3 If α_n, β_n and γ_n are define by equation (2.1) then (for $n > 1$)

$$\gamma_{n+9} = \left(\prod_{i=n+1}^{n+6} \alpha_i\right)\left(\prod_{j=n+2}^{n+4} \beta_j\right)\left(\prod_{k=n+3}^{n+4} \beta_k\right)\beta_{n+4} \tag{2.6}$$

Proof: Theorem can be proved by mathematical induction method on n

For $n = 1$ by equations (2.1) and (2.6)

$$\left(\prod_{i=2}^{7} \alpha_i\right)\left(\prod_{j=3}^{5} \beta_j\right)\left(\prod_{k=4}^{5} \beta_k\right)\beta_5 = (\alpha_2\,\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7)(\beta_3\beta_4\beta_5)(\beta_4\beta_5)\beta_5$$

by using equation (2.1) repeatedly we have

$$\left(\prod_{i=2}^{7} \alpha_i\right)\left(\prod_{j=3}^{5} \beta_j\right)\left(\prod_{k=4}^{5} \beta_k\right)\beta_5 = \gamma_{10}$$

which proves for $n = 1$

Suppose the theorem is true for $n = m$, so by equation (2.6)

$$\gamma_{m+9} = \left(\prod_{i=m+1}^{m+6} \alpha_i\right)\left(\prod_{j=m+2}^{m+4} \beta_j\right)\left(\prod_{k=m+3}^{m+4} \beta_k\right)\beta_{m+4} \tag{2.7}$$

Now to prove for $n = m + 1$, by using equation (2.1) and (2.6)

$$\left(\prod_{i=(m+1)+1}^{(m+1)+6} \alpha_i\right)\left(\prod_{j=(m+1)+2}^{(m+1)+4} \beta_j\right)\left(\prod_{k=(m+1)+3}^{(m+1)+4} \beta_k\right)\beta_{(m+1)+4}$$
$$= (\alpha_{m+2}\,\alpha_{m+3}\alpha_{m+4}\alpha_{m+5}\alpha_{m+6}\alpha_{m+7})(\beta_{m+3}\beta_{m+4}\beta_{m+5})(\beta_{m+4}\beta_{m+5})\beta_{m+5}$$

by using equation (2.1) we have

$$\left(\prod_{i=(m+1)+1}^{(m+1)+6} \alpha_i\right)\left(\prod_{j=(m+1)+2}^{(m+1)+4} \beta_j\right)\left(\prod_{k=(m+1)+3}^{(m+1)+4} \beta_k\right)\beta_{(m+1)+4} = \gamma_{(m+1)+9}$$

which proves the theorem.

References

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